Chapter 12

WAVE GUIDES

Wave guides or hollow metal tubes are now employed extensively for transmission of the very high frequencies where the attenuation caused by the wave guide is smaller than that for any other form of transmission line. In order to obtain an understanding or physical picture of electric wave transmission through hollow metal tubes, the previous chapter analyzed the directing of electromagnetic waves between infinite parallel metal planes. It is now possible to complete the picture by adding two more sides to form a complete tube. It will be demonstrated that the approach of the previous chapter was helpful, since many of the wave properties will be unaffected and unchanged by the presence of the sides on the guide.

The guides will first be assumed made of perfect metallic conductors. If it is desired to determine the attenuation due to finite conductivity, the methods of the preceding section may be employed.

12-1. Application of Maxwell's equations to the rectangular wave guide

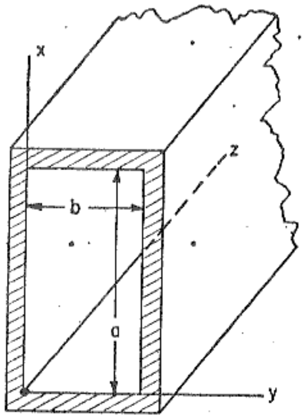

Fig. 12-1. Cross section of the rectangular wave guide.

Conducting side planes may be added to the parallel planes of the preceding chapter to form a rectangular wave guide, as in Fig. 12-1. Note that the guide is oriented with two faces in the planes of the axes, and that the dimension a is the height in the x direction, b being the width in the y direction. The guide is assumed made of perfectly conducting metal walls and to be filled with a good dielectric with constants μ_1 and ϵ_1, and with $\sigma_1 = 0$.

It is desired that propagation of energy take place in the z direction, as before, with the length of the guide being infinite in the z direction.

Thus it is reasonable to assume that all fields which may be present will vary with time and distance z according to

$$\epsilon^{-\gamma z} \sin \omega t$$

where γ is again complex in general. Using this form of variation the maximum values of the fields, indicated by the carets, may be written from Maxwell's equations as

$$\left.\begin{aligned} \frac{\partial \hat{H}_z}{\partial y} + \gamma \hat{H}_y &= (\sigma_1 + j\omega\epsilon_1)\hat{\mathcal{E}}_x \\ -\gamma \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} &= (\sigma_1 + j\omega\epsilon_1)\hat{\mathcal{E}}_y \\ \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} &= (\sigma_1 + j\omega\epsilon_1)\hat{\mathcal{E}}_z \end{aligned}\right\} \qquad (12\text{-}1)$$

$$\left.\begin{aligned} \frac{\partial \hat{\mathcal{E}}_z}{\partial y} + \gamma \hat{\mathcal{E}}_y &= -j\omega\mu_1 \hat{H}_x \\ -\gamma \hat{\mathcal{E}}_x - \frac{\partial \hat{\mathcal{E}}_z}{\partial x} &= -j\omega\mu_1 \hat{H}_y \\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} - \frac{\partial \hat{\mathcal{E}}_x}{\partial y} &= -j\omega\mu_1 \hat{H}_z \end{aligned}\right\} \qquad (12\text{-}2)$$

After selection of a field configuration having certain field components, these equations will give the other field components that must exist in the chosen configuration for a propagating wave to exist. The result will then be a wave propagating in the z direction and varying sinusoidally with time as assumed. As in the preceding chapter, two possible arbitrary field configurations will meet the imposed conditions: one with an entirely transverse electric field and one with an entirely transverse magnetic field. For the transverse electric wave it is then possible to require that $\mathcal{E}_z = 0$, and for the transverse magnetic wave that $H_z = 0$. Because of lack of advance knowledge on the effects of the two sides added to the parallel planes, it is not possible to impose any other field restrictions. The boundary restriction that $\mathcal{E} = 0$ along any of the perfectly conducting planes is the remaining physical information of use in defining the fields. To this may be added intuition based on the analysis of the parallel-plane case.

12-2. The $TM_{m,n}$ wave in the rectangular guide

The TM wave is obtained if it be assumed that the magnetic field is wholly transverse to the z direction of propagation and thus $H_z = 0$. Maxwell's equations for propagation in the z direction, as given in Eqs. 12-1 and 12-2, may then be modified for TM transmission in a good dielectric inside the guide where $\sigma_1 = 0$, under the assumption $H_z = 0$, and in terms of maximum values, as

$$\left.\begin{aligned} \gamma\hat{H}_y &= j\omega\epsilon_1\hat{\mathcal{E}}_x \\ -\gamma\hat{H}_x &= j\omega\epsilon_1\hat{\mathcal{E}}_y \\ \frac{\partial\hat{H}_y}{\partial x} - \frac{\partial\hat{H}_x}{\partial y} &= j\omega\epsilon_1\hat{\mathcal{E}}_z \end{aligned}\right\} \tag{12-3}$$

$$\left.\begin{aligned} \frac{\partial\hat{\mathcal{E}}_z}{\partial y} + \gamma\hat{\mathcal{E}}_y &= -j\omega\mu_1\hat{H}_x \\ -\gamma\hat{\mathcal{E}}_x - \frac{\partial\hat{\mathcal{E}}_z}{\partial x} &= -j\omega\mu_1\hat{H}_y \\ \frac{\partial\hat{\mathcal{E}}_y}{\partial x} - \frac{\partial\hat{\mathcal{E}}_x}{\partial y} &= 0 \end{aligned}\right\} \tag{12-4}$$

These equations indicate the possible presence of field components H_x, H_y, $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{E}_z$, in the TM wave, representing a consistent set of

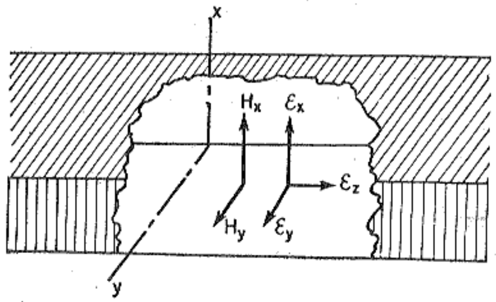

Fig. 12-2. Possible fields present in the TM wave.

physically realizable fields that will propagate in the z direction, with sinusoidal variation in time, and with $H_z = 0$ in the guide. Assumption of a value for any one field component and insertion in the above equations lead to values for the accompanying field components.

In the previous discussion of parallel conducting planes it was

found that the fields varied as a sinusoidal function of distance between the planes. The tangential electric field components were also required to have zero values at the boundaries. From Fig. 12-2 it can be seen that the requirement of zero values at the boundaries must be placed on $\mathcal{E}_z$ in both x and y directions. If the field is to vary sinusoidally in the dielectric between the boundaries, it seems reasonable that $\mathcal{E}_z$ may be expressed as

$$\mathcal{E}_z = \mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-\gamma z}\sin\omega t \tag{12-5}$$

where m and n are integers having values 0, 1, 2, 3, . . . , and a and b are the guide dimensions shown in Fig. 12-1. With the origin located as in that figure, Eq. 12-5 would satisfy the boundary conditions that $\mathcal{E}_z = 0$ at the guide walls or at $x = a$ or $y = b$.

Equations 12-3 and 12-4 may be manipulated to give

$$\frac{\partial^2\hat{\mathcal{E}}_z}{\partial x^2} + \frac{\partial^2\hat{\mathcal{E}}_z}{\partial y^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{\mathcal{E}}_z \tag{12-6}$$

The value assumed for $\mathcal{E}_z$ may then be substituted in the above and found to satisfy the equation and in turn to satisfy Maxwell's equations if

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + \omega^2\mu_1\epsilon_1 \tag{12-7}$$

If these conditions are met, the assumption for $\mathcal{E}_z$ is a possible solution (actually the only one) for the z-directed electric component of a physically realizable wave propagating as required in a TM mode in the rectangular guide.

From Eq. 12-7,

$$\gamma_{m,n} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu_1\epsilon_1} \tag{12-8}$$

and thus the propagation constant for the z direction is determined in terms of frequency, guide dimensions, and arbitrary integers m and n. As in the case of parallel planes, $\gamma_{m,n}$ may be either wholly real or imaginary, dependent on the value of ω. If it is real, the value of $\mathcal{E}_z$ in Eq. 12-5 attenuates according to the factor $\epsilon^{-\gamma_{m,n}z}$; if it is imaginary, the field propagates without attenuation in the z direction along the guide according to the factor $\epsilon^{-j\beta_{m,n}z}$. The rectangular wave guide acts as a high-pass filter, in a manner similar to the parallel-plane case and in the propagation region

$$\alpha_{m,n} = 0$$

$$\gamma_{m,n} = j\beta_{m,n} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{12-9}$$

The cutoff frequency for TM waves occurs when $\beta_{m,n} = 0$ and is

$$f_c = \frac{v_1}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{12-10}$$

and the cutoff wavelength may be obtained as

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

The minimum cutoff frequency or maximum cutoff wavelength occurs for $m = n = 1$. As will be shown, values of $m = 0$ or $n = 0$ are not possible with the TM mode.

