while that of a round conductor with alternating current flowing in a skin of thickness δ is

$$R_{ac} = \frac{K}{2\pi a \delta} \tag{7-7}$$

Therefore the ratio of resistance to alternating current to resistance to direct current is

$$\frac{R_{ac}}{R_{dc}} = \frac{a\sqrt{\pi f \mu \sigma}}{2} \tag{7-8}$$

which becomes, for copper,

$$\frac{R_{ac}}{R_{dc}} = 7.53a\sqrt{f} \tag{7-9}$$

where a is in meters and f in cycles per second. This equation shows that the increase in resistance with increasing frequency is greater for large-radius than for small-radius conductors.

The resistance of an open-wire line of copper, with spacing greater than $20a$, can be computed as, for copper,

$$R_{ac} = \frac{8.33 \times 10^{-8}\sqrt{f}}{a} \text{ ohms/meter of line} \tag{7-10}$$

For spacings closer than $20a$, the effect of the proximity of the two currents in crowding to the sides of the conductors further increases the resistance.

7-2. Parameters of the coaxial line at high frequencies

Just as for the parameters of the open-wire line in the previous section, the parameters of the coaxial line are also modified by the presence of high-frequency currents on the line. Because of skin effect, the current flows on the outer surface of the inner conductor and the inner surface of the outer conductor. This phenomenon eliminates flux linkages due to internal conductor flux, and the inductance of the coaxial line is given by Eqs. 5-23 and 5-24 as

$$L = \frac{\mu_v}{2\pi}\ln\frac{b}{a} = 2 \times 10^{-7}\ln\frac{b}{a} \text{ henrys/m} \tag{7-11}$$

$$= 4.60 \times 10^{-7}\log\frac{b}{a} \text{ henrys/m} \tag{7-12}$$

The capacitance of the coaxial line is not affected by frequency

(except as frequency may alter the relative permittivity of the dielectric), so that

$$C = \frac{2\pi\epsilon}{\ln\frac{b}{a}} \quad \text{farads/m}$$

$$= \frac{55.5\epsilon_r}{\ln\frac{b}{a}} \quad \mu\mu\text{f/m} \tag{7-13}$$

$$= \frac{24.14\epsilon_r}{\log\frac{b}{a}} \quad \mu\mu\text{f/m} \tag{7-14}$$

By use of the reasoning developed in the preceding section, the resistance of a coaxial line with appreciable skin effect may be considered as due to current in two thin-walled tubes and an expression for the resistance of a copper coaxial line obtained,

$$R_{ac} = 4.16 \times 10^{-8}\sqrt{f}\left(\frac{1}{b} + \frac{1}{a}\right) \text{ ohms/m of line} \tag{7-15}$$

where a and b are outer radius of the inner conductor and inner radius of the outer conductor in meters, respectively.

The shunt losses of air dielectric lines are zero, but many coaxial lines employ solid dielectric materials, and the conductance losses may have to be considered in some applications, especially at very high frequencies. The quality of the insulating material may be measured in terms of the *power factor* of the material, when employed as the dielectric in a capacitor C. The shunt susceptance is

$$y = g + j\omega C$$

and the power factor is then expressable from the susceptance triangle of Fig. 7-1 as the cosine of θ:

$$\text{pf} = \frac{g}{\sqrt{g^2 + \omega^2 C^2}}$$

The conductance of usual good insulating materials is very small, so that $g \ll \omega C$, and

$$\text{pf} = \frac{g}{\omega C}$$

$$g = \omega C \times \text{pf} \tag{7-16}$$

The quality of the dielectric may also be expressed in terms of the *dissipation factor*, which is the ratio of energy dissipated to energy stored in the dielectric per cycle, and is proportional to the tangent of angle φ of Fig. 7-1. For good dielectrics with small power-factor angles, where the approximation $g \ll \omega C$ holds, the dissipation factor and power factor are equal in magnitude.

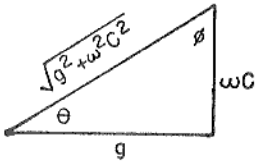

Fig. 7-1. The loss triangle for dielectrics.

Power factor or dissipation factor values for a few insulating materials are given in Table 5.

