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Chapter 11

GUIDED WAVES BETWEEN PARALLEL PLANES

Maxwell's equations indicate the most general conditions necessary for propagation of an electromagnetic wave in an arbitrary medium. In their normal form they may apply to conditions in a radio wave in a free region remote from other media, but for much transmission it is necessary to confine and *guide* the wave energy from point to point as desired. In such applications the fields are confined or restricted by boundaries of materials different from those of the transmission path, and the waves are said to be guided by these materials. Hence it is necessary to apply to Maxwell's equations certain mathematical restrictions or *boundary conditions* in order to fit the general equations to the particular physical problem in hand. After insertion of the boundary requirements, it may be possible to obtain a solution showing the form and type of wave transmission that can occur in the confined region.

The ordinary coaxial line is an example of such confining of the fields by the conductor boundaries, with transmission of energy wholly within the outer conductor by reason of the traveling fields that are present. The open-wire line is also a form of wave-guiding system, with the field guided by the two wires. Another example is the tube or *wave guide,* in which the field energy propagates inside a rectangular or cylindrical tube, without a central conductor.

The study of the wave guide is most easily approached by first merely confining the fields between two parallel planes of perfectly conducting material, determining the conditions necessary for propagation and the forms of the fields that may be present. Most of the physical ideas and much of the mathematical formulation developed for such a simple configuration may then be directly applied to the completely enclosed form of wave guide.

It should be noted that the method is that of applying successive restrictions to the general form of Maxwell's equations: first, choosing a particular form of time variation; second, requiring propagation in the z direction; and third, restricting the wave to travel between parallel conducting planes parallel to the y,z plane. In Chapter 10 parallel conducting planes parallel to the x,z plane are added to restrict the wave further, forming the complete tube or wave guide.

For simplicity the planes first will be assumed as perfect conductors and later modified to develop the case of finite conductivity.

11-1. Application of the restrictions to Maxwell's equations

The wave equations were obtained in Chapter 9 as

$$\mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = \nabla^2 \mathcal{E}, \qquad \mu\epsilon \frac{\partial^2 H}{\partial t^2} = \nabla^2 H$$

and these show that time variation of the electromagnetic field is necessary for propagation of electric field waves to occur. Because of the wide use of sinusoidal variation with time in electrical theory,

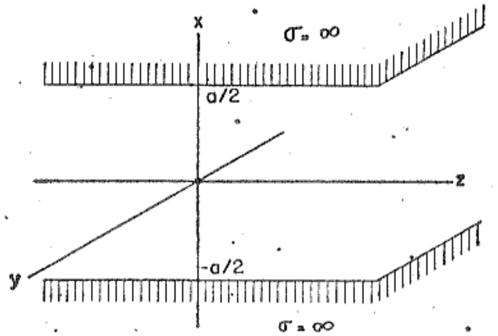

Fig. 11-1. Placement of the conducting planes as wave boundaries.

it will be required here. In a uniform or homogeneous dielectric medium the $\mathcal{E}$ and H fields will also be in time phase, so that, as initial assumed field conditions for a propagating plane wave,

$$\mathcal{E} = \hat{\mathcal{E}} \sin \omega t \tag{11-1}$$

$$H = \hat{H} \sin \omega t \tag{11-2}$$

Variations in time phase may arise later as these waves are introduced into a nonhomogeneous region.

It is desired to study the transmission of this electromagnetic field when it is constricted between two parallel sheets of a perfect conductor, infinite in extent, parallel to the y,z plane, and intersecting the x axis at distance $a/2$ and $-a/2$ from the origin.

It is desirable to orient the axes so that propagation, if there be any, will occur in some desired direction. Thus it is convenient to require the wave energy to propagate in the positive-z direction in Fig. 11-1, since this will correspond to the direction of propagation of some of the field waves discussed in the preceding chapter. This assumption causes no loss of generality in this case, since the axes may be oriented at will, and the conducting planes of the figure were assumed infinite in extent in both y and z directions.

If a propagation constant γ, possibly complex in nature, is assumed to determine the manner of variation of the wave in the z direction, then Eqs. 11-1 and 11-2 become

$$\mathcal{E} = \hat{\mathcal{E}} \epsilon^{-\gamma z} \sin \omega t \tag{11-3}$$

$$H = \hat{H} \epsilon^{-\gamma z} \sin \omega t \tag{11-4}$$

It may be shown that this is just another way of indicating that there is a $(z - vt)$ or $(z + vt)$ function, or that a traveling wave is present.

It should not be immediately assumed that because of the z-directed propagation the electric and magnetic fields will lie wholly in the x,y plane, as was found true in the previous chapter for plane waves in a homogeneous region. The region under discussion is no longer homogeneous, as a result of the introduction of the conducting planes. Therefore it is much safer to permit the solution of the equations to indicate the direction of the fields present.

Now that the fields have been restricted to sinusoidal time variation and to propagation of energy in the z direction, and have been bounded in the x direction by infinite perfectly conducting planes, it is time to insert these conditions in Maxwell's field equations and to determine the field components that may exist in a propagating wave. Writing Maxwell's two field equations in terms of the ordinary values, $\mathcal{E}$ and H of the fields, results in

$$\nabla \times \mathcal{E} = -\mu \frac{\partial H}{\partial t} \tag{11-5}$$

$$\nabla \times H = \sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t} \tag{11-6}$$

Because of the infinite extent of the planes in the y direction and to the assumed z-propagation it may be reasoned that there will be no variation of any of the field magnitudes with y. Consequently,

$$\frac{\partial H_z}{\partial y} = \frac{\partial H_x}{\partial y} = \frac{\partial \mathcal{E}_z}{\partial y} = \frac{\partial \mathcal{E}_x}{\partial y} = 0$$

and Eqs. 11-5 and 11-6 become

$$\left.\begin{aligned} -\frac{\partial H_y}{\partial z} &= \sigma \mathcal{E}_x + \epsilon \frac{\partial \mathcal{E}_x}{\partial t} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \sigma \mathcal{E}_y + \epsilon \frac{\partial \mathcal{E}_y}{\partial t} \\ \frac{\partial H_y}{\partial x} &= \sigma \mathcal{E}_z + \epsilon \frac{\partial \mathcal{E}_z}{\partial t} \end{aligned}\right\} \qquad (11\text{-}7)$$

$$\left.\begin{aligned} -\frac{\partial \mathcal{E}_y}{\partial z} &= -\mu \frac{\partial H_x}{\partial t} \\ \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} &= -\mu \frac{\partial H_y}{\partial t} \\ \frac{\partial \mathcal{E}_y}{\partial x} &= -\mu \frac{\partial H_z}{\partial t} \end{aligned}\right\} \qquad (11\text{-}8)$$

If the appropriate derivatives of Eqs. 11-3 and 11-4 are taken and with $\hat{\mathcal{E}}$ and $\hat{H}$ as maximum time values, there results

$$\left.\begin{aligned} \gamma \hat{H}_y &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_x \\ -\gamma \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_y \\ \frac{\partial \hat{H}_y}{\partial x} &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_z \end{aligned}\right\} \qquad (11\text{-}9)$$

$$\left.\begin{aligned} \gamma \hat{\mathcal{E}}_y &= -j\omega\mu \hat{H}_x \\ -\gamma \hat{\mathcal{E}}_x - \frac{\partial \hat{\mathcal{E}}_z}{\partial x} &= -j\omega\mu \hat{H}_y \\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} &= -j\omega\mu \hat{H}_z \end{aligned}\right\} \qquad (11\text{-}10)$$

These equations are the result of introduction of restrictions on time, and on the form and direction of propagation, as limited by the perfectly conducting infinite planes. It remains to find the form of the fields and the types of propagation possible under the restrictions.