It is possible to write $\gamma_{m,n}$ in the propagation or pass band as

$$\gamma_{m,n} = j\beta_{m,n} = j\,\frac{2\pi f_c}{v_1}\sqrt{\frac{f^2}{f_c^2} - 1} \tag{12-11}$$

The phase velocity for TM waves can be obtained from $\omega/\beta_{m,n}$ and is

$$v_p = \frac{v_1}{\sqrt{1 - \frac{f_c^2}{f^2}}} \tag{12-12}$$

Since it has been demonstrated that the phase and group velocities are related according to $\sqrt{v_p v_g} = v_1$, the group velocity can be readily written as

$$v_g = v_1\sqrt{1 - \frac{f_c^2}{f^2}} \tag{12-13}$$

The same result could have been obtained by taking $1/(d\beta/d\omega)$. By analogy with the case of the parallel planes, it may be reasoned that propagation inside the wave guide takes place by successive glancing reflections of the waves from the conducting side walls, the actual wave path being zigzag in form. Another viewpoint may be that standing-wave patterns are set up on the x and y axes due to reflections from the conducting side walls and that these standing-wave patterns then propagate down the guide in the z direction.

Use of the value for $\mathcal{E}_z$ in Eqs. 12-3 and 12-4 and some manipulation allow the complete field components of the $\text{TM}_{m,n}$ wave to be written for the pass band where $\gamma_{m,n} = j\beta_{m,n}$ as

$$H_x = \frac{n\pi}{b}\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-14}$$

$$H_y = -\frac{m\pi}{a}\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-15}$$

$$H_z = 0 \tag{12-16}$$

$$\mathcal{E}_x = -\frac{m\pi}{a}\frac{\beta_{m,n}}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-17}$$

$$\mathcal{E}_y = -\frac{n\pi}{b}\frac{\beta_{m,n}}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-18}$$

$$\mathcal{E}_z = \mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\sin\omega t \tag{12-19}$$

It should be noted that if either m or n is zero, all the fields are zero. Thus a $\text{TM}_{m,0}$ or a $\text{TM}_{0,n}$ mode of propagation cannot exist in the rectangular wave guide. This result also rules out a TEM mode in a guide.

For the parallel planes, a Z_0 or characteristic wave impedance was defined as the ratio of transverse electric to transverse magnetic field. If a similar definition is used for the rectangular wave guide

$$\frac{\mathcal{E}}{H} = \frac{\sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2}}{\sqrt{H_x^2 + H_y^2}} = \frac{\beta_{m,n}}{\omega\epsilon_1} = Z_{0(TM)} \tag{12-20}$$

Use of Eq. 12-11 for $\beta_{m,n}$ allows the expression for the characteristic impedance of a guide with the $\text{TM}_{m,n}$ mode existing to be written

$$Z_{0(TM)} = \eta_1\sqrt{1 - \frac{f_c^2}{f^2}} \tag{12-21}$$

The field configurations for the rectangular guide for the $TM_{1,1}$ and $TM_{2,1}$ modes are shown in Fig. 12-3. The pattern as shown in the longitudinal section is drawn for the center line of the guide and parallel to the x,z plane. The field pattern may be visualized as traveling along the infinite length guide in the z direction. Should the guide be of finite length and not properly terminated in its Z_0

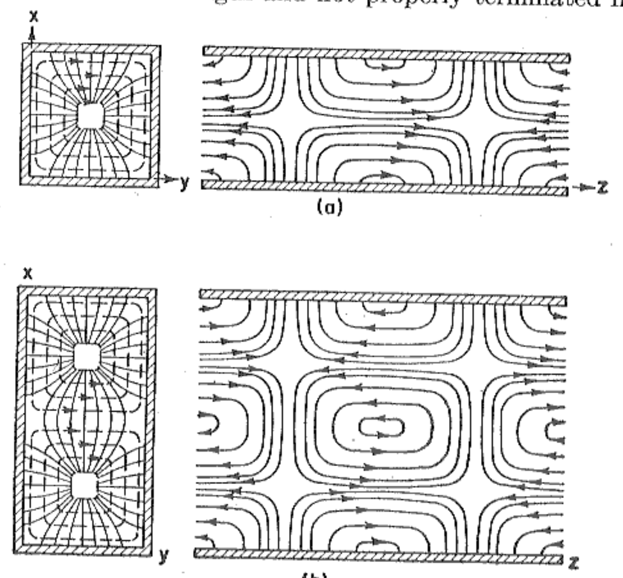

Fig. 12-3. (a) Configuration of $TM_{1,1}$ fields; (b) configuration of $TM_{2,1}$ fields.

value, then the pattern will be set up as a standing wave because of reflection of part of the incident energy.

Consideration of the field patterns will give meaning to the n and m subscripts employed. The m subscript indicates the number of maxima of field intensity in the x direction between the walls, whereas n indicates the number of such field-intensity maxima between the walls in the y direction. Modes in which the subscripts are complementary, such as $TM_{1,2}$ and $TM_{2,1}$, simply imply that the guide has been turned on its side.

It is customary to assume the y coordinate coincident with the smaller transverse dimension, the x coordinate coincident with the larger transverse dimension.

12-3. The $TE_{m,n}$ wave in the rectangular guide

If it be assumed that the electric field is wholly transverse to the z direction of propagation, the TE mode is obtained in the rectangular guide. For this case, $\mathcal{E}_z = 0$. Considering the dielectric in the guide to have $\sigma_1 = 0$. Eqs. 12-1 and 12-2 become, for the TE mode,

$$\left.\begin{aligned} \frac{\partial \hat{H}_z}{\partial y} + \gamma \hat{H}_y &= j\omega\epsilon_1 \hat{\mathcal{E}}_x \\ -\gamma \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} &= j\omega\epsilon_1 \hat{\mathcal{E}}_y \\ \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} &= 0 \end{aligned}\right\} \qquad (12\text{-}22)$$

$$\left.\begin{aligned} \gamma \hat{\mathcal{E}}_y &= -j\omega\mu_1 \hat{H}_x \\ -\gamma \hat{\mathcal{E}}_x &= -j\omega\mu_1 \hat{H}_y \\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} - \frac{\partial \hat{\mathcal{E}}_x}{\partial y} &= -j\omega\mu_1 \hat{H}_z \end{aligned}\right\} \qquad (12\text{-}23)$$

As before, these equations indicate fields with components H_x, H_y, H_z, $\mathcal{E}_x$, and $\mathcal{E}_y$, representing a consistent set with propagation in the z direction down the guide.

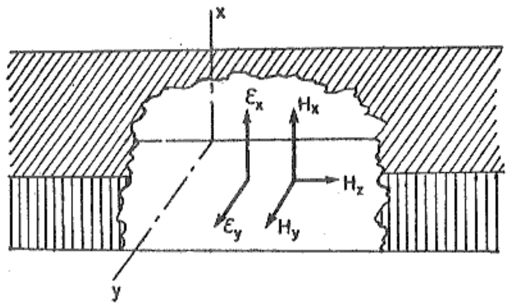

Fig. 12-4. Possible fields present in the TE wave.

In the previous TM case it was convenient to assume a value for $\mathcal{E}_z$, since it was symmetrically affected by both sets of guide walls. For the same reason it will be convenient to assume a value for H_z for the TE case. It appears reasonable to assume that

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \epsilon^{-\gamma z} \sin \omega t \qquad (12\text{-}24)$$

the use of cosine functions being dictated by the physical knowledge that since the currents flow in the side walls, the fields will be largest when near the currents. If the intensities for the various components obtained under this assumption satisfy the boundary requirements set up by the conducting walls, the assumption will have been proved correct.

Equations 12-22 and 12-23 may be manipulated to give

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{H}_z \qquad (12\text{-}25)$$

into which Eq. 12-24 for H_z may be substituted. The assumed H_z value will satisfy this equation, and thus will satisfy Maxwell's equations, if

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + \omega^2\mu_1\epsilon_1 \qquad (12\text{-}26)$$

thus leading to the same propagation condition as for TM waves.

The propagation constant $\gamma_{m,n}$ for TE waves is the same as for TM waves:

$$\gamma_{m,n} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu_1\epsilon_1}$$

and becomes, in the propagation or pass-band region,

$$\gamma_{m,n} = j\beta_{m,n} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad (12\text{-}27)$$

with $\alpha_{m,n} = 0$

Obviously, the rectangular wave guide again behaves as a high-pass filter. The cutoff frequency, cutoff wavelength, and phase and group velocities will be the same for TE waves as TM waves.

Insertion of the assumed H_z value into Eqs. 12-22 and 12-23

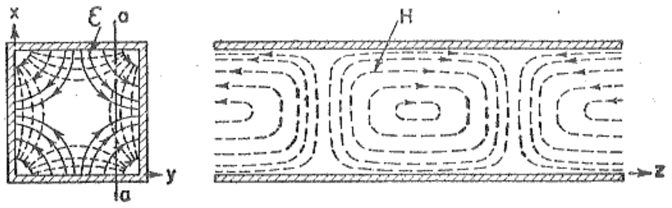

Fig. 12-5. Configuration of the $TE_{1,1}$ field.

permits the TE field components to be written for the pass band as

$$H_x = \frac{m\pi}{a}\frac{\beta_{m,n}}{\omega_c^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \qquad (12\text{-}28)$$

$$H_y = \frac{n\pi}{b}\frac{\beta_{m,n}}{\omega_c^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \qquad (12\text{-}29)$$

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\sin\omega t \qquad (12\text{-}30)$$

$$\mathcal{E}_x = \frac{n\pi}{b}\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \qquad (12\text{-}31)$$

$$\mathcal{E}_y = -\frac{m\pi}{a}\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \qquad (12\text{-}32)$$

$$\mathcal{E}_z = 0 \qquad (12\text{-}33)$$

A field configuration for the $TE_{1,1}$ mode in the rectangular guide is drawn in Fig. 12-5.