TABLE 5

DIELECTRIC LOSSES

	ϵ_r	Power factor at frequency of (cycles)			
		60	10^3	10^6	3×10^9
Quartz	3.78	0.0009	0.00075	0.0002	0.00006
Steatite	5.77	...	0.003	0.0007	0.00089
Polyethylene	2.26	<0.0002	<0.0002	<0.0002	0.00031
Polystyrene	2.56	<0.00005	<0.00005	0.00007	0.00033
Teflon	2.1	<0.0005	<0.0003	<0.0002	0.00016

7-3. Constants for the line of zero dissipation

For transmission of energy at high frequencies, where the power efficiency is high, it is possible to assume negligible losses or zero dissipation in the analysis of performance of transmission lines. This assumption of a perfect line is justified by the fact that ω is large, making ωL large. A line of a few wavelengths may be physically short, possibly only a few centimeters, but it is electrically long. The resistance of such a short line is very small compared with the reactance; and with G assumed zero because of the small number of insulators, the assumption of completely negligible losses may be made. The chief advantage of the assumption of zero dissipation is in the easy analysis and physical interpretation of line performance made possible by the method. Actually, such an assumption of a perfect line, though close to fact, may at times lead to absurd or impossible results, in which case an analysis must be made according

to the methods at the end of this chapter, using the proper small value of R.

The line parameters for the line of zero dissipation are

$$Z = j\omega L$$
$$Y = j\omega C$$

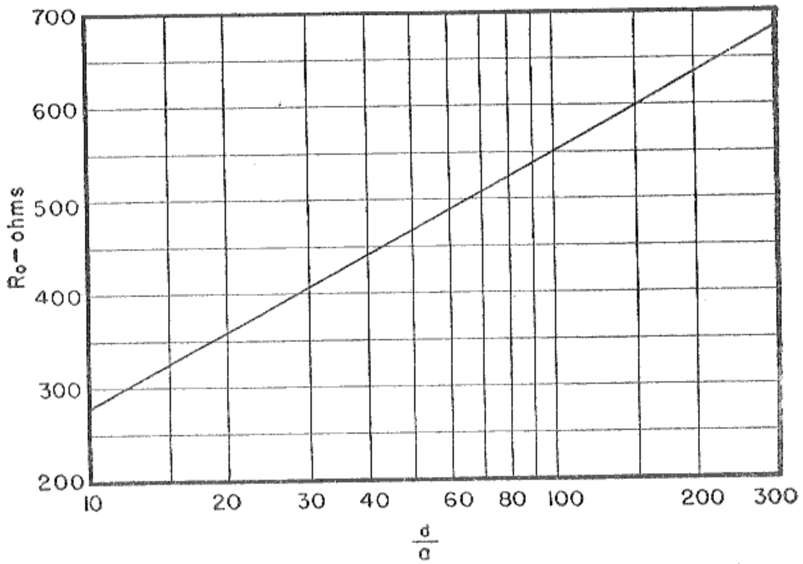

Fig. 7-2. Variation of R_0 with d/a ratio for an open-wire line.

so that the characteristic impedance, Z_0, may be written

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega L}{j\omega C}}$$
$$= \sqrt{\frac{L}{C}} \text{ ohms} \qquad (7\text{-}17)$$

This value is wholly resistive and may be given the symbol R_0 where

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} \qquad (7\text{-}18)$$

Using Eq. 7-1 for the inductance and Eq. 7-3 for the capacitance of the open-wire line at high frequency, the value of the characteristic impedance of the *open-wire line* can be found directly from line dimensions as