11-2. Types of propagation; TM, TE, and TEM waves

Equations 11-9 and 11-10 indicate that, in general, electric and magnetic field components may exist along all axes. The equations may be further simplified, without loss of generality, by orientation of the fields or by appropriate excitation from the wave source, such that either the magnetic field lies wholly along the y axis or the electric field lies wholly along the y axis.

In the first case, $\hat{H}_x = \hat{H}_z = 0$, and the magnetic field is seen to be wholly *transverse* to the direction of propagation on z in Fig. 11-2(a); hence the electric field has components $\hat{\mathcal{E}}_x$ and $\hat{\mathcal{E}}_z$, with

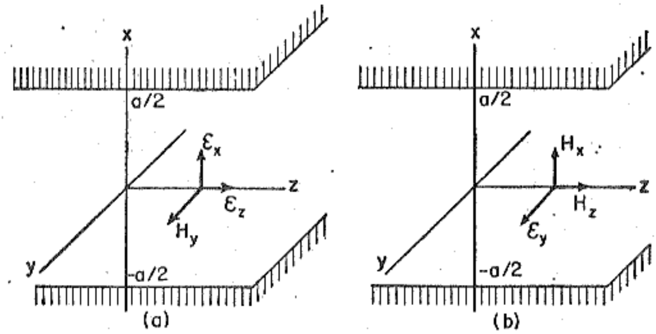

Fig. 11-2. (a) Fields in the transverse-magnetic (TM) wave; (b) fields in the transverse-electric (TE) wave.

$\hat{\mathcal{E}}_y = 0$, since no electric field can exist in the direction of the magnetic field. Such a wave is said to be *transverse magnetic* and is known as a TM wave. Equations 11-9 and 11-10 then reduce to

$$\left.\begin{aligned} \gamma \hat{H}_y &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_x && \text{(a)} \\ \frac{\partial \hat{H}_y}{\partial x} &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_z && \text{(b)} \\ \gamma \hat{\mathcal{E}}_x + \frac{\partial \hat{\mathcal{E}}_z}{\partial_x} &= j\omega\mu \hat{H}_y && \text{(c)} \end{aligned}\right\} \text{TM waves} \qquad (11\text{-}11)$$

In the second case, illustrated by Fig. 11-2(b), $\hat{\mathcal{E}}_x = \hat{\mathcal{E}}_z = 0$, and the electric field is made wholly transverse to the direction of propagation on z. The magnetic field has components $\hat{H}_x$ and $\hat{H}_z$, and $H_y = 0$. The wave so oriented is said to be *transverse electric* and is known as a TE wave. For the TE wave, Eqs. 11-9 and 11-10 reduce to

$$\left.\begin{aligned} \gamma\hat{H}_x + \frac{\partial \hat{H}_z}{\partial x} &= -(\sigma + j\omega\epsilon)\hat{\mathcal{E}}_y \quad &(a)\\ \gamma\hat{\mathcal{E}}_y &= -j\omega\mu\hat{H}_x \quad &(b)\\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} &= -j\omega\mu\hat{H}_z \quad &(c) \end{aligned}\right\}\text{TE waves} \qquad (11\text{-}12)$$

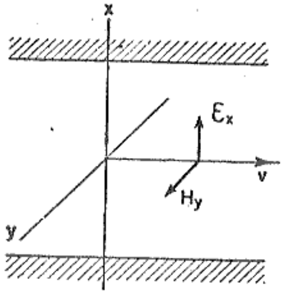

Fig. 11-3. The transverse-electromagnetic (TEM) field vectors.

A third case may exist in which the magnetic field is wholly along y while the electric field is wholly along x, no fields existing in the z direction. In this case, which represents a special form of TM wave with $\mathcal{E}_z = 0$, both the electric and magnetic field components are transverse to the direction of propagation on z, and the wave is said to be *transverse electromagnetic* or of the TEM type.

Equations 11-9 and 11-10 reduce, for the TEM condition, to

$$\left.\begin{aligned} \gamma\hat{H}_y &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_x \quad &(a)\\ \frac{\partial \hat{H}_y}{\partial x} &= 0 \quad &(b)\\ \gamma\hat{\mathcal{E}}_x &= j\omega\mu\hat{H}_y \quad &(c) \end{aligned}\right\}\text{TEM waves} \qquad (11\text{-}13)$$

11-3. Transmission of TM waves between parallel planes

The three differential equations describing TM waves (Eq. 11-11) may be readily solved for expressions for the three field components present. Differentiation of Eq. 11-11(b) and substitution of the result and Eq. 11-11(a) into Eq. 11-11(c) yields an expression containing H_y only:

$$\frac{\partial^2 \hat{H}_y}{\partial x^2} = [-\gamma^2 + (\sigma + j\omega\epsilon)j\omega\mu]\,\hat{H}_y$$

If the space between the parallel planes is considered a good dielectric of properties μ_1 and ϵ_1, and with σ_1 very small with respect to $\omega\epsilon_1$,

$$\frac{\partial^2 \hat{H}_y}{\partial x^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{H}_y \qquad (11\text{-}14)$$

This equation may be readily solved; and the solution may be written

$$\hat{H}_y = A_1\epsilon^{j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x} + A_2\epsilon^{-j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x}$$

where A_1 and A_2 are arbitrary constants whose values are determined by the source of field excitation.

The two exponential functions of x indicate the presence of an incident and reflected wave system for H_y, the two waves propagating in the plus and minus x directions. This result is not surprising, because of the bounding of the field in the x directions by the perfectly conducting planes. The above equation can be put in a form in which the field may be more easily visualized by use of the expressions $\epsilon^{j\theta} = \cos\theta + j\sin\theta$ and $\epsilon^{-j\theta} = \cos\theta - j\sin\theta$. Then

$$\hat{H}_y = B_1\sin(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x + B_2\cos(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x \qquad (11\text{-}15)$$

By insertion of this result in Eq. 11-11(a), the value of $\hat{\mathcal{E}}_x$ can be obtained as

$$\hat{\mathcal{E}}_x = \frac{-j\gamma}{\omega\epsilon_1}[B_1\sin(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x + B_2\cos(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x] \qquad (11\text{-}16)$$

The $\hat{\mathcal{E}}_z$ component of field may be obtained from Eq. 11-11(b) as

$$\hat{\mathcal{E}}_z = \frac{-j\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}}{\omega\epsilon_1}[B_1\cos(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x - B_2\sin(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x] \qquad (11\text{-}17)$$

For each of these fields it is obvious that the value along the x axis is the resultant of two fields traveling in opposite directions, and that because of the lack of attenuation in this direction, a true standing-wave pattern of field distribution in the x direction will result.