The characteristic wave impedance of the rectangular guide for TE waves can be readily found as the ratio of the transverse electric field to the transverse magnetic field:

$$\frac{\mathcal{E}}{H} = \frac{\sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2}}{\sqrt{H_x^2 + H_y^2}} = \frac{\omega\mu_1}{\beta_{m,n}} = Z_{0(TE)} \qquad (12\text{-}34)$$

and substitution for β gives

$$Z_{0(TE)} = \frac{\eta_1}{\sqrt{1 - f_c^2/f^2}} \qquad (12\text{-}35)$$

which is *not* identical with that obtained for TM waves. In fact, comparison with Eq. 12-21 shows

$$\eta_1 = \sqrt{Z_{0(TM)}Z_{0(TE)}}$$

or that the intrinsic impedance in the dielectric in the unbounded case is the geometric mean of the wave impedance to TE and TM propagation in the rectangular guide.

It can be seen from these equations that waves of the $TE_{m,0}$ type are possible except for the $TE_{0,0}$ mode in which case all fields go to zero. The $TE_{1,0}$ wave is of very considerable interest because of its simplicity. For this wave, $m = 1$, $n = 0$.

Since $\gamma_{1,0} = j\beta_{1,0} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{\pi}{a}\right)^2}$

the field expressions for the $TE_{1,0}$ wave reduce to

$$H_x = \frac{a}{\pi}\beta_{m,n}H_0 \sin\left(\frac{\pi x}{a}\right)\epsilon^{-j\beta_{1,0}z}\cos\omega t \tag{12-36}$$

$$H_z = H_0 \cos\left(\frac{\pi x}{a}\right)\epsilon^{-j\beta_{1,0}z}\sin\omega t \tag{12-37}$$

$$\mathcal{E}_y = -\frac{a}{\pi}\omega\mu_1 H_0 \sin\left(\frac{\pi x}{a}\right)\epsilon^{-j\beta_{1,0}z}\cos\omega t \tag{12-38}$$

all other fields being zero. None of the field components vary with dimension b, so that guides of different b dimensions but equal a

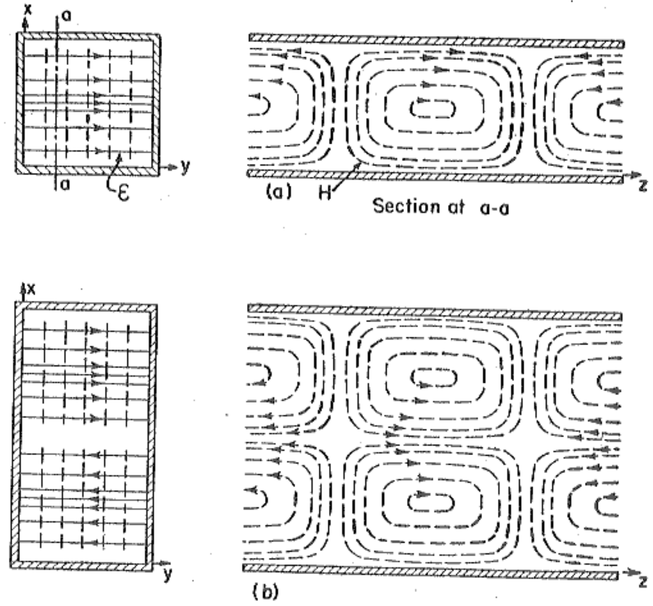

Fig. 12-6. (a) The arrangement of the $TE_{1,0}$ fields in the rectangular guide; (b) the $TE_{2,0}$ field.

dimensions will have the same value of cutoff frequency. The field configuration for a $TE_{1,0}$ wave is shown in Fig. 12-6(a), wherein the simplicity of the field may be noted.

The $TE_{1,0}$ critical frequency is given by

$$f_c = \frac{v_1}{2a}$$

and it can be seen that at the cutoff wavelength the a dimension is just one-half wavelength, or the value of λ_c is

$$\frac{\lambda_c}{2} = a$$

bearing out the physical concept developed for the parallel-plane cutoff wavelength.

Since a $TM_{1,0}$ wave cannot exist, the $TE_{1,0}$ wave is the simplest possible form and has the lowest cutoff frequency for a given guide

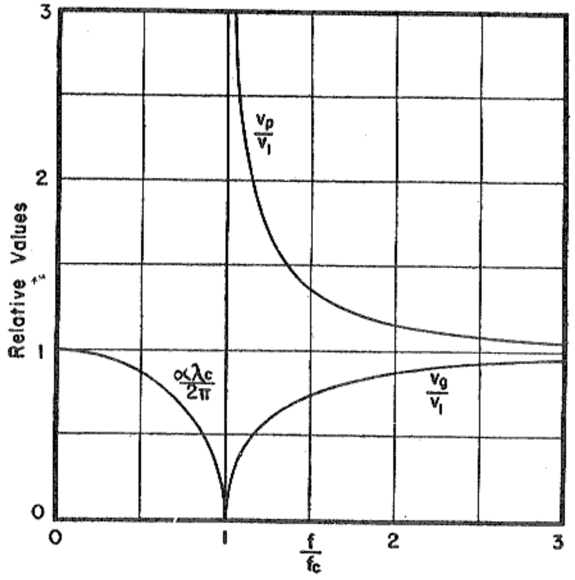

Fig. 12-7. Variation of attenuation below cutoff, and of the velocities above cutoff, for the rectangular wave guide.

dimension in the x direction. The $TE_{1,0}$ mode is called the *dominant mode* for the rectangular guide, the dominant mode being the only wave which will convey energy through the guide when the excitation frequency is between the lowest cutoff and that due to the next higher mode. Ordinarily it is desired that only one mode be propagated, and the guide is so designed that for a $TE_{1,0}$ wave for example, dimension a is longer than half a wavelength at the operating frequency, and shorter than two half wavelengths. The dimension is thus too small to support a $TE_{2,0}$ wave. For dominant-mode operation the frequency range of a rectangular guide is restricted to less than 2 to 1. For circular guides this is 1.25 to 1 or less. Ordinarily a rectangular guide is used with the $TE_{1,0}$ mode and the circular guide with the $TE_{1,1}$ mode excited.

The smaller dimension of the rectangular guide is kept definitely less than a half wavelength at any operating frequency in order to avoid exciting the $TM_{1,1}$ or $TE_{1,1}$ mode. Where rotational symmetry is required, as in coupling a guide to a rotating antenna, higher modes may be employed. For the $TE_{1,0}$ mode the lowest cutoff frequency in a rectangular guide is

$$f_c = \frac{v_1}{2a}$$

Since all tangential components of electric field and all normal components of magnetic field go to zero at the perfectly conducting guide walls, the derived fields satisfy the boundary conditions and the original assumption of a value for H_z is proved satisfactory.

12-4. Cylindrical wave guides

In order to determine the conditions for propagation of waves inside a hollow, perfectly conducting cylinder as in Fig. 12-8, it is convenient to employ cylindrical coordinates r, ϕ, and z, instead of the previously used rectangular coordinates x, y, and z. As shown in Appendix B, Maxwell's equations may be converted to cylindrical coordinates, resulting in

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} &= \sigma \mathcal{E}_r + \epsilon \frac{\partial \mathcal{E}_r}{\partial t} \\ \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= \sigma \mathcal{E}_\phi + \epsilon \frac{\partial \mathcal{E}_\phi}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} &= \sigma \mathcal{E}_z + \epsilon \frac{\partial \mathcal{E}_z}{\partial t} \end{aligned}\right\} \quad (12\text{-}39)$$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi} - \frac{\partial \mathcal{E}_\phi}{\partial z} &= -\mu \frac{\partial H_r}{\partial t} \\ \frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} &= -\mu \frac{\partial H_\phi}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{E}_\phi) - \frac{1}{r}\frac{\partial \mathcal{E}_r}{\partial \phi} &= -\mu \frac{\partial H_z}{\partial t} \end{aligned}\right\} \quad (12\text{-}40)$$

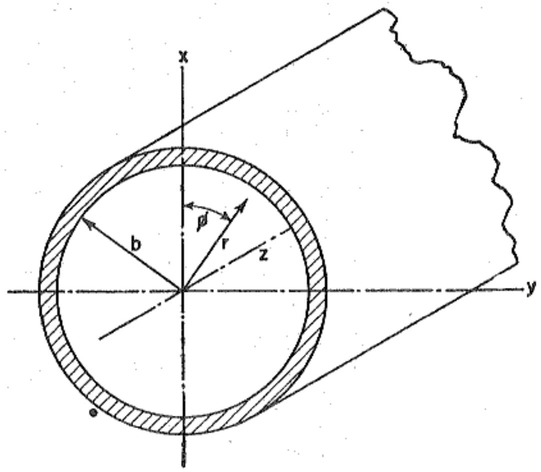

Fig. 12-8. The cylindrical wave guide.