$$R_0 = 120 \ln \frac{d}{a} \quad \text{(ohms)} \tag{7-19}$$

$$R_0 = 276 \log \frac{d}{a} \quad \text{(ohms)} \tag{7-20}$$

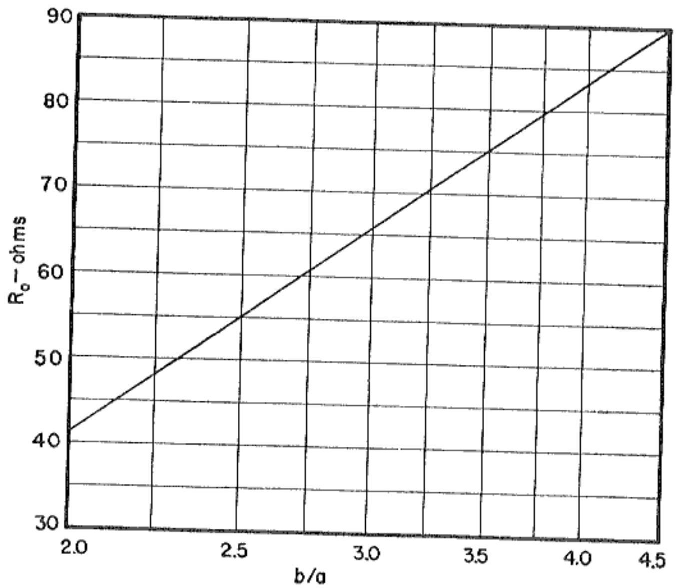

Fig. 7-3. Variation of R_0 with b/a ratio for a coaxial line.

Proximity effect has been neglected here, so that these expressions become less exact for d/a less than 10.

The characteristic impedance of the *coaxial line* can likewise be computed through use of the line dimensions as

$$R_0 = \frac{60}{\sqrt{\epsilon_r}} \ln \frac{b}{a} \quad \text{ohms} \tag{7-21}$$

$$R_0 = \frac{138}{\sqrt{\epsilon_r}} \log \frac{b}{a} \quad \text{ohms} \tag{7-22}$$

the value of ϵ_r being 1 for air-spaced lines.

The values of characteristic impedance are plotted against line dimensions in Figs. 7-2 and 7-3.

The propagation constant γ is

$$\gamma = \sqrt{ZY} = \sqrt{-\omega^2 LC}$$
$$= \alpha + j\beta = j\omega\sqrt{LC}$$

from which

$$\alpha = 0, \qquad \beta = \omega\sqrt{LC} \text{ radians/m} \tag{7-23}$$

The velocity of propagation can then be calculated as

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \text{ m/sec} \tag{7-24}$$

With the values of L and C for an open-wire line from Section 7-1, the velocity becomes

$$v = 3 \times 10^8 \text{ m/sec}$$

Thus the velocity of propagation for the air-spaced open-wire dissipationless line is the same as the velocity of light in space. For the coaxial line

$$v = \frac{3 \times 10^8}{\sqrt{\epsilon_r}} \text{ m/sec} \tag{7-25}$$

in which case the velocity may be reduced due to the presence of a dielectric other than air between the conductors.

7-4. Voltages and currents on the dissipationless line

The voltage at any point distant s units from the receiving end of a transmission line is

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}(\epsilon^{\gamma s} + K\epsilon^{-\gamma s})$$

For the line of zero dissipation, the attenuation constant α is zero and $Z_0 = R_0$, so that

$$E = \frac{E_R(Z_R + R_0)}{2Z_R}(\epsilon^{j\beta s} + K\epsilon^{-j\beta s}) \tag{7-26}$$

The term varying with $\epsilon^{j\beta s}$ has previously been identified as a wave progressing from the source toward the load, and the term involving $\epsilon^{-j\beta s}$ as the reflected wave moving from load back toward the source.