One further bit of boundary information may now be employed to gain further knowledge of the arbitrary constants B_1 and B_2. Because of the perfect conductivity of the planes, it is known that at $x = a/2$ and $x = -a/2$ the tangential component of electric field $\mathcal{E}_z$ must be zero. Examination of Eq. 11-17 shows that this value may occur either if $B_1 = B_2 = 0$ (which is of no importance since the field would be zero everywhere), or if the angles have values of $m\pi/2$ at $a/2$ and $-a/2$. Accordingly it can be said that the conditions at the boundary planes require that

$$(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})\frac{a}{2} = \frac{m\pi}{2} \qquad (m = 0, 1, 2, 3, \ldots)$$

from which

$$\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1} = \frac{m\pi}{a} \tag{11-18}$$

and the expression for $\hat{\mathcal{E}}_z$ may be written

$$\hat{\mathcal{E}}_z = \frac{-jm\pi}{a\omega\epsilon_1}\left[B_1 \cos\left(\frac{m\pi x}{a}\right) - B_2 \sin\left(\frac{m\pi x}{a}\right)\right] \tag{11-19}$$

This expression may be studied further. When the field is examined at the planes or at $x = a/2$ or $x = -a/2$, three possible conditions leading to zero field intensity are discovered. These are:

If $m = 0$, then $\mathcal{E}_z = 0$ everywhere.
If m is even, then $B_1 = 0$.
If m is odd, then $B_2 = 0$.

Thus the configuration of the fields between the planes becomes dependent on the value of the integer m. To designate the particular type of wave under discussion, it is customary to refer to waves having $m = 0$, $m = 1$, $m = 2$, . . . , as TM_0, TM_1, TM_2, . . . , waves. These various configurations of the field intensities in the guide are called *modes of propagation*. The field distributions are sinusoidal or cosinusoidal in the x direction, the number of maxima depending on the value of m.

As previously required, the fields are to propagate in the positive z direction according to the relation $\epsilon^{-\gamma z}$. The constant γ is complex in general, and it has been customary to express it in terms of the attenuation constant α and phase constant β as

$$\gamma = \alpha + j\beta$$

The value of γ for the particular field under consideration may be found from Eq. 11-18 as

$$\gamma_m = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu_1\epsilon_1} \tag{11-20}$$

Since all the values under the radical are real, the value of γ must be

wholly real or wholly imaginary, depending on the relative magnitudes of $m\pi/a$ or $\omega^2\mu_1\epsilon_1$. Thus two conditions arise, with α equal to zero or with β equal to zero. It is possible to have propagation with zero attenuation or high attenuation with β equal to zero. The angular velocity ω or the frequency chosen is the factor determining whether or not propagation takes place for a given spacing a of the planes. The critical frequency f_c at which the attenuation condition changes to the propagation condition can be determined from

$$\sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega_c^2\mu_1\epsilon_1} = 0$$

$$f_c = \frac{m}{2a}\sqrt{\frac{1}{\mu_1\epsilon_1}} \tag{11-21}$$

A value for m/a in terms of f_c permits Eq. 11-20 for γ_m to be written as

$$\gamma_m = \frac{m\pi}{a}\sqrt{1 - \frac{f^2}{f_c^2}} \tag{11-22}$$

$$= \omega_c\sqrt{\mu_1\epsilon_1}\sqrt{1 - \frac{f^2}{f_c^2}} \tag{11-23}$$

For frequencies *below cutoff* where $f < f_c$ and γ_m is real, the fields are attenuated and $\gamma_m = \alpha_m$, $\beta_m = 0$. The phase angles will be constant, and the field amplitudes will decrease very rapidly with distance z, because of the attenuation term $\epsilon^{-\gamma_m z} = \epsilon^{-\alpha_m z}$.

At frequencies *above cutoff*, $f > f_c$, γ_m is imaginary, and $\alpha_m = 0$. Thus propagation will occur and

$$\gamma_m = j\beta_m = j\frac{m\pi}{a}\sqrt{\frac{f^2}{f_c^2} - 1} \tag{11-24}$$

$$= j\omega_c\sqrt{\mu_1\epsilon_1}\sqrt{\frac{f^2}{f_c^2} - 1} \tag{11-25}$$

A condition of propagation without attenuation exists in the z direction; this condition is normally the one of interest. In the practical case, owing to the finite conductivity of the plates, some attenuation accompanies propagation at frequencies above cutoff.

The action of the planes in only propagating frequencies above a certain cutoff value bears a definite resemblance to the operation of the high-pass filter. This similarity can be further borne out by study of relations for cutoff frequency, Z_0, attenuation in the stop band, as functions of f/f_c.

The critical frequency of cutoff may be given some physical significance by writing the expression for f_c as

$$f_c = \frac{v_1}{2a/m}$$

from which the critical wavelength can be seen as

$$\lambda_c = \frac{2a}{m} \tag{11-26}$$

The critical wavelength is related to the plate spacing a as

$$\frac{m\lambda_c}{2} = a$$

so that the critical wavelength is fixed as that at which the distance between the planes becomes exactly m half waves. For waves longer than this value (lower frequencies), the planes serve to attenuate the fields. Shorter wavelengths are propagated without loss. The integer m can be interpreted as the number of field maximums occurring in the x direction between the planes.

The expressions developed above now permit the three field components to be written by inclusion of the time and z-propagation function $\epsilon^{-\gamma z} \sin \omega t$ as

$$H_y = \left[B_1 \sin\left(\frac{m\pi x}{a}\right) + B_2 \cos\left(\frac{m\pi x}{a}\right)\right] \epsilon^{-j\beta_m z} \sin \omega t \tag{11-27}$$

$$\mathcal{E}_x = \frac{\beta_m}{\omega\epsilon_1}\left[B_1 \sin\left(\frac{m\pi x}{a}\right) + B_2 \cos\left(\frac{m\pi x}{a}\right)\right] \epsilon^{-j\beta_m z} \sin \omega t \tag{11-28}$$

$$\mathcal{E}_z = \frac{-m\pi}{a\omega\epsilon_1}\left[B_1 \cos\left(\frac{m\pi x}{a}\right) - B_2 \sin\left(\frac{m\pi x}{a}\right)\right] \epsilon^{-j\beta_m z} \cos \omega t \tag{11-29}$$

for the TM_m waves under the conditions of propagation. The form is obtained by use of the value for $\gamma_m = j\beta_m$ in the propagation region and consideration of the meaning of the j coefficient appearing before the magnitude function in some cases.

11-4. Transmission of TE waves between parallel planes

The TE wave equations given as Eq. 11-12 may be combined to yield

$$\frac{\partial^2 \hat{\mathcal{E}}_y}{\partial x^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{\mathcal{E}}_y \tag{11-30}$$

for the case of a good dielectric between the planes. This equation is of the same form as Eq. 11-14 for $\hat{H}_y$ in the TM wave case, and a similar solution follows directly, giving

$$\hat{\mathcal{E}}_y = A_3\epsilon^{j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x} + A_4\epsilon^{-j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x}$$

As for the TM case, the two exponential x functions indicate two $\mathcal{E}_y$ field wave components traveling oppositely in the x direction, as a result of reflections from the parallel conducting planes.