If it is required that the waves propagate in the z direction and be a sinusoidal time function, then

$$\mathcal{E} = \hat{\mathcal{E}}\epsilon^{-\gamma z} \sin \omega t$$
$$H = \hat{H}\epsilon^{-\gamma z} \sin \omega t$$

The proper derivatives may be taken, after which Maxwell's equations for a good dielectric ($\sigma_1 = 0$), in terms of maximum values, become

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \hat{H}_z}{\partial \phi} + \gamma \hat{H}_\phi &= j\omega\epsilon_1 \hat{\mathcal{E}}_r \\ -\gamma \hat{H}_r - \frac{\partial \hat{H}_z}{\partial r} &= j\omega\epsilon_1 \hat{\mathcal{E}}_\phi \\ \frac{1}{r}\frac{\partial}{\partial r}(r\hat{H}_\phi) - \frac{1}{r}\frac{\partial \hat{H}_r}{\partial \phi} &= j\omega\epsilon_1 \hat{\mathcal{E}}_z \end{aligned}\right\} \quad (12\text{-}41)$$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \hat{\mathcal{E}}_z}{\partial \phi} + \gamma \hat{\mathcal{E}}_\phi &= -j\omega\mu_1 \hat{H}_r \\ -\gamma \hat{\mathcal{E}}_r - \frac{\partial \hat{\mathcal{E}}_z}{\partial r} &= -j\omega\mu_1 \hat{H}_\phi \\ \frac{1}{r}\frac{\partial}{\partial r}(r\hat{\mathcal{E}}_\phi) - \frac{1}{r}\frac{\partial \hat{\mathcal{E}}_r}{\partial \phi} &= -j\omega\mu_1 \hat{H}_z \end{aligned}\right\} \quad (12\text{-}42)$$

As before, it is possible to have TE modes with $\mathcal{E}_z = 0$, and TM modes with $H_z = 0$.

Solution of Maxwell's equations in rectangular coordinates gave fields in terms of trigonometric functions of the tube dimensions. Operation on Maxwell's equations in cylindrical coordinate systems leads to Bessel's equation and field solutions in terms of Bessel functions. After considerable manipulation and the use of an assumed field value

$$\mathcal{E}_z = \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi \epsilon^{-\gamma z} \sin \omega t \qquad (12\text{-}43)$$

it is possible to write as the field values for the TM wave that will fulfill the boundary conditions of the cylindrical guide:

$$\mathcal{E}_r = -\frac{\gamma}{\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}} \mathcal{E}_0 J_n'(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-\gamma z} \sin \omega t \qquad (12\text{-}44)$$

$$\mathcal{E}_\phi = \frac{n\gamma}{r(\gamma^2 + \omega^2\mu_1\epsilon_1)} \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \sin n\phi\, \epsilon^{-\gamma z} \sin \omega t \qquad (12\text{-}45)$$

$$\mathcal{E}_z = \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-\gamma z} \sin \omega t \qquad (12\text{-}46)$$

$$H_r = -\frac{n\omega\epsilon_1}{r(\gamma^2 + \omega^2\mu_1\epsilon_1)} \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \sin n\phi\, \epsilon^{-\gamma z} \cos \omega t \qquad (12\text{-}47)$$

$$H_\phi = -\frac{\omega\epsilon_1}{\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}} \mathcal{E}_0 J_n'(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-\gamma z} \cos \omega t \qquad (12\text{-}48)$$

$$H_z = 0$$

The fields $\mathcal{E}_r$ and H_r are directed along r. The $\mathcal{E}_\phi$ and H_ϕ fields have positive directions in the direction of increasing ϕ and are therefore at right angles to the radially directed fields. The z components are at right angles to both the r and ϕ components.

So that $\mathcal{E}_z$ shall have the same value for $\cos n\phi = \cos n(\phi + 2\pi)$, that is, that $\mathcal{E}_z$ be single-valued at a given space point, it is necessary that n be an integer. Therefore the Bessel function J_n and its derivative J_n' are only of integer orders. That is, $n = 0, 1, 2, 3, \ldots$.

For the TM wave it is necessary that $\mathcal{E}_z$ and $\mathcal{E}_\phi$ be zero at the pipe wall where $r = b$, because of the infinite conductivity of the wall. This requirement fixes the value of the Bessel function

$$J_n(b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) = 0$$

and a listing of a few of the roots of J_n is given in Table 10. The manner of variation of J_0, J_1, J_2, and J_3 is indicated in Fig. 12-9,

TABLE 10

ROOTS OF $J_n = 0$

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	2.40	3.83	5.14	6.38	7.59
2	5.52	7.02	8.42	9.76	11.06
3	8.65	10.17	11.62	13.02	14.37
4	11.79	13.32	14.80	16.22	17.62

which gives significance to the value m as designating the particular root meant, since there are an infinite number of roots. The subscripts n and m then have definite meaning in terms of the field

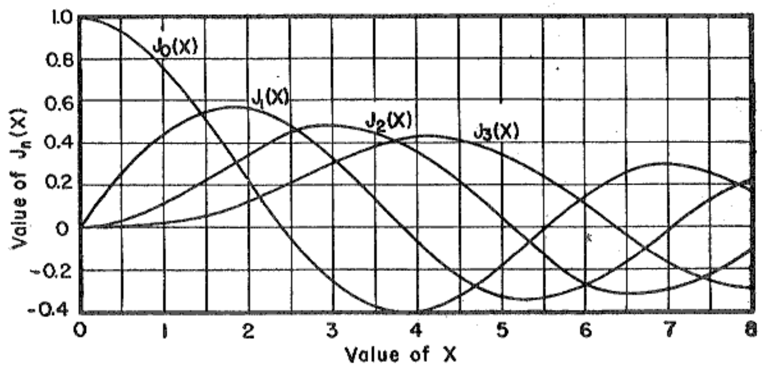

FIG. 12-9. Variation of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, $J_3(x)$.

configuration and are used to designate the mode of field propagation as in $\text{TM}_{n,m}$. Thus the root of $J_n = 0$ for $n = 1$, $m = 2$ is equal to 7.02, and this corresponds to a $\text{TM}_{1,2}$ wave. Physically, Eq. 12-43 shows that n is the number of cycles of variation of $\mathcal{E}_z$ found as ϕ varies around the complete cylinder, or through 2π radians.

The subscript m indicates the number of zeros of electric field along a radial path from the center to the inside surface of the outer guide wall.

If $\tau_{n,m}$ is a root of $J_n = 0$, then

$$b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1} = \tau_{n,m}$$

and

$$\gamma_{n,m} = \sqrt{\left(\frac{\tau_{n,m}}{b}\right)^2 - \omega^2\mu_1\epsilon_1} \qquad (12\text{-}49)$$

Following the analysis of preceding sections, for propagation to occur it is necessary that $\gamma_{n,m}$ be imaginary or that

$$\gamma_{n,m} = j\beta_{n,m} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{\tau_{n,m}}{b}\right)^2} \qquad (12\text{-}50)$$

which makes the attenuation $\alpha_{n,m}$ equal to zero in the pass band. The critical or cutoff frequency, above which transmission takes place, is

$$f_c = \frac{\tau_{n,m}v_1}{2\pi b} \qquad (12\text{-}51)$$

and the critical wavelength is

$$\lambda_c = \frac{2\pi b}{\tau_{n,m}}$$

Since $2\pi b$ is the inner circumference of the outer wall of the guide, the critical wavelength is that at which the circumference is equal to $\tau_{n,m}$ wavelengths.

The phase velocity is

$$v_p = \frac{\omega}{\beta_{n,m}} = \frac{v_1}{\sqrt{1 - \dfrac{f_c^2}{f^2}}} \qquad (12\text{-}52)$$

and the group velocity is

$$v_g = \frac{1}{\dfrac{d\beta_{n,m}}{d\omega}} = v_1\sqrt{1 - \frac{f_c^2}{f^2}} \qquad (12\text{-}53)$$

as for the rectangular guide. The characteristic impedances of the cylindrical guide may also be found identical with those of the rectangular guide for both TM and TE modes.

Choice of an assumed field value

$$H_z = H_0J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})\cos n\phi\,\epsilon^{-\gamma z}\sin\omega t$$

for H_z, a value selected because of similarity to the preceding TM case, and its use in Eqs. 12-41 and 12-42 permit an equation for H_r in the TE field to be written as

$$H_r = \frac{-\beta_{n,m}}{\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}} H_0J_n'(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})\cos n\phi\,\epsilon^{-j\beta_{n,m}z}\cos\omega t \qquad (12\text{-}54)$$

It is known that at $r = b$, the H_r or normal field value must be zero, which shows that

$$J_n'(b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) = 0$$

is a condition of propagation for the TE mode. If $\tau'_{n,m}$ is a root of $J_n' = 0$, the field equations for the TE case in the cylindrical guide may be written

$$H_r = \frac{-\beta_{n,m}}{\left(\dfrac{\tau'_{n,m}}{b}\right)} H_0J_n'\left(r\frac{\tau'_{n,m}}{b}\right)\cos n\phi\,\epsilon^{-j\beta_{n,m}z}\cos\omega t \qquad (12\text{-}55)$$

$$H_\phi = \frac{n\beta_{n,m}}{r\left(\dfrac{\tau'_{n,m}}{b}\right)^2} H_0J_n\left(r\frac{\tau'_{n,m}}{b}\right)\sin n\phi\,\epsilon^{-j\beta_{n,m}z}\cos\omega t \qquad (12\text{-}56)$$

$$H_z = H_0J_n\left(r\frac{\tau'_{n,m}}{b}\right)\cos n\phi\,\epsilon^{-j\beta_{n,m}z}\sin\omega t \qquad (12\text{-}57)$$

$$\mathcal{E}_r = \frac{n\omega\mu_1}{r\left(\dfrac{\tau'_{n,m}}{b}\right)^2} H_0J_n\left(r\frac{\tau'_{n,m}}{b}\right)\sin n\phi\,\epsilon^{-j\beta_{n,m}z}\cos\omega t \qquad (12\text{-}58)$$

$$\mathcal{E}_\phi = \frac{\omega\mu_1}{\left(\dfrac{\tau'_{n,m}}{b}\right)} H_0J_n'\left(r\frac{\tau'_{n,m}}{b}\right)\cos n\phi\,\epsilon^{-j\beta_{n,m}z}\cos\omega t \qquad (12\text{-}59)$$

$$\mathcal{E}_z = 0 \qquad (12\text{-}60)$$

The values for $\beta_{n,m}$, $\alpha_{n,m}$, f_c, v_p, and v_g are the same as those obtained for the TM mode except for the use of $\tau'_{n,m}$ in place of $\tau_{n,m}$.