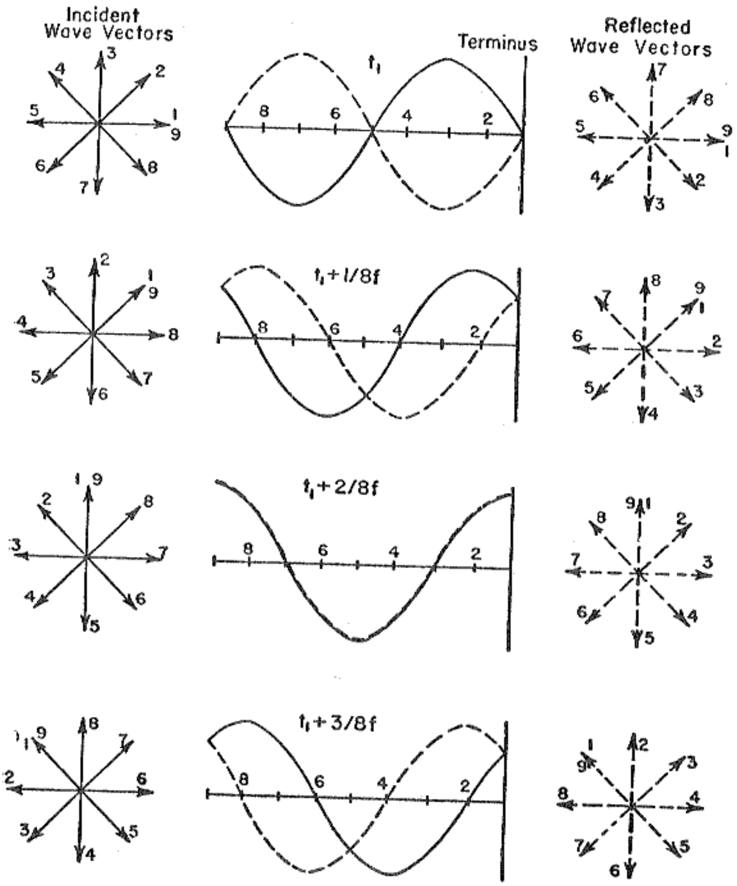

Fig. 7-4. Incident and reflected voltage-wave phasors and values along the dissipationless line for successive instants of time, for an open-circuited line.

The magnitude of the reflected wave is dependent on the value of K, the reflection coefficient. The significant difference between this analysis and that of Chapter 6 is in the absence of attentuation, the rotating vectors that may be considered as generating either the incident or reflected waves remaining constant in magnitude at all points on the line.

The successive positions of the incident and reflected waves on an

open-circuited line are shown in Fig. 7-4 for values of time differing by one-eighth of a cycle. The progression toward or away from the load is readily seen. The actual voltage at any point on the transmission line is the sum of the incident and reflected wave voltages at that point. This sum voltage is plotted in Fig. 7-5(a) for the open-circuited line conditions of Fig. 7-4. It can be seen that the resultant total voltage wave appears to stand still on the line, oscillating in magnitude with time but having fixed positions of maxima and minima. Such a wave is known as a *standing wave*. If the line voltages are measured with an effective-reading voltmeter, the magnitudes will appear as in Fig. 7-5(b), since the voltmeter does not distinguish between positive and negative values.

The standing-wave condition may be better understood if Eq. 7-26 is reduced to

$$E = \frac{E_R}{Z_R}\left[Z_R\frac{(\epsilon^{j\beta s}+\epsilon^{-j\beta s})}{2} + jR_0\frac{(\epsilon^{j\beta s}-\epsilon^{-j\beta s})}{2j}\right]$$

$$= E_R\cos\beta s + jI_RR_0\sin\beta s \qquad (7\text{-}27)$$

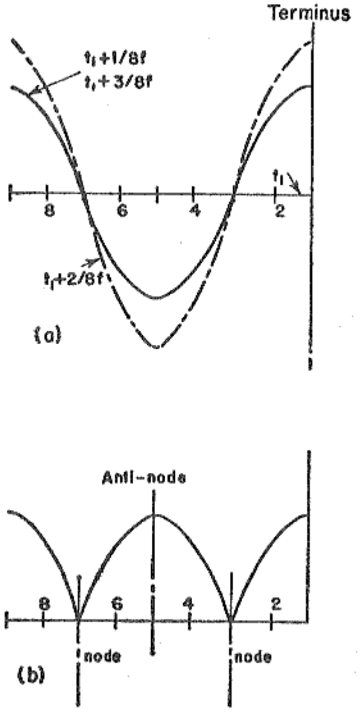

Fig. 7-5. (a) Waves of Fig. 7-4 superposed; (b) voltage values as read on line of (a) by an effective-reading voltmeter.