The above equation may be written in terms of sines and cosines as

$$\hat{\mathcal{E}}_y = B_3 \sin(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x + B_4 \cos(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x \tag{11-31}$$

indicating a standing-wave form of distribution in the x direction between the planes.

Use of Eq. 11-12(b) gives for $\hat{H}_x$,

$$\hat{H}_x = \frac{j\gamma}{\omega\mu_1}[B_3 \sin(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x + B_4 \cos(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x] \tag{11-32}$$

and, after performing the indicated differentiation, Eq. 11-12(c) gives

$$\hat{H}_z = \frac{j\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}}{\omega\mu_1}[B_3 \cos(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x - B_4 \sin(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})x] \tag{11-33}$$

The three field intensities for the TE waves between parallel planes are thus determined.

The perfect conductivity of the bounding parallel planes requires that the tangential electric field, $\mathcal{E}_y$, be zero at $x = a/2$ and $x = -a/2$. This condition may be satisfied by proper choice of the angle involved as

$$(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1})\frac{a}{2} = \frac{m\pi}{2} \qquad (m = 0, 1, 2, 3, \ldots) \tag{11-34}$$

and by assignment of either B_3 or B_4 as zero.

The propagation constant γ_m for TE transmission may be determined from Eq. 11-34 as

$$\gamma_m = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu_1\epsilon_1} \tag{11-35}$$

Since this result is identical with that for TM waves, the conditions of propagation are the same and the parallel planes become a high-pass filter with cutoff frequency

$$f_c = \frac{m}{2a}\sqrt{\frac{1}{\mu_1\epsilon_1}} \tag{11-36}$$

and with constants in the pass-band of frequencies as

$$\gamma_m = j\beta_m = j\,\frac{m\pi}{a}\sqrt{\frac{f^2}{f_c^2} - 1} \tag{11-37}$$

$$\alpha_m = 0$$

Use of the relations developed above and the time and propagation function permits simplification of the field equations for TE_m waves between parallel planes to

$$\mathcal{E}_y = \left[B_3 \sin\left(\frac{m\pi x}{a}\right) + B_4 \cos\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z}\sin\omega t \tag{11-38}$$

$$H_x = \frac{-\beta_m}{\omega\mu_1}\left[B_3 \sin\left(\frac{m\pi x}{a}\right) + B_4 \cos\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z}\sin\omega t \tag{11-39}$$

$$H_z = \frac{m\pi}{a\omega\mu_1}\left[B_3 \cos\left(\frac{m\pi x}{a}\right) - B_4 \sin\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z}\cos\omega t \tag{11-40}$$

Further consideration of the boundary conditions imposed on $\mathcal{E}_y$ indicates that three possible conditions may arise, dependent on the choice of m:

If $m = 0$, then $B_4 = 0$ and all fields are zero.
If m is even, then $B_4 = 0$.
If m is odd, then $B_3 = 0$.

The configuration of the fields between the planes is seen as dependent on the value chosen for m; and for values of $m = 1$, $m = 2$, . . . , the various field modes designated as TE_1, TE_2, . . . , arise.

11-5. Transmission of TEM waves between parallel planes

The TEM form of field was arrived at in Section 11-2 as a special case of TM propagation in which $\mathcal{E}_z$ was zero. In Section 11-3 it was found that this condition on $\mathcal{E}_z$ was obtained if m were made zero. Accordingly, the TEM wave becomes a TM wave with $m = 0$, and the field equations may be written from Eqs. 11-27, 11-28, and 11-29 as

$$H_y = B_2\epsilon^{-j\beta_0 z}\sin\omega t \tag{11-41}$$

$$\mathcal{E}_x = \frac{\beta_0}{\omega\epsilon_1}B_2\epsilon^{-j\beta_0 z}\sin\omega t \tag{11-42}$$

$$\mathcal{E}_z = 0$$

for conditions of TEM propagation. The same results could have been obtained from the differential equations for TEM waves, Eq. 11-13, under the good-dielectric assumption. One additional piece of information may be gained from Eq. 11-13(b), which states

$$\frac{\partial \hat{H}_y}{\partial x} = 0$$

This equation states that $\hat{H}_y$ is a constant for all values of x or for all positions between the planes, for a particular z value.

From Eq. 11-13 it is also possible to write that for the good-dielectric condition,

$$\frac{\gamma^2}{j\omega\mu_1}\hat{\mathcal{E}}_x = j\omega\epsilon_1\hat{\mathcal{E}}_x$$

from which

$$\gamma^2 + \omega^2\mu_1\epsilon_1 = 0 \tag{11-43}$$

which is Eq. 11-18, previously obtained for TM waves, with $m = 0$. With the above equation, the propagation constant γ for TEM waves is

$$\gamma = j\beta = j\omega\sqrt{\mu_1\epsilon_1} \tag{11-44}$$

and

$$\alpha = 0$$

Since $m = 0$, the cutoff frequency given for TM waves becomes zero, so that TEM waves propagate without attenuation between the perfectly conducting plates for all frequencies above that of zero.

Use of Eq. 11-44 permits the expression for $\mathcal{E}_x$ to be reduced to

$$\mathcal{E}_x = \sqrt{\frac{\mu_1}{\epsilon_1}}\, B_2 \epsilon^{-j\beta z} \sin \omega t$$

$$= \eta_1 B_2 \epsilon^{-j\beta z} \sin \omega t \qquad (11\text{-}45)$$

Comparison with Eq. 11-41 for H_y,

$$H_y = B_2 \epsilon^{-j\beta z} \sin \omega t$$

shows $\mathcal{E}$ and H to be related by the factor η_1, the intrinsic impedance of the dielectric, which is a condition analogous to that occurring in wave propagation in free space. The value of β also corresponds to free-space transmission. Since there is no component of electric field tangential to the conducting planes, it can be seen that these planes have no effect except to limit the area of the wave.

Because of the relation of TEM waves to the previously studied space waves and in turn the analogy of space waves to transmission-line waves, it would seem possible to deduce an analogy between TEM waves and transmission-line waves. In fact, it is found that TEM waves are actually those which would appear in the fields present along a dissipationless transmission line.

Equations 11-41 and 11-45 show that for the TEM wave,

$$\mathcal{E}_x = \eta_1 H_y$$

The power propagating in the z direction between the planes, per unit width in the y direction, is given by the *Poynting radiation theorem from field concepts* as

$$P = \tfrac{1}{2}\eta_1 \hat{H}_y^2 \times a \times 1 \text{ watts/meter of width} \qquad (11\text{-}46)$$

the area of the path being $a \times 1$ meter. The Poynting vector direction confirms the positive-z direction of this power flow. The area under consideration is outlined in Fig. 11-4. The maximum value of potential *drop* between the bottom and top plates is given by the integral of the electric field as

$$\hat{V} = \int \hat{\mathcal{E}} \cos \theta\, dl = \int_{-a/2}^{a/2} \eta_1 B_2\, dx$$

$$\hat{V} = \eta_1 B_2 a \text{ volts} \qquad (11\text{-}47)$$

The current flowing in either top or bottom plate per unit y width

may be found by consideration of the path $ABCD$ in Fig. 11-4. Because of the perfect conductivity, all current will flow on the plate surface and a field intensity H_y will exist between A and B in the dielectric. Since H_y does not vary with x, the exact location of the path AB is immaterial. Since the magnetic field will not penetrate into the perfect conductor, no magnetic-field intensity will exist