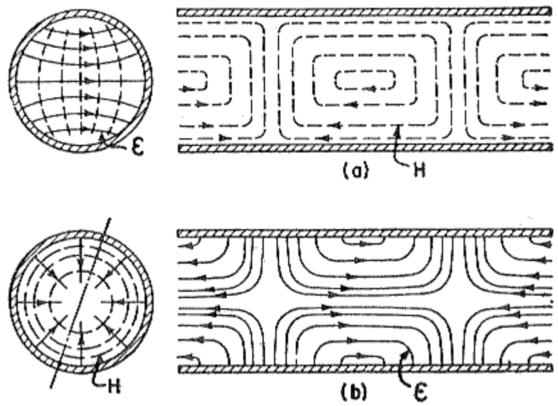

Fig. 12-10. (a) The $TE_{1,1}$ field in the cylindrical guide; (b) the $TM_{0,1}$ field in the cylindrical guide.

The derivative J_n' with respect to r of the Bessel function may be obtained from

$$\frac{\partial J_n(kr)}{\partial r} = J_n'(kr) = \frac{n}{kr} J_n(kr) - J_{n+1}(kr)$$

and a few roots of $J_n' = 0$ are given in Table 11.

TABLE 11

VALUES OF $J_n' = 0$

m	$n = 0$	$n = 1$	$n = 2$
1	3.83	1.84	3.05
2	7.02	5.33	6.71
3	10.17	8.54	9.97

Since there is no root $r_{0,0}$ of the Bessel function J_0, $TM_{0,0}$ waves cannot exist. Waves of order $TM_{1,0}$ and $TE_{1,0}$ are possible, as well as higher orders. The field patterns for several modes are shown in Fig. 12-10.

12-5. The TEM wave in the coaxial line

The TEM mode (sometimes called the *principal mode*) is the approximate form of propagation encountered on parallel-wire and

coaxial lines at low frequencies. Actually, it exists only in theory on the dissipationless form of those lines; but since the losses are small, the true condition closely approximates the TEM field.

To analyze the TEM wave in the coaxial line, it is desirable to start with all fields varying in time and propagating in the z direction according to

$$H = \hat{H}\epsilon^{-\gamma z} \sin \omega t$$

$$\mathcal{E} = \hat{\mathcal{E}}\epsilon^{-\gamma z} \sin \omega t$$

The TEM mode is a special case of the TM mode with $\mathcal{E}_z = H_z = 0$. It may also be established that $\mathcal{E}_\phi = H_r = 0$. Under these conditions, Eqs. 12-41 and 12-42 reduce to

$$\gamma \hat{H}_\phi = j\omega\epsilon_1\mathcal{E}_r \qquad (12\text{-}61)$$

$$\frac{\partial}{\partial r}(r\hat{H}_\phi) = 0 \qquad (12\text{-}62)$$

$$\gamma\hat{\mathcal{E}}_r = j\omega\mu_1\hat{H}_\phi \qquad (12\text{-}63)$$

$$\frac{\partial\hat{\mathcal{E}}_r}{\partial\phi} = 0 \qquad (12\text{-}64)$$

and it is then possible to write

$$\hat{H}_\phi = \frac{j\omega\epsilon_1}{\gamma}\hat{\mathcal{E}}_r = \frac{\gamma}{j\omega\mu_1}\hat{\mathcal{E}}_r$$

indicating that $\gamma^2 = -\omega^2\mu_1\epsilon_1$

or that $\gamma = j\beta = j\omega\sqrt{\mu_1\epsilon_1}$ (12-65)

which is identical with the propagation constant obtained for the dissipationless line from transmission-line theory. It shows that under the special TEM conditions, propagation will occur in the z direction with $\alpha = 0$.

The cutoff frequency is found by setting β equal to zero. Such an operation indicates that

$$f_c = 0$$

or that all frequencies are propagated on the coaxial line by the TEM mode.

The Poynting radiation vector resulting from $\hat{\mathcal{E}}_r$ and $\hat{H}_\phi$ also shows that energy will propagate along the positive-z axis. The electric field is totally radial and the magnetic field is concentric to the inner conductor. Equation 12-64 shows that $\mathcal{E}_r$ does not vary with ϕ, or that circular symmetry of radial electric field exists.

Equation 12-62 indicates that

$$\hat{H}_\phi = \frac{k}{r}$$

where k is not a function of r, and thus the H_ϕ field is an inverse function of r between the conductors. This is exactly the situation to be expected from the static field configuration in the coaxial cable. However,

$$\oint H \cdot dl = i$$

by Ampere's law, where i is the instantaneous current enclosed by the path and H_ϕ is the corresponding component instantaneous magnetic field. If the circumference of the inner conductor of radius a is chosen as a path, then if I_0 is the value of instantaneous current in the inner conductor,

$$2\pi a H_\phi = I_0$$

Evaluating k at $r = a$,

$$k = \frac{I_0}{2\pi} \tag{12-66}$$

and thus the magnetic field intensity is expressible in terms of the inner conductor current as

$$H_\phi = \frac{I_0}{2\pi r} \tag{12-67}$$

which is certainly true by reason of the definition of H, the field intensity along a path of radius r.

From Eqs. 12-63 and 12-65,

$$\mathcal{E}_r = \sqrt{\frac{\mu_1}{\epsilon_1}}\, H_\phi$$

so that

$$\mathcal{E}_r = \eta_1 \frac{I_0}{2\pi r} \tag{12-68}$$

showing the electric field also to be inversely proportional to r, as expected from the cylindrical symmetry of the situation.

The voltage *drop* from the center conductor to the outer conductor of the coaxial line may be written in terms of the maximum value $\hat{V}$ as

$$\hat{V} = \int_a^b \hat{\mathcal{E}}_r\, dr = \eta_1 \frac{\hat{I}_0}{2\pi} \ln \frac{b}{a}$$

The characteristic impedance of such an infinite length line is the ratio of V/I by circuit methods, so that

$$Z_0 = \frac{1}{2\pi} \eta_1 \ln \frac{b}{a} \text{ ohms} \tag{12-69}$$

which, for a dielectric of relative permittivity ϵ_r between the conductors, is

$$Z_0 = \frac{60}{\sqrt{\epsilon_r}} \ln \frac{b}{a} \text{ ohms}$$

and this checks the value obtained through circuit considerations in Eq. 7-21.

Inserting the z-propagation and time functions gives the instantaneous voltage as

$$V = \eta_1 \frac{\hat{I}_0 \ln (b/a)}{2\pi} \epsilon^{-j\beta z} \sin \omega t \tag{12-70}$$

Taking the z derivative of the voltage,

$$\frac{\partial V}{\partial z} = -j\beta\eta_1 \hat{I}_0 \ln \frac{b}{a} \epsilon^{-j\beta z} \sin \omega t$$

$$= -\frac{j\omega\mu_1}{2\pi} I_0 \ln \frac{b}{a} \tag{12-71}$$

in terms of the effective current and voltage. Since $I_0 = \hat{I}_0 \epsilon^{-j\beta z} \sin \omega t$,

$$\frac{\partial I_0}{\partial z} = -j\beta \hat{I}_0 \epsilon^{-j\beta z} \sin \omega t$$

$$= -\frac{j2\pi\omega\epsilon_1}{\ln (b/a)} V \tag{12-72}$$

It has been previously shown that

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a} \qquad C = \frac{2\pi\epsilon}{\ln (b/a)}$$

in Chapter 7 for the high-frequency line, so that Eqs. 12-71 and 12-72 become

$$\frac{\partial V}{\partial z} = -j\omega LI \tag{12-73}$$

$$\frac{\partial I}{\partial z} = -j\omega CV \tag{12-74}$$

which, obtained from field theory, are identical with the transmission-line equations as previously obtained for a dissipationless line by use of circuit parameters and current and voltage concepts. This analysis then serves to show once more that circuit concepts and field concepts are really one and the same thing.

In addition to the TEM wave in the coaxial line, it is possible for higher-order forms of TM and TE waves to exist with components of electric or magnetic field in the direction of the line axis. However, for the usual coaxial lines the dimensions are small enough that the lines are operating at frequencies far below cutoff for these modes, and these modes will not be considered here.