A similar derivation may be used for the current on the line, starting with

$$I = \frac{I_R(Z_R+Z_0)}{2Z_0}(\epsilon^{j\beta s} - K\epsilon^{-j\beta s}) \qquad (7\text{-}28)$$

and resulting in the current at any point on a dissipationless line as

$$I = I_R\cos\beta s + j\frac{E_R}{R_0}\sin\beta s \qquad (7\text{-}29)$$

From the definition for velocity of propagation,

$$\beta = \frac{2\pi f}{v}$$

it is seen that

$$\beta = \frac{2\pi}{\lambda}$$

after which the current and voltage expressions may be written

$$E = E_R \cos\frac{2\pi s}{\lambda} + jI_R R_0 \sin\frac{2\pi s}{\lambda} \tag{7-30}$$

$$I = I_R \cos\frac{2\pi s}{\lambda} + j\frac{E_R}{R_0}\sin\frac{2\pi s}{\lambda} \tag{7-31}$$

The voltage or current distribution is then seen as the sum of cosine and quadrature sine distributions. If the line is open-circuited, I_R equals zero, and

$$E_{oc} = E_R \cos\frac{2\pi s}{\lambda} \tag{7-32}$$

$$I_{oc} = \frac{jE_R}{R_0}\sin\frac{2\pi s}{\lambda} \tag{7-33}$$

The voltage and current *magnitude* distributions for an open-circuited line 3/2 wavelengths long are plotted in Fig. 7-5(a). The current and voltage are in quadrature everywhere; thus no power is transmitted along the line.

If the line is short-circuited, E_R equals zero, and Eqs. 7-29 and 7-30 become

$$E_{sc} = jI_R R_0 \sin\frac{2\pi s}{\lambda} \tag{7-34}$$

$$I_{sc} = I_R \cos\frac{2\pi s}{\lambda} \tag{7-35}$$

and these *magnitude* distributions are plotted in Fig. 7-6(e) for a short-circuited line 3/2 wavelengths long. Again the current and voltage are in quadrature, but the current and voltage waves have shifted $\lambda/4$ from the positions for the open-circuit case. It should be noted that although the curves show points of zero voltage or current along the line, because of the losses present on even the best

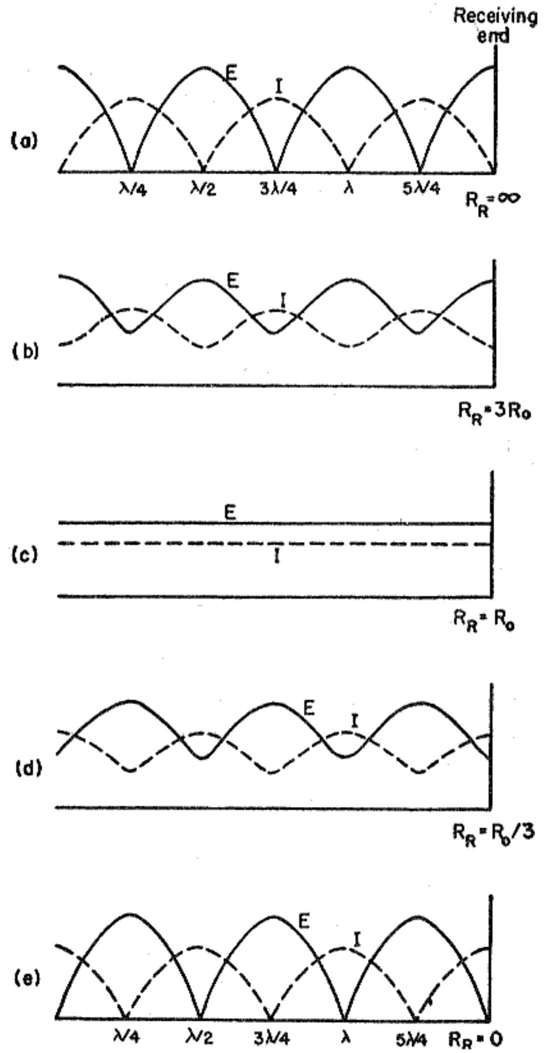

Fig. 7-6. (a) Voltages and currents on an open-circuited dissipationless line; (b) same if load $R_R = 3\,R_0$; (c) same if $R_R = R_0$; (d) same if $R_R = R_0/3$; (e) same if $R_R = 0$, or a short circuit.

lines, the voltages and currents do not reach zero but have small minimum values at the usual zero points.

Returning to Eq. 7-26 for the incident and reflected voltage waves, if the line is terminated in $Z_R = R_0$, the reflection coefficient and reflected wave become zero, and the voltage on the line is expressed by

$$E = E_R \epsilon^{j\beta s} \tag{7-36}$$

which represents a constant voltage magnitude (no attenuation) with continuously varying phase angle along the line. A similar expression may be obtained for the current as

$$I = I_R \epsilon^{j\beta s} \tag{7-37}$$

Such current or voltage distributions are represented in magnitude by the straight horizontal lines in Fig. 7-6(c).