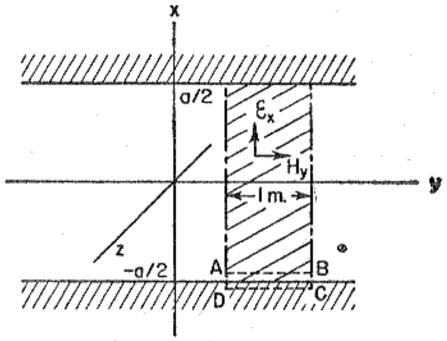

Fig. 11-4. The cross section considered for energy flow in the TEM wave.

along the path DC in the perfect conductor. Hence the current flowing per meter of width in the y direction is the current enclosed by the path $ABCD$. Consequently,

$$I = \oint \boldsymbol{H} \cdot \boldsymbol{dl} = \int_A^B H_y\, dy + \int_B^C 0\, dx + \int_C^D 0\, dy + \int_D^A 0\, dx$$

there being no normal H component. The maximum value of current per unit width of plate then is

$$\hat{I} = B_2$$

Since the planes are lossless, the power being transmitted along the planes and supplied to the load is

$$P = \tfrac{1}{2}\hat{V}\hat{I} = \tfrac{1}{2}\eta_1 B_2^2 a \qquad (11\text{-}48)$$

per unit width in the y direction. Since the maximum value of H_y is B_2 the power delivered to the load, *in terms of current and voltage concepts*, is

$$P = \tfrac{1}{2}\eta_1 H_y^2 a \text{ watts/meter of width} \qquad (11\text{-}49)$$

and this is identical with the power shown to be conveyed by the electromagnetic fields by use of Poynting's vector, given by Eq. 11-46. It may then be concluded that the energy transmitted along a lossless conductive path is conveyed by the electric and magnetic fields in the region, the voltage and current being present merely as evidence of existence of the fields.

11-6. Manner of wave travel

Propagation of TM and TE waves has been shown to be similar, except for polarization of the fields. The presence of the z component of field introduces an apparent inconsistency, since propagation has been assumed to occur in the z direction; yet Fig. 11-2 would

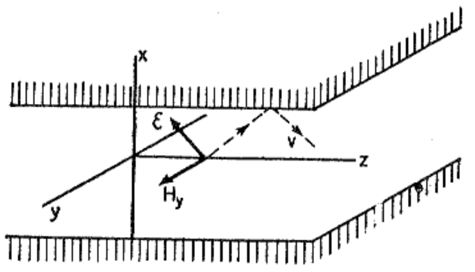

Fig. 11-5. TM field components.

seem to indicate propagation at an angle, because of the presence of the z-directed field component. This point can be readily clarified.

The electric field components of an arbitrarily chosen TM wave are obtainable from Eqs. 11-28 and 11-29 as

$$\mathcal{E}_x = \frac{\beta_m}{\omega\epsilon_1}\left[B_1 \sin\left(\frac{m\pi x}{a}\right) + B_2 \cos\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z} \sin \omega t$$

$$\mathcal{E}_z = \frac{-m\pi}{a\omega\epsilon_1}\left[B_1 \cos\left(\frac{m\pi x}{a}\right) - B_2 \sin\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z} \cos \omega t$$

and the wave is illustrated in Fig. 11-5.

For simplicity, an m = odd mode of operation may be selected, making $B_2 = 0$. The electric field components then become

$$\mathcal{E}_x = \frac{\beta_m}{\omega\epsilon_1} B_1\epsilon^{-j\beta_m z} \sin \frac{m\pi x}{a} \sin \omega t \qquad (11\text{-}50)$$

$$\mathcal{E}_z = \frac{-m\pi}{a\omega\epsilon_1} B_1\epsilon^{-j\beta_m z} \cos \frac{m\pi x}{a} \cos \omega t \qquad (11\text{-}51)$$

By use of the trigonometric identities,

$$\sin a \sin b = \tfrac{1}{2}[\cos (a - b) - \cos (a + b)]$$

$$\cos a \cos b = \tfrac{1}{2}[\cos (a - b) + \cos (a + b)]$$

the electric fields may be converted to

$$\mathcal{E}_x = \frac{\beta_m}{2\omega\epsilon_1} B_1\epsilon^{-j\beta_m z}\left[\cos\left(\omega t - \frac{m\pi x}{a}\right) - \cos\left(\omega t + \frac{m\pi x}{a}\right)\right] \qquad (11\text{-}52)$$

$$\mathcal{E}_z = \frac{-m\pi}{2a\omega\epsilon_1} B_1\epsilon^{-j\beta_m z}\left[\cos\left(\omega t - \frac{m\pi x}{a}\right) + \cos\left(\omega t + \frac{m\pi x}{a}\right)\right] \qquad (11\text{-}53)$$

If the directions of the field components are indicated by unit vectors, $\bar{a}_x$ for the x direction and $\bar{a}_z$ for the z direction, then the total electric field intensity is given by the vector sum of $\mathcal{E}_x$ and $\mathcal{E}_z$ as

$$\boldsymbol{\mathcal{E}} = \bar{a}_x\mathcal{E}_x + \bar{a}_z\mathcal{E}_z$$

Then

$$\boldsymbol{\mathcal{E}} = \frac{B_1}{2\omega\epsilon_1}\epsilon^{-j\beta_m z}\left\{\left[\bar{a}_x\beta_m - \bar{a}_z\frac{m\pi}{a}\right]\cos\left(\omega t - \frac{m\pi x}{a}\right) - \left[\bar{a}_x\beta_m + \bar{a}_z\frac{m\pi}{a}\right]\cos\left(\omega t + \frac{m\pi x}{a}\right)\right\} \qquad (11\text{-}54)$$

The H_y component of the field may be written from Eq. 11-27 for the TM_m (m = odd) mode as

$$H_y = B_1\epsilon^{-j\beta_m z} \sin \frac{m\pi x}{a} \sin \omega t$$

which may be converted to

$$H_y = \frac{B_1}{2}\epsilon^{-j\beta_m z}\left[\cos\left(\omega t - \frac{m\pi x}{a}\right) - \cos\left(\omega t + \frac{m\pi x}{a}\right)\right] \qquad (11\text{-}55)$$

Equations 11-54 and 11-55 show that propagation is the resultant of two waves at an angle in the region between the plates as represented in Fig. 11-6, one wave at a particular z value having an upward-directed x component of velocity, $\cos(\omega t - m\pi x/a)$ and the other at the same point having a downward-directed x component of velocity, $\cos(\omega t + m\pi x/a)$.