12-6. Attenuation in the coaxial line

For the TEM mode in the coaxial line, the tangential magnetic field is

$$H_\phi = \frac{\hat{I}_0}{2\pi r} \epsilon^{-j\beta z} \sin \omega t$$

from the preceding section. Again it may be assumed that the currents flowing in conductors of finite conductivity are essentially the same as those for the perfect-conductor case, or that the losses are small and the fields and currents are affected only slightly thereby. At some particular point of observation along the z axis, the average power entering through the surface of the inner conductor of the coaxial cable of radius a, due to the tangential magnetic field H evaluated at $r = a$, is by Eqs. 12-67 and 12-68,

$$P_{M1} = \frac{1}{2}\left(\frac{\hat{I}_0}{2\pi a}\right)^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2$$

The area of surface of inner conductor is $2\pi a$ square meters per meter of length, so that the power supplied to the inner conductor per

meter of length is

$$P_{M1} = \frac{1}{2}\frac{\hat{I}_0^2}{2\pi a}\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \tag{12-75}$$

The power supplied to the outer conductor of radius b per meter of length is

$$P_{M2} = \frac{1}{2}\frac{\hat{I}_0^2}{2\pi b}\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \tag{12-76}$$

The total power loss into the conductors, per meter of line, is

$$P_M = \frac{1}{4\pi}\hat{I}_0^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}}\left(\frac{1}{a}+\frac{1}{b}\right) \text{ watts} \tag{12-77}$$

The power transmitted past a given point in the guide is readily obtained by integrating the Poynting radiation vector or product of the $\mathcal{E}_r$ and H_ϕ fields over the space between conductors and is

$$P_T = \int_0^{2\pi}\int_a^b \frac{1}{2}\hat{\mathcal{E}}_r \hat{H}_\phi r \, d\phi \, dr$$

By use of the field relations of the preceding section, the above may be written

$$P_T = \frac{\hat{I}_0^2}{4\pi}\eta_1 \ln\frac{b}{a} \text{ watts} \tag{12-78}$$

It has already been shown that the attenuation is given by

$$\alpha = \frac{1}{2}\frac{P_M}{P_T}$$

so that for the coaxial line, under approximate TEM conditions, the attenuation due to side-wall, or conductor, losses is

$$\alpha = \frac{\sqrt{\pi\mu_m/\sigma_m}\,(1/a + 1/b)}{2\eta_1 \ln(b/a)}\sqrt{f} \text{ nepers/m} \tag{12-79}$$

showing that this attenuation is proportional to the square root of frequency.

Equation 12-77, for the total power conveyed into the metal by the fields, may be written for a line with copper walls as

$$P_M = \frac{\hat{I}_0^2}{2}\left[4.16 \times 10^{-8}\sqrt{f}\left(\frac{1}{a}+\frac{1}{b}\right)\right] \text{ watts}$$

The term in brackets is the resistance expression for a coaxial line as previously obtained in Section 7-2. Since $\hat{I}_0$ is the maximum value of current flowing in the conductors, it can be seen that the power delivered from the field to the side walls is exactly the amount of power which is dissipated as I^2R losses in the conductor. It has previously been shown that the power transmitted along a line to the load is conveyed in the fields; and now it is evident that as the power is transmitted along the line, the fields supply power to the conductors, this power then appearing as the heat losses in the conductors.

12-7. Attenuation in guides due to imperfect conductors

For the TM mode in the rectangular guide the H_x field is the tangential component on the sides of the guide parallel to the x,z plane, or of length a, and the H_y component is tangential on the side parallel to the y,z plane, or of length b. For the $\text{TM}_{m,n}$ mode in the rectangular guide, the maximum values of these fields at $y = 0$ or $y = b$ and at $x = 0$ or $x = a$, respectively, are

$$\hat{H}_x = \frac{n\pi}{b}\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\epsilon^{-j\beta_{m,n}z} \tag{12-80}$$

$$\hat{H}_y = -\frac{m\pi}{a}\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z} \tag{12-81}$$

As previously stated, the power conveyed into a metal of finite conductivity by an electromagnetic field is

$$P_M = \frac{1}{2}\hat{H}^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2$$

Assuming that the rectangular guide walls are conductors of finite conductivity and that since the metal losses are small, the fields are approximately the same as for the case of perfect conductivity, an assumption supported by the inequality $\sigma_m \gg \omega\epsilon_m$, then the power delivered to the guide walls by the above $\text{TM}_{m,n}$ fields is

$$P_M = \sqrt{\frac{\omega\mu_m}{2\sigma_m}}\left(\int_0^a H_x^2\,dx + \int_0^b H_y^2\,dy\right) \text{ watts/meter of length}$$

the $\frac{1}{2}$ factor being removed by the fact that there are two faces acted upon by each field.

Substituting for the fields gives

$$P_M = \sqrt{\frac{\omega\mu_m}{2\sigma_m}}\left(\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0\right)^2\left[\left(\frac{n\pi}{b}\right)^2\int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx + \left(\frac{m\pi}{a}\right)^2\int_0^b \sin^2\left(\frac{n\pi y}{b}\right)dy\right]$$

$$= \sqrt{\frac{\omega\mu_m}{2\sigma_m}}\frac{v_1^2f^2\mathcal{E}_0^2}{8\eta_1^2f_c^4}\left(\frac{n^2a}{b^2} + \frac{m^2b}{a^2}\right) \tag{12-82}$$

This is the loss in watts per meter of length of guide.

The power transmitted along the guide may be considered as due to two waves traveling in the z direction simultaneously. One wave is composed of the $\mathcal{E}_x$ and H_y fields, the other of the $-\mathcal{E}_y$ and H_x fields. Consequently, the power passing any point in the guide is

$$P = \frac{1}{2}(\hat{\mathcal{E}}_x\hat{H}_y + \hat{\mathcal{E}}_y\hat{H}_x)$$

and the total power transmitted may be obtained by integration of this result over the cross-sectional area of the guide. With the field values for the $\text{TM}_{m,n}$ wave from Eqs. 12-14 to 12-19, the power transmitted past a given point by each of the two component waves is

$$P_1 = \frac{1}{2}\left(\frac{m\pi\mathcal{E}_0}{a\omega_c^2\mu_1\epsilon_1}\right)^2\omega\epsilon_1\beta_{m,n}\int_0^a\int_0^b \cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$= \frac{1}{2}\left(\frac{m\pi\mathcal{E}_0}{a\omega_c^2\mu_1\epsilon_1}\right)^2\omega\epsilon_1\beta_{m,n}\frac{ab}{4} \tag{12-83}$$

$$P_2 = \frac{1}{2}\left(\frac{n\pi\mathcal{E}_0}{b\omega_c^2\mu_1\epsilon_1}\right)^2\omega\epsilon_1\beta_{m,n}\int_0^a\int_0^b \sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)dx\,dy$$

$$= \frac{1}{2}\left(\frac{n\pi\mathcal{E}_0}{b\omega_c^2\mu_1\epsilon_1}\right)^2\omega\epsilon_1\beta_{m,n}\frac{ab}{4} \tag{12-84}$$

The total power transmitted through the guide being the sum of P_1 and P_2, it may be written, after insertion of values for $\beta_{m,n}$ and f_c, as

$$P_T = \frac{ab\mathcal{E}_0^2}{8\eta_1}\left(\frac{f}{f_c}\right)^2\sqrt{1 - \frac{f_c^2}{f^2}} \text{ watts} \tag{12-85}$$

this being the value for the $\text{TM}_{m,n}$ mode in the *rectangular guide.*

Similar methods give the following value for the TE mode in the *rectangular guide:*

$$P_T = \frac{ab\eta_1 H_0^2}{8}\left(\frac{f}{f_c}\right)^2\sqrt{1-\frac{f_c^2}{f^2}}\ \text{watts} \qquad (12\text{-}86)$$

Likewise for the TM mode, *cylindrical guide:*

$$P_T = \frac{\pi b^2 \mathcal{E}_0^2}{4\eta_1}[J_n'(\tau_{m,n})]^2\left(\frac{f}{f_c}\right)^2\sqrt{1-\frac{f_c^2}{f^2}}\ \text{watts} \qquad (12\text{-}87)$$

For the TE mode, *cylindrical guide:*

$$P_T = \frac{\pi b^2\eta_1 H_0^2}{4}[J_n(\tau_{m,n})]^2\left(1-\frac{n^2}{\tau_{m,n}^2}\right)\left(\frac{f}{f_c}\right)^2\sqrt{1-\frac{f_c^2}{f^2}}\ \text{watts} \qquad (12\text{-}88)$$

Since the attenuation is given by

$$\alpha = \frac{1}{2}\frac{P_M}{P_T}$$

then the attenuation for the TM mode in the *rectangular guide* with side walls of finite conductivity is

$$\alpha_{TM} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{v_1^2\sqrt{f}}{2\eta_1 f_c^2}\frac{(n^2/b^3+m^2/a^3)}{\sqrt{1-f_c^2/f^2}}\ \text{nepers/m} \qquad (12\text{-}89)$$

By the same methods, the attenuations for other forms of guide are as follows: Rectangular guide, $TE_{0,n}$ mode:

$$\alpha_{TE(0,n)} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{\sqrt{f}}{a\eta_1\sqrt{1-f_c^2/f^2}}\left[1+\frac{2a}{b}\left(\frac{f_c}{f}\right)^2\right]\ \text{nepers/m} \qquad (12\text{-}90)$$

Rectangular guide, $TE_{m,n}$ mode, $(m \neq 0)$:

$$\alpha_{TEm,n} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{2\sqrt{f}}{b\eta_1\sqrt{1-f_c^2/f^2}}$$

$$\left\{\left[1-\left(\frac{f_c}{f}\right)^2\right]\frac{bm^2/an^2+1}{bm^2/an^2+a/b}+\left(1+\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2\right\}\text{nepers/m} \qquad (12\text{-}91)$$

Cylindrical guide, $TE_{n,m}$ mode:

$$\alpha_{TE} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{\sqrt{f}}{b\eta_1\sqrt{1-f_c^2/f^2}}\left[\left(\frac{f_c}{f}\right)^2+\frac{n^2}{\tau_{m,n}^2-n^2}\right]\ \text{nepers/m} \qquad (12\text{-}92)$$