If the line is terminated in a resistance R_R greater than R_0, the reflection coefficient K will be positive and the voltage and current conditions on the line will be intermediate to the open-circuit and R_0-terminated conditions. If, for example, $R_R = 3R_0$, the value of K is $\frac{1}{2}$ and the incident wave has an amplitude twice that of the reflected wave. The resultant voltage and current *magnitudes* are plotted in (b), indicating that there is a finite value of voltage or current at all points on the line, the zeros of (a) being replaced with minima in (b). Since both voltage and current have values other than zero at the load, some power is being transmitted.

If the load condition is made such that $R_R = R_0/3$, the value of K is $-\frac{1}{2}$ and the phase of the reflected wave is reversed. The resultant voltage and current magnitudes are plotted in (d) and appear as in (b) except for the reversed maxima and minima points. Power is again being transmitted.

In general, for resistive loads greater than R_0, the current and voltage distributions somewhat resemble those of an open-circuited line. For resistive loads less than R_0, the distributions take on some of the properties of the short-circuited line.

For loads other than resistance, the reflection coefficient K has an angle, and the reflected wave is shifted in phase with respect to the incident wave. Points of maximum or minimum voltage will not then fall at the end of the line but will be moved back up the line by an amount equal to half the angle of K.

7-5. Standing waves; nodes; standing-wave ratio

If voltage magnitudes are measured along the length of a line terminated in a load other than R_0, the plotted values will appear as in Fig. 7-7. Figure 7-7(a) is drawn for a resistive load of value not equal to R_0, and (b) is the case for either open or short circuit.

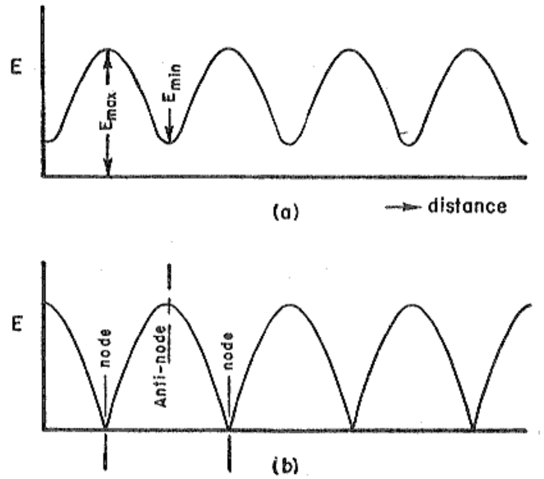

Fig. 7-7. (a) Standing waves on a dissipationless line terminated in a load not equal to R_0; (b) standing waves on a line having open- or short-circuit termination.

Current magnitudes might be plotted and would be similar except for a $\lambda/4$ shift in position of maxima and minima. Maximum and minimum values on a line are labeled as in (a), whereas the nodes and antinodes are indicated in (b). *Nodes* are points of zero voltage or current in the standing wave systems, *antinodes* or loops are points of maximum voltage or current. A line terminated in R_0 has no standing waves, and thus no nodes or loops, and is called a *smooth* line.

For *open circuit*, Fig. 7-6(a) shows that voltage nodes occur at distances $\lambda/4$, $3\lambda/4$, $5\lambda/4$, and so on, from the open end of the line. Under the same conditions, current nodes occur at distances 0, $\lambda/2$, λ, $3\lambda/2$, and so on, from the open termination. For *short circuit*, these nodal points shift by a distance of $\lambda/4$, and voltage nodes occur at 0, $\lambda/2$, λ, and so on, with current nodes at $\lambda/4$, $3\lambda/4$, $5\lambda/4$, and so on. For resistive loads greater than R_0, the voltage and current minima occur at the voltage and current nodal points for an open-circuited line. For resistive loads less than R_0, the voltage and current minima occur at the voltage and current nodal points for a short-circuited line. For pure reactive terminations the standing-wave patterns are similar to those discussed above but are shifted along the line by an angle determined by the angle of the reflection coefficient divided by 2.