When the electric field having a magnitude $(\bar{a}_x\beta_m - \bar{a}_z m\pi/a)$ and upward-directed velocity is combined with its companion H_y term, Poynting's vector indicates a power-density flow up to the right, as in Fig. 11-6(b). Consideration of the electric field component having a magnitude $-(\bar{a}_x\beta_m + \bar{a}_z m\pi/a)$ and downward-directed x velocity

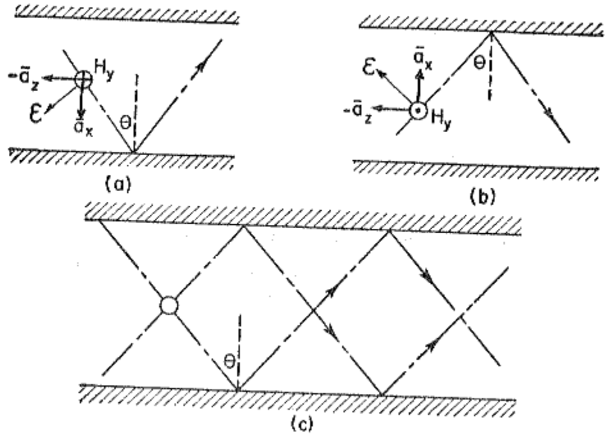

Fig. 11-6. (a) Component wave with velocity component in downward x direction; (b) component wave with velocity component in upward x direction; (c) path of both waves between the parallel planes.

component combined with the appropriate H_y field (which is negative) gives a Poynting vector indicating a power flow down to the right as in Fig. 11-6(a). These fields represent simultaneous propagation of crossed waves between the plane guides, the wave paths making an angle of $\pi - 2\theta$ with each other as in (c). Both waves are propagated equally in the z direction according to $\epsilon^{j\beta_m z}$.

From the figures it is apparent that the angle of incidence θ on the guiding planes is related to the magnitude of the field vectors in the $\bar{a}_x$ and $\bar{a}_z$ directions as

$$\tan\theta = \frac{\beta_m}{m\pi/a} = \frac{a\beta_m}{m\pi}$$

$$= \sqrt{\frac{f^2}{f_c^2} - 1} \qquad (11\text{-}56)$$

It may be shown that this result, though derived for TM modes with m = odd, is perfectly general for any TM or TE modes. As the

frequency is reduced and nears the critical frequency, $\tan\theta$ and θ approach zero. At $\theta = 0°$ there is no transmission in the z direction, since $\beta_m = 0$, and the waves simply bounce back and forth between the upper and lower planes. For frequencies much above f_c the angle of incidence θ becomes large and the waves propagate between the guiding planes by a succession of glancing reflections.

11-7. Velocities of the waves

The velocity of propagation of the component waves *along* their respective paths in Fig. 11-6 is determined by the dielectric material between the plates and is

$$v_1 = \frac{1}{\sqrt{\mu_1\epsilon_1}} \text{ m/sec}$$

In Eq. 11-56 the angle of incidence of the bouncing waves on the planes was found to be specified by

$$\tan\theta = \sqrt{\frac{f^2}{f_c^2} - 1}$$

The other trigonometric functions of θ may be calculated as

$$\sin\theta = \frac{\sqrt{f^2/f_c^2 - 1}}{f/f_c} = \sqrt{1 - \frac{f_c^2}{f^2}} \qquad (11\text{-}57)$$

$$\cos\theta = \frac{1}{f/f_c} = \frac{f_c}{f} \qquad (11\text{-}58)$$

In Eq. 10-100 the phase velocity of a wave was found to be

$$v_p = \frac{v_1}{\sin\theta} \qquad (11\text{-}59)$$

By use of the value of $\sin\theta$ from Eq. 11-57, this may be written

$$v_p = \frac{v_1}{\sqrt{1 - f_c^2/f^2}} \qquad (11\text{-}60)$$

Equation 11-60 indicates that the phase velocity is above v_1, the velocity of light along the wave path, and that at cutoff the phase velocity becomes infinite. Since the phase velocity measures the rate of change of phase along the surface of a conducting plane and the wave at cutoff is bouncing back and forth between the planes with $\theta = 0$ deg, the phase angle changes simultaneously at all points along one of the conducting planes and an infinite phase velocity is indicated.

The phase constant β is given by

$$\beta = \frac{m\pi}{a}\sqrt{\frac{f^2}{f_c^2} - 1}$$

The derivative $d\beta/d\omega$ may be taken as

$$\frac{d\beta}{d\omega} = \frac{m\pi}{a}\frac{\omega/\omega_c^2}{\sqrt{\omega^2/\omega_c^2 - 1}} = \frac{m}{2af_c\sqrt{1 - f_c^2/f^2}} \tag{11-61}$$

By taking the reciprocal the group velocity is obtained as

$$v_g = \frac{1}{d\beta/d\omega} = \frac{2af_c}{m}\sqrt{1 - \frac{f_c^2}{f^2}} \tag{11-62}$$

Using the expression for f_c,

$$f_c = \frac{mv_1}{2a}$$

the group velocity becomes

$$v_g = v_1\sqrt{1 - \frac{f_c^2}{f^2}} \tag{11-63}$$

$$= v_1 \sin\theta \tag{11-64}$$

By reference to Fig. 11-7, the expression $v_1 \sin\theta$ can be seen as simply the component of v_1 in the z direction, and thus the group velocity is merely the *average rate of travel* of energy in the z direction for the wave guided between the conducting planes.

The wave-front configuration indicated in Fig. 11-7 for one of the crossed waves can be supported through use of the expression for $\cos\theta$

$$\cos\theta = \frac{f_c}{f} = \frac{m}{2af\sqrt{\mu_1\epsilon_1}}$$

which, for $m = 1$, becomes

$$\cos\theta = \frac{\lambda/2}{a} \tag{11-65}$$

a statement for which Fig. 11-7 is drawn.

It should be noted that the dielectric between the planes is a dispersive medium as indicated by the fact that β is a function of frequency. The region is lossless by assumption, however, so that the implications of Section 10-10 concerning dispersive and nondispersive media are supported. In general, it may be stated that a

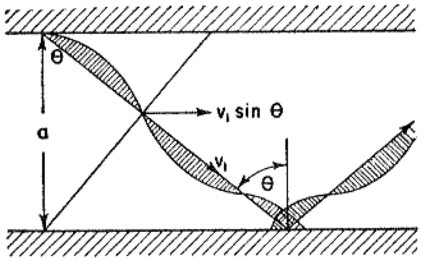

Fig. 11-7. Manner of travel of one wave component between parallel planes.

dissipationless medium will appear dispersive when the phase and group velocities are measured in directions different from the direction of travel of the wave.

From Eqs. 11-59 and 11-64 for v_p and v_g, it is possible to write

$$v_1 = \sqrt{v_p v_g} \tag{11-66}$$

showing that v_1 is always the geometric mean of v_p and v_g.

11-8. Characteristic impedance of the planes

The *characteristic wave impedance* or intrinsic impedance of a traveling electromagnetic field has been defined as a result of the transmission-line analogy. In Eq. 10-18 it was found that this Z_0 or η was related to γ of the medium as

$$Z_0 = \eta = \frac{j\omega\mu_1}{\gamma} = \frac{\gamma}{\sigma + j\omega\epsilon_1} \tag{11-67}$$

This result was found to be the ratio between $\hat{\mathcal{E}}$ and $\hat{H}$ in the medium. Reference to Eqs. 11-15 and 11-16 for $\mathcal{E}_x$ and H_y, the transverse field components of the TM wave, gives for the ratio of their maximum values

$$\frac{\hat{\mathcal{E}}_x}{\hat{H}_y} = \frac{\gamma}{j\omega\epsilon_1}$$

which is equivalent to the result of Eq. 11-67, since $\sigma = 0$. It is thus reasonable to define the characteristic impedance of the guiding planes as the ratio of the transverse electric to the transverse magnetic field. Use of the value of $\gamma_m = j\beta_m$ from Eq. 11-25 allows Z_0 for TM waves to be modified to

$$Z_0 = \eta_1 \sqrt{1 - \frac{f_c^2}{f^2}} \qquad \text{TM waves} \qquad (11\text{-}68)$$

where $\eta_1 = \sqrt{\mu_1/\epsilon_1}$, the intrinsic impedance of the dielectric.