Cylindrical guide, $TM_{n,m}$ mode:

$$\alpha_{TM} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{\sqrt{f}}{b\eta_1\sqrt{1-f_c^2/f^2}}\ \text{nepers/m} \qquad (12\text{-}93)$$

From these expressions the attenuation due to imperfect conductors in the guide walls can be readily found. Attenuation due to

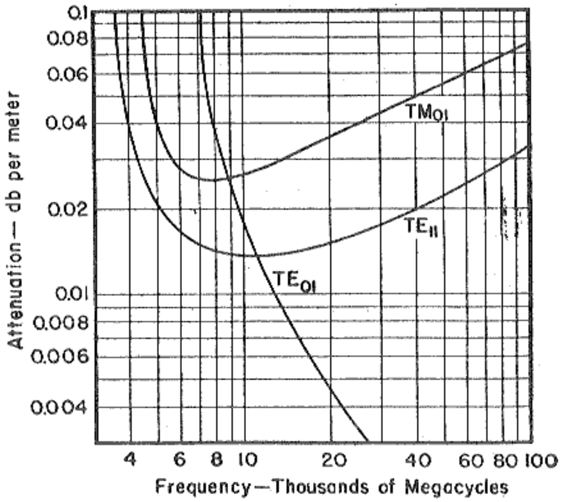

Fig. 12-11. Variation in attenuation of various modes in a cylindrical copper guide of 12.7 cm diameter.

imperfect dielectrics inside the guide may also occur, although most ·ides operate with gaseous dielectric of zero conductivity.

The values of attenuation for the $TE_{0,1}$, $TM_{0,1}$, and $TE_{1,1}$ waves in cylindrical metal guides are plotted in Fig. 12-11 for a guide diameter of 12.7 cm. It can be seen here, and also from Eq. 12-92 that the attenuation of the $TE_{0,1}$ wave decreases indefinitely with increasing frequency. This statement is true only if the guide cross section is perfectly circular, a very difficult situation to maintain.

12-8. Excitation of wave guides

The form or mode of propagation is determined by the type and location of the exciting device. Although either loops (magnetic) or rods (electric) may be used as excitation sources, the rod antenna is normally preferred because of its simplicity. If a guide is closed at one end by a conducting wall and an appropriate exciting antenna rod is inserted through the end or side of the guide, as in Fig. 12-12, the end of the guide serves as a reflector, and if the distance between rod and end wall is properly adjusted, the reflected wave arrives at

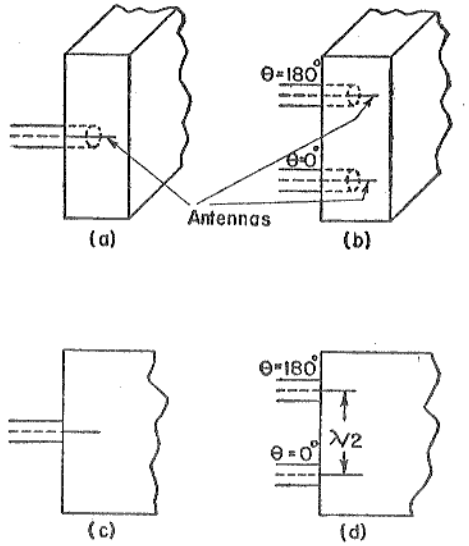

Fig. 12-12. Methods of exciting various modes by setting up the appropriate electric fields: (a) $TE_{1,0}$; (b) $TE_{2,0}$; (c) $TM_{1,1}$; (d) $TM_{2,1}$.

the antenna in phase with the emitted wave, and the two propagate down the guide as one wave.

It can be seen that the rods should coincide with the positions of maximum electric intensity in the fields that they are intended to excite, with attention being given to the proper phasing of the potentials supplied to the rods in accordance with the phasing of the fields to be excited. If current loops are introduced for excitation, the plane of the loop will be made normal to the magnetic field and the loop will be located at a point of maximum magnetic field intensity.

These sources will not usually excite just the waves desired but will also excite higher-order modes. By choice of guide dimensions it is possible to have only the desired wave above cutoff frequency,

the other waves then being attenuated and not propagated. A set of customary guide dimensions is given in Table 12.

Table 12

Usual Wave-Guide Dimensions

Dimensions, in.	λ_c, cm	Useful range, λ-cm	Attenuation—brass walls, db per meter
1.5 × 3.0	14.4	7.6–11.8	0.038 at 10 cm λ
1.0 × 2.0	9.5	5.0– 7.6	0.068 at 6 cm λ
0.75 × 1.5	6.97	3.7– 5.7	0.118 at 5 cm λ
0.625 × 1.25	5.7	3.0– 4.7	0.164 at 3.6 cm λ
0.5 × 1.0	4.57	2.4– 3.7	0.25 at 3.2 cm λ

12-9. Guide terminations

A wave guide is a form of transmission line and must be properly terminated to avoid reflection losses. The termination should provide a wave impedance equal to that of the transmission mode in the guide. Since loads are not always of suitable values, various forms of couplings have been developed for transformation of the impedances to the desired values.

One of the most common forms of loads that are coupled to guides is free space. That is, it is desired that the energy being conveyed by the guide be radiated or transmitted as a space wave, usually in a narrow, well-defined beam. This requirement involves coupling the guide impedance to a wave impedance of 377 ohms and is usually done by expanding the area of the guide in an appropriate direction. The terminating transformer section thus approximates an acoustical horn and in properties is similar to the exponential line. At the throat of the horn the impedance approximates that of the guide, whereas at the mouth the wave impedance is that of free space. If the horn is several wavelengths long, good transformation without reflection is obtained.

Dissipative loads or nonreflecting terminations are also available. They again employ the principle of the tapered or exponential line and use dielectrics having considerable conductivity to provide the power-absorbing properties. The simplest form is that of a wave guide entering a tank of water at a small angle, the guide then being partially filled with a wedge of water. The wedge shape provides the approximate exponential taper, and the water furnishes dielectric loss and a ready means for conveying away the generated heat.

Materials consisting of graphite-bearing porcelains, or graphite- or metallic-bearing plastics, are also available in wedge shapes to replace the water as other types of power-dissipating or nonreflecting loads.

12-10. Resonant cavities

If an excited wave guide is closed by a perfectly conducting sheet at some point, a standing-wave pattern will be set up along the axis of propagation. Since the tangential electric intensity must be zero at the end-wall closure, it will also be zero at half wavelength points back along the guide, wavelength being measured along the guide wall and thus in terms of phase velocity. Since the electric field is zero at any half-wave point, a perfectly conducting plane may be inserted across the guide at any such point without changing the field distribution present between the plane and the end wall, the exciting antenna being considered as present between the shorting planes. A volume or cavity is then formed and is discovered to have resonant properties similar to those of the parallel-resonant circuit, since only definite frequencies can produce the necessary standing-wave field conditions. The *dominant mode* is that field configuration having the lowest resonant frequency. Since such a cavity has other dimensions also terminated in conducting planes, there is a possibility of resonance at other frequencies, depending on excitation and orientation of the cavity.

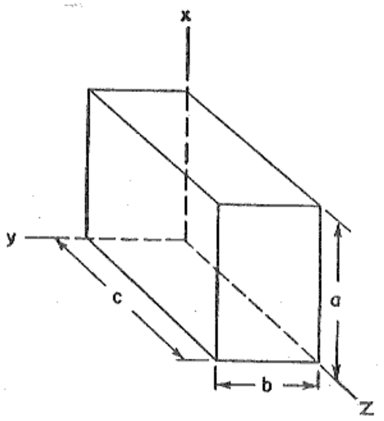

Fig. 12-13. Dimensions of the resonant cavity.

Assume that initially there existed in the rectangular cavity of Fig. 12-13 a $\text{TE}_{m,n}$ field propagating in the z direction. The cavity has walls that are perfect conductors and is filled with a

good dielectric. The initial electric fields would be

$$\hat{\mathcal{E}}_x = \frac{n\pi}{b}\,\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \epsilon^{-j\beta_{m,n}z} \tag{12-94}$$

$$\hat{\mathcal{E}}_y = -\frac{m\pi}{a}\,\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \epsilon^{-j\beta_{m,n}z} \tag{12-95}$$

$$\hat{\mathcal{E}}_z = 0 \tag{12-96}$$

Because of reflections at the walls after the guide is closed off to become a cavity, the field components present will consist of incident and reflected waves, and

$$\mathcal{E}_{xs} = (1 + K)\mathcal{E}_x$$
$$\mathcal{E}_{ys} = (1 + K)\mathcal{E}_y$$

For reflection from perfect conductors, $K = -1$ so that, after considering the reversed sign in the exponential $\epsilon^{-j\beta z}$ for the reflected wave, the standing-wave fields $\mathcal{E}_{xs}$ and $\mathcal{E}_{ys}$ are

$$\hat{\mathcal{E}}_{xs} = \frac{n\pi}{b}\,\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) [\epsilon^{-j\beta_{m,n}z} - \epsilon^{j\beta_{m,n}z}] \tag{12-97}$$

$$\hat{\mathcal{E}}_{ys} = -\frac{m\pi}{a}\,\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) [\epsilon^{-j\beta_{m,n}z} - \epsilon^{j\beta_{m,n}z}] \tag{12-98}$$

If part of the coefficient is collected as H_0', then

$$\mathcal{E}_{xs} = \frac{n\pi}{b} H_0' \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\beta_{m,n}z) \sin\omega t \tag{12-99}$$

$$\mathcal{E}_{ys} = -\frac{m\pi}{a} H_0' \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin(\beta_{m,n}z) \sin\omega t \tag{12-100}$$

$$\mathcal{E}_{zs} = 0$$

At $z = 0$ and $z = c$, the boundary conditions require that $\mathcal{E}_y = 0$. To satisfy this condition it is necessary that

$$\beta_{m,n} = \frac{p\pi}{c}$$

where p is an integer and c is the dimension of the cavity in the z direction.