The ratio of the maximum to minimum magnitudes of current or voltage on a line having standing waves is called the *standing-wave ratio*, S. That is,

$$S = \left|\frac{E_{max}}{E_{min}}\right| = \left|\frac{I_{max}}{I_{min}}\right| \qquad (7\text{-}38)$$

The maxima of voltage along the line occur at points at which the incident and reflected waves are in phase and add directly. From the voltage equation,

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}(\epsilon^{j\beta s} + K\epsilon^{-j\beta s})$$

at the points where the incident and reflected waves are in phase,

$$E_{max} = \frac{E_R(Z_R + Z_0)}{2Z_R}(1 + |K|) \qquad (7\text{-}39)$$

Likewise, the voltage minima occur at points at which the incident and reflected waves are out of phase; thus

$$E_{min} = \frac{E_R(Z_R + Z_0)}{2Z_R}(1 - |K|) \qquad (7\text{-}40)$$

The standing-wave ratio then may be defined in terms of the reflection coefficient as

$$S = \frac{1 + |K|}{1 - |K|} \qquad (7\text{-}41)$$

This relation may be rearranged as

$$|K| = \frac{S - 1}{S + 1} \qquad (7\text{-}42)$$

$$= \frac{|E_{max}| - |E_{min}|}{|E_{max}| + |E_{min}|} \qquad (7\text{-}43)$$

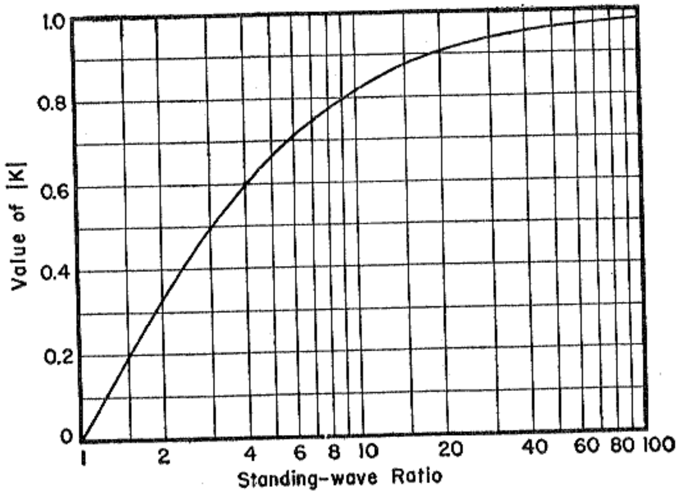

Fig. 7-8. Relation between the standing-wave ratio S and the magnitude of the reflection coefficient.

From Eqs. 7-41 and 7-43 it is possible to calculate values of $|K|$ and S from measurements of maximum and minimum voltages on the line. Exactly similar relations could be derived in terms of the maximum and minimum current values. Figure 7-8 is a plot of Eq. 7-42 that permits obtaining the magnitude of K from a knowledge of the standing-wave ratio.

Standing-wave-ratio measurements are readily made on open-wire lines. For coaxial lines it is necessary to use a length of line in which a longitudinal slot, one-half wavelength or more long, has been cut. A wire probe is inserted into the air dielectric of the line as a pickup device, a vacuum-tube voltmeter or other detector being connected between probe and sheath as an indicator. If the meter provides linear indications, S is readily determined. If the indicator is nonlinear, corrections must be applied to the readings obtained. A probe voltmeter is indicated diagrammatically in Fig. 7-9.

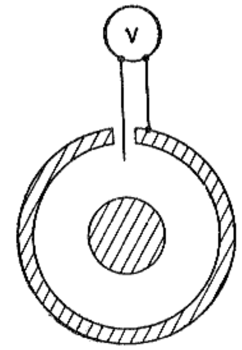

Fig. 7-9. Diagrammatic of a slotted-line section and a probe voltmeter for coaxial line measurements.

The same equipment and techniques may also be used to measure the wavelength on the line, the distance between successive voltage or current maxima or minima being equal to a half wavelength. Such measurements when made on an open line are called *Lecher measurements*, after the man who performed many early high-frequency-line experiments.