For TE waves the transverse fields may be obtained from Eqs. 11-31 and 11-32, giving Z_0 for TE waves between the planes

$$Z_0 = \frac{\hat{\mathcal{E}}_y}{\hat{H}_x} = \frac{j\omega\mu_1}{\gamma}$$

and again using the appropriate value for γ permits Z_0 for TE waves to be stated as

$$Z_0 = \frac{\eta_1}{\sqrt{1 - f_c^2/f^2}} \qquad \text{TE waves} \qquad (11\text{-}69)$$

The ratio of the transverse fields for TEM waves is easily obtained from Eqs. 11-41 and 11-42 as

$$Z_0 = \frac{\hat{\mathcal{E}}_x}{\hat{H}_y} = \frac{\beta}{\omega\epsilon_1}$$

Use of the value $\beta = \omega\sqrt{\mu_1\epsilon_1}$ for TEM waves gives

$$Z_0 = \eta_1 \qquad \text{TEM waves} \qquad (11\text{-}70)$$

which is just that of the same waves propagating in an unbounded medium. This result further confirms the previous statement that the planes merely serve to confine and limit the area of the fields in transmission of TEM waves.

11-9. Attenuation with planes of finite conductivity—TEM case

To this point the effects of the guiding planes have been studied by considering their conductivity to be infinite and the attenuation zero above cutoff frequency. The conductivities of silver, gold, or copper, which might be the plane materials, are high; and losses of energy that occur in the planes will be small compared with the energy transmitted in the dielectric. Because of the smallness of the losses, it is reasonable to assume that the field intensities near the conducting surfaces will be essentially unchanged because of the small component of energy flow into the planes. That is, the ratio of the power transmitted between the planes to the power transmitted into the planes as losses is very great. Field magnitudes required to transmit the power along the guide are large; those required to transmit the losses into the planes are very small. Therefore the fields due to the transmission of power into the planes will have no appreciable effect on the magnitude of the fields transmitting the power along the planes.

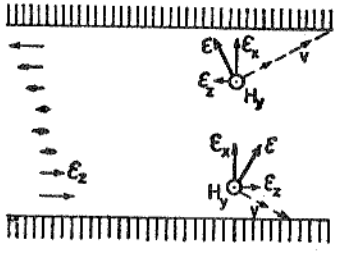

Fig. 11-8. Effect of finite wall conductivity in adding an $\mathcal{E}_z$ component in the TEM wave.

The magnitude field existing at the surface of the planes for the TEM wave has been determined as

$$H_y = B_2\epsilon^{-\gamma z} \sin \omega t$$

and this is assumed unchanged in magnitude by the finite plane conductivity. There will now exist a small $\mathcal{E}_z$ component of field due to the value of J/σ present in the metal. This will cause the wave to tip and no longer lie entirely in the x,y plane. The $\mathcal{E}_z$ component will be directed on $+z$ on one plate and $-z$ on the other plate. The tipping of the wave toward the plane indicates that a component of the field is incident on, and entering, the metal plane. It is this component of field that conveys to the metal the energy required to supply the conduction losses.

Since H_y is wholly tangential, it is continuous across the boundary and represents the magnetic field of the wave entering the metal. The power conveyed by this field into the metal per unit area at some value of z is given by the average power expression for metals:

$$P_M = \frac{1}{2}\hat{H}^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

$$= \frac{1}{2}B_2^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \qquad (11\text{-}71)$$

The loss in both planes will be twice this value.

The power being transmitted in the z direction by the TEM wave past some plane was stated in Eq. 11-46 as

$$P_T = \tfrac{1}{2}\eta_1 \hat{H}_y{}^2 a$$
$$= \tfrac{1}{2}\eta_1 B_2{}^2 a \qquad (11\text{-}72)$$

It is now necessary to restate the definition of the attenuation factor α. This factor was originally defined by

$$\frac{|I_T|}{|I_1|} = \epsilon^{-\alpha z}$$

where α was given as nepers per meter of line with I_T the current at z, and I_1 the current at $z = 0$. For equal impedances a power ratio may be stated as

$$\frac{P_T}{P_1} = \epsilon^{-2\alpha z}$$

or

$$P_T = P_1 \epsilon^{-2\alpha z}$$

The rate of decrease of power with distance is

(11-73)

and this rate represents the power loss per unit length, where P_T is the power being transmitted past any point. In the case of the guided wave, P_M is the power loss per unit length in the z direction per meter of width in the y direction. P_T is the power being transmitted between the planes per meter of width in the y direction. Then

$$P_M = 2\alpha P_T$$
$$\alpha = \frac{1}{2}\frac{P_M}{P_T} \text{ nepers/m} \qquad (11\text{-}74)$$

Thus α is defined in terms of the ratio of the power lost to the power transmitted per unit width in the y direction. With this definition for α and the values of P_M and P_T from Eqs. 11-71 and 11-72, the attenuation for the TEM wave traveling between parallel planes is found to be

$$\alpha_{TEM} = \frac{1}{a\eta_1}\sqrt{\frac{\pi f \mu_m}{\sigma_m}} \qquad (11\text{-}75)$$

in terms of nepers per meter in the z direction, per meter of width in the y direction.

11-10. Attenuation with planes of finite conductivity—TM case

As an example of attenuation with planes of finite conductivity confining a TM wave, consider that m = odd, for which the value of H_y in the TM_m wave can be written

$$\hat{H}_y = B_1 \sin\left(\frac{m\pi x}{a}\right)$$

At the upper plate, for which $x = a/2$, this is

$$\hat{H}_y = B_1 \qquad (11\text{-}76)$$

The power entering the metal plane per square meter at some z value is then obtainable from

$$P = \frac{1}{2}\hat{H}_y{}^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

as

$$P_M = \frac{1}{2} B_1{}^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2 \qquad (11\text{-}77)$$

The power lost in both planes is twice this value per unit area of one plane.