Maxwell's field equations, as given in Eqs. 12-1 and 12-2, give the wave equation in terms of $\mathcal{E}_x$, or any other component. In terms of $\hat{\mathcal{E}}_x$ this might be

$$\frac{\partial^2 \hat{\mathcal{E}}_x}{\partial x^2} + \frac{\partial^2 \hat{\mathcal{E}}_x}{\partial y^2} + \frac{\partial^2 \hat{\mathcal{E}}_x}{\partial z^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{\mathcal{E}}_x \tag{12-101}$$

Use of the $\mathcal{E}_{xs}$ value in the cavity (Eq. 12-99) in this form of the wave equation gives as a requirement for satisfaction of the wave equation, and in turn for the existence of the fields, that

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2 = \gamma^2 + \omega^2\mu_1\epsilon_1$$

It is then seen that for the fields to exist, the value of γ must be given by

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2 - \omega^2\mu_1\epsilon_1} \tag{12-102}$$

which is closely related to the value for the wave guide. For the fields to propagate in the cavity, γ must be imaginary, or

$$\gamma_{m,n,p} = j\beta_{m,n,p} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{p\pi}{c}\right)^2}$$

The resonant frequency f_r of the cavity is that frequency at which β is zero, so from the above radical,

$$f_r = \frac{v_1}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2} \tag{12-103}$$

This is analogous to the cutoff frequency in the wave guide.

Use of Maxwell's equations permits the accompanying magnetic field components for the TE mode in the resonant rectangular cavity to be written as

$$H_{xs} = -\frac{m\pi}{a}\frac{p\pi}{c}\frac{H_0'}{\omega\mu_1}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)\cos\omega t \tag{12-104}$$

$$H_{ys} = \frac{n\pi}{b}\frac{p\pi}{c}\frac{H_0'}{\omega\mu_1}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)\cos\omega t \tag{12-105}$$

$$H_{zs} = -\frac{H_0'}{\omega\mu_1}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{c}\right)\cos\omega t \tag{12-106}$$

Any one of the integers m, n, or p may be zero but not two, since in that case all the fields would go to zero. Because of the form of the resonant-frequency expression, a wide variety of resonant frequencies are possible, depending on the choice of integers. For a cubical resonator with all dimensions equal, the resonant wavelength is equal to a diagonal of one of the cube faces for the $TE_{0,1,1}$ mode.

Exactly similar expressions for γ, and f_r can be obtained for the TM mode of oscillation in the cavity. When field expressions are written it is found that any zero subscript mode is impossible.

Since the cavity behaves and is employed as a resonant circuit or tuning element, its frequency selectivity is of interest and may be evaluated in terms of its Q at resonance. The Q value was originally defined according to

$$Q = 2\pi \times \frac{\text{energy stored per cycle}}{\text{energy lost per cycle}}$$

The energy loss is due to finite conductivity of the cavity walls and is therefore proportional to the area of the walls. The energy stored is considered as scattered through the cavity volume and consequently is proportional to that volume. If it be assumed that the tangential magnetic field at the walls is twice the average value of magnetic-field intensity in the cavity volume, it is possible to obtain an approximate and simple expression for the Q of a rectangular cavity as

$$\begin{aligned} Q &= \frac{\pi f\mu_1 \times \text{volume}}{\sqrt{\pi f\mu_m/\sigma_m} \times \text{area}} \\ &= \sqrt{\frac{\pi f\sigma_m}{\mu_m}}\,\frac{\mu_1 \times \text{volume}}{\text{area}} \end{aligned} \tag{12-107}$$

Cylindrical cavities are frequently used because of the ease with which they may be manufactured, and their ease of adjustment to resonance by a sliding circular piston. The resonant frequency of such a cavity is given by

$$f_r = \frac{v_1}{2\pi}\sqrt{\left(\frac{p\pi}{c}\right)^2 + \frac{\tau^2}{b^2}} \tag{12-108}$$

where τ means any of the roots of $J_n(b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) = 0$ for TM modes and of $J_n'(b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})$ for TE modes. The mode designations become $TE_{m,n,p}$ and $TM_{m,n,p}$. The $TM_{1,0,0}$ mode is the simplest of the TM type, whereas $TE_{1,0,1}$ is the simplest for the TE modes, since p cannot be zero.

The Q of a cylindrical resonator in the $TM_{m,0,0}$, mode is

$$Q = \mu_1\eta_1 \sqrt{\frac{\pi f_r \sigma_m}{\mu_m}} \left(\frac{bc}{b+c}\right) \tag{12-109}$$

By careful design and plating of internal surfaces with silver or gold, it is possible to obtain Q values up to 100,000, far above those obtained by coils at the lower frequencies or by lines at intermediate frequencies. Tuning means may be supplied by the use of sliding pistons in one end of the cavity, provided good spring contact is made between the end and side walls at all points of the periphery

PROBLEMS

12-1. (a) Find the cutoff frequencies for a $TE_{1,0}$ wave in air in a rectangular guide measuring 5 cm by 2.5 cm.

(b) Calculate the phase and group velocities for the above waves at a frequency of 6000 megacycles.

(c) Find the attenuation to be expected for a frequency at 0.95 times the cutoff value. Neglect conductivity.

12-2. (a) Find the attenuation of a 900-megacycle wave if applied to a square guide with $f_c = 1000$ megacycles, $TE_{1,0}$ mode, air dielectric.

(b) What would be the dimensions of such a guide?

12-3. A wave guide is 8 cm on a side. Tabulate all modes which may be propagated at frequencies of 1500, 3000, and 6000 megacycles.

12-4. Find the characteristic wave impedance for the TM and TE modes in the cylindrical guide.

12-5. In a coaxial line of dimensions $a = 0.5$ cm, $b = 3$ cm, the radial electric field in TEM transmission is equal to 1000 v per centimeter at the surface of the inner conductor. If the dielectric is air, find the magnetic flux density at the surface of the inner conductor, with $f = 10$ megacycles.

(a) Compute the voltage across the line, the conductor currents and the power being transmitted.

(b) By integration of the value of $\mathcal{E} \times H$, or Poynting's vector, over the cross-sectional area of the line, check the value of power transmitted as obtained in (a), above.

12-6. (a) If the line of Prob. 12-5 is made of copper, find the power loss per meter when transmitting the field intensity $\mathcal{E}$ of that problem, with a frequency of 1.4×10^8 c.

(b) Find the attenuation in decibels per meter.

(c) If the line is silver-plated to a depth considerably greater than the depth of penetration, compute the attenuation in decibels per meter.

12-7. Prove that Eq. 12-86 for power transmitted by the TE mode, in a rectangular guide, is correct.

12-8. Compute the power loss for the $TE_{m,0}$ mode in a rectangular guide with walls of finite conductivity.

12-9. (a) A square guide 10 cm on a side is transmitting a 5000-megacycle signal. If it is of silver, air-filled, find the attenuation in decibels per 100 m of guide.

(b) Repeat, if of copper.

(c) Repeat for iron of relative permeability = 100.

12-10. A cylindrical copper tube of inside diameter 3 cm is air-filled. Calculate the cutoff frequencies in the $TE_{1,0}$, $TM_{1,0}$, $TE_{1,1}$, and $TM_{2,1}$ modes.

12-11. If a frequency of 10^{10} c is transmitted in the tube of Prob. 12-10 in the $TE_{1,1}$ mode calculate the attenuation in decibels per 100 ft.

12-12. A metal box is $3 \times 4 \times 5$ cm in dimensions. If it is filled with air, find the resonant frequencies for $TE_{1,0,2}$ fields in the various possible directions.

12-13. Show that for all $TM_{m,n}$ waves in a rectangular guide the minimum attenuation arising from imperfect side-wall conductors occurs at a frequency $f = \sqrt{3}f_c$.

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Chapter 13

RADIATION INTO SPACE

In a previous chapter it was shown that it was possible to set up a traveling electromagnetic wave carrying energy into space. This chapter will deal with a few of the practical means for coupling circuits with space, or the *antennas* for radiating such waves, and with methods of calculating the power radiated, the input impedance, the effect of the earth, and methods for controlling the direction in which the energy is radiated.

A collection of antennas is shown in Fig. 13-1, only a few of which will be discussed here.

13-1. Vector potential

In static electric fields it is relatively easy to determine the scalar potentials present, and to obtain the field about a configuration of charges by differentiating the potential, or taking the gradient. An analogous method is useful in finding the magnetic fields about currents, since in general it is easier to obtain the magnetic potentials from the known currents than it is to write the fields directly.

The static electric potential in a system of charges having density ρ is given by

$$\phi = \frac{1}{\epsilon}\int_v \frac{\rho\, dv}{4\pi r}$$

and by analogy a *magnetic vector potential* is assumed as

$$A = \mu \int_v \frac{J\, dv}{4\pi r} \tag{13-1}$$

where J is the current density throughout the volume dv. This expression is a vector because it depends on the orientation of the current density J.

In a thin filament of cross sectional area da,

$$dv = da\, dl$$

and

$$i\, dl = J\, dv$$