The z-directed power flowing in the wave past a point in the dielectric between the planes is given by

$$P_T = \tfrac{1}{2}\hat{\mathcal{E}}_x \hat{H}_y$$
$$= \frac{1}{2}\frac{\beta_m}{\omega\epsilon_1} B_1{}^2 \sin^2\left(\frac{m\pi x}{a}\right) \qquad (11\text{-}78)$$

using the field values from Eqs. 11-27 and 11-28. The total power flowing in the z direction through a strip extending from top to bottom plane and one meter wide in the y direction is

$$P_T = \frac{1}{2}\frac{\beta_m}{\omega\epsilon_1} B_1{}^2 \int_{-a/2}^{a/2} \sin^2\left(\frac{m\pi x}{a}\right) dx$$
$$= \frac{1}{2}\frac{\beta_m}{\omega\epsilon_1} B_1{}^2 \frac{a}{2} \qquad (11\text{-}79)$$

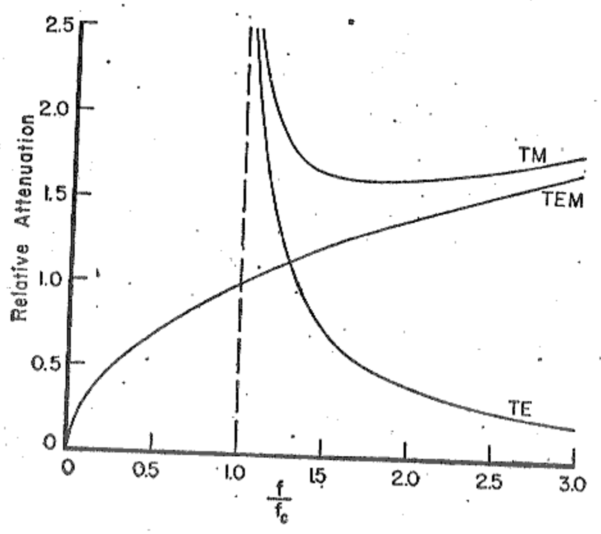

Fig. 11-9. Variation of attenuation of TM, TE, and TEM modes between parallel planes ($m = 1$).

The attenuation factor, expressed in terms of power lost, P_M, and power transmitted, P_T, is

$$\alpha_{TM} = \frac{1}{2}\frac{P_M}{P_T} = \frac{2\omega\epsilon_1}{a\beta_m}\sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

Introduction of Eq. 11-21 for f_c and considerable manipulation give

$$\alpha_{TM} = \frac{\sqrt{m}}{a^{3/2}}\sqrt{\frac{2\pi\mu_m}{\sigma_m\mu_1\eta_1}}\frac{\sqrt{f/f_c}}{\sqrt{1 - f_c^2/f^2}} \qquad (11\text{-}80)$$

for the attenuation in the pass band for the TM_m mode between parallel planes of finite conductivity, m being odd. The result, a rather involved function of the ratio f/f_c, is plotted in Fig. 11-9.

It may be shown that a similar result is obtainable with m even.

11-11. Attenuation with planes of finite conductivity—TE case

Attenuation of the wave for planes of finite conductivity for TE waves may be investigated for m = odd, for example, in a manner similar to that of the preceding section. The tangential component of magnetic field in the TE_m wave is given by Eq. 11-40.

$$\mathcal{H}_z = -\frac{m\pi}{a\omega\mu_1}B_4\sin\left(\frac{m\pi x}{a}\right)$$

Evaluation of the fields at the bounding planes and use of the Poynting radiation vector give, for the power loss per unit area in both planes,

$$P_M = \left(\frac{m\pi}{a\omega\mu_1}\right)^2 B_4^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2$$

The power flowing through the dielectric at any point is given by

$$P_T = \tfrac{1}{2}\hat{\mathcal{E}}_y\mathcal{H}_x$$

and since

$$\hat{\mathcal{E}}_y = B_4\cos\left(\frac{m\pi x}{a}\right)$$

$$\mathcal{H}_x = -\frac{\beta_m}{\omega\mu_1}B_4\cos\left(\frac{m\pi x}{a}\right)$$

the power transmitted through a plane strip extending from plane to plane and one meter wide in the y direction is obtained by integration as

$$P_T = \frac{1}{4}\frac{\beta_m}{\omega\mu_1}B_4^2 a \qquad (11\text{-}81)$$

The attenuation then may be written, after some labor,

$$\alpha_{TE} = \frac{\sqrt{m}}{a^{3/2}}\sqrt{\frac{2\pi\mu_m}{\sigma_m\mu_1\eta_1}}\frac{(f_c/f)^{3/2}}{\sqrt{1 - f_c^2/f^2}} \text{ nepers/m} \qquad (11\text{-}82)$$

This is also an involved function of f/f_c, as well as being dependent on m and on the metal plane parameters. The same expression is obtained for m = even.

The variation in value of attenuation with frequency for TM, TE, and TEM modes is plotted in Fig. 11-9 as a function of f/f_c. The minimum in the attenuation curve for the TM mode may be shown to occur at $f = \sqrt{3}\,f_c$.

It may be noted that

$$\alpha_{TE} = \left(\frac{f_c}{f}\right)^2 \alpha_{TM}$$

and that the attenuation of the TE mode decreases with increasing frequency, becoming zero for infinite frequency.

PROBLEMS

11-1. Sketch the field distributions for $\mathcal{E}$ and H between the parallel planes for the TE_1 and TM_1 modes, for one wavelength in the z direction.

11-2. Sketch the field distribution for $\mathcal{E}$ and H between the parallel planes for the TE_4 and TM_4 modes, for one wavelength in the z direction.

11-3. By analogy between Z_0 and γ for the TM_1 wave and for a high-pass filter, find an analogous electric circuit for the TM_1 wave between parallel planes.

11-4. A pair of perfectly conducting planes are separated 8 cm in air. For a frequency of 5000 megacycles with the TM_1 mode excited, find the following:

(a) Cutoff frequency.
(b) Characteristic impedance.
(c) β.
(d) Attenuation constant for $f = 0.95f_c$.
(e) Phase and group velocity.
(f) Wavelength measured along the guiding walls.

11-5. A parallel plane guide is transmitting an average power of 1000 kw per meter of width. The plane separation is 4 cm and the frequency is 10,000 megacycles, with TE_1 transmission. Compute the maximum values of electric and magnetic intensity in the space between planes and show where these maxima occur.

11-6. (a) If the planes of Prob. 11-5 are made of copper, find the values and direction of the total current flowing in the planes per meter of width for both TM_1 and TE_1 propagation, with power as stated flowing.

(b) Find the power loss per unit area of both planes, for TE_1.
(c) Compute the attenuation.
(d) What percentage of reduction in power loss may be achieved by changing to silver planes for the TE_1 case?

11-7. Prove that Eq. 11-80 is the attenuation for m = even with the TM_m mode.

11-8. Prove that Eq. 11-82 applies equally well for attenuation with m = even, for the TE_m mode.

11-9. Find the frequency of minimum attenuation for the TM mode, in terms of f_c.

11-10. For a frequency of 6000 megacycles and plane separation = 7 cm, find the following for the TE_1 mode:

(a) Cutoff frequency.
(b) Angle of incidence on the planes.
(c) Phase and group velocity.
(d) Is it possible to propagate the TE_3 mode?

11-11. Show that the propagation of the TE_1 wave takes place by reason of two crossed waves and find the value of the angle between the waves.

11-12. Consider the possibilities inherent in the use of an infinite slab of solid good dielectric, of permittivity ϵ_1, oriented parallel to the x,z plane in space and of thickness a, as a wave-guiding medium. Discover the cutoff frequency and expressions for phase and group velocities.

11-13. A field $\mathcal{E}_z = A \sin(\pi y/b) \sin(\omega t - \beta z)$ is the only electric component in a region. Show that div $\mathcal{E} = 0$.

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