For the special case of a resistive load, Eq. 7-41 becomes

$$S = \frac{1 + |K|}{1 - |K|} = \frac{1 + \left(\dfrac{R_R - R_0}{R_R + R_0}\right)}{1 - \left(\dfrac{R_R - R_0}{R_R + R_0}\right)}$$

$$S = \frac{R_R}{R_0} \quad (\text{for } R_R > R_0) \tag{7-44}$$

$$S = \frac{R_0}{R_R} \quad (\text{for } R_R < R_0) \tag{7-45}$$

7-6. Directional coupler

For direct indication and measurement of standing waves a device known as a *directional coupler* is available. Shown diagrammatically in Fig. 7-10, it consists of a section of coaxial transmission line,

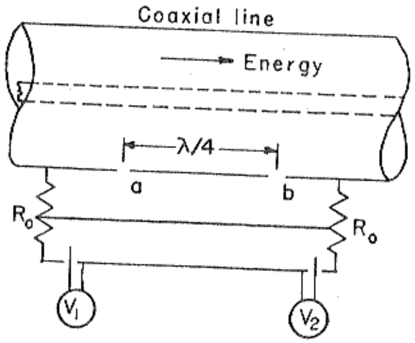

Fig. 7-10. A directional coupler on a coaxial line.

having two small holes in the outer sheath spaced by $\frac{1}{4}$ wavelength. Clamped over these holes is a small section of line, terminated in its R_0 value at both ends to prevent reflections.

Some energy will leak through the holes, and will set up a wave traveling to both left and right in the second line. If the main line is transmitting energy to the right, then a wave entering the secondary line through hole a and traveling to the right will be in phase with

and reinforce a wave entering at hole b, setting up a wave traveling to the right in the secondary line.

Energy entering hole a and traveling to the left will be out of phase and will cancel the wave which entered at b and traveled to the left, since the wave reaching a from hole b will have traveled an additional distance of $\lambda/2$.

Thus a wave in the main line traveling to the right will produce a wave traveling to the right in the secondary line, and give an indication on V_2, but will not produce a wave to the left, nor will it give an indication on V_1. In a similar manner, a wave in the main line traveling to the left will give an indication on V_1, but not on V_2.

In the main line, the wave to the right might be an incident wave, and that to the left a reflected wave. The ratio of the indications of V_1 and V_2 will therefore be the ratio of incident to reflected wave in the main line. For a flat main line, with energy travel to the right, V_1 would read zero. The device then gives a continuous indication of the flatness or degree of reflection present in a line and due to any connected load.

7-7. Input impedance of the dissipationless line

The input impedance of a dissipationless line, a useful quantity, may be written from Eqs. 7-27 and 7-29 as

$$Z_s = \frac{E_s}{I_s} = \frac{E_R \cos \beta s + jI_R R_0 \sin \beta s}{I_R \cos \beta s + j \dfrac{E_R}{R_0} \sin \beta s}$$

and since $I_R Z_R = E_R$,

$$Z_s = R_0 \left(\frac{Z_R + jR_0 \tan \beta s}{R_0 + jZ_R \tan \beta s}\right) \tag{7-46}$$

It can be seen that the impedance is complex in general and is periodic with variation of βs, the period being π or $s = \lambda/2$.

Another convenient form for the input impedance may be obtained by use of Eqs. 7-26 and 7-28,

$$Z_s = \frac{E_s}{I_s} = \frac{\dfrac{I_R(Z_R + R_0)}{2}(\epsilon^{j\beta s} + K\epsilon^{-j\beta s})}{\dfrac{I_R(Z_R + R_0)}{2R_0}(\epsilon^{j\beta s} - K\epsilon^{-j\beta s})}$$

$$= R_0 \left(\frac{1/\beta s + |K|/\phi - \beta s}{1/\beta s - |K|/\phi - \beta s}\right) \tag{7-47}$$

where ϕ is the angle of the reflection coefficient K. This may be further simplified by dividing both numerator and denominator by $1/\underline{\beta s}$, giving

$$Z_s = R_0\left(\frac{1 + |K|/\underline{\phi - 2\beta s}}{1 - |K|/\underline{\phi - 2\beta s}}\right) \tag{7-48}$$

In effect, the above operation has merely shifted the reference by an angle $-\beta s$.

The expression above in parentheses is shown as a phasor diagram in Fig. 7-11, the numerator and denominator being shown as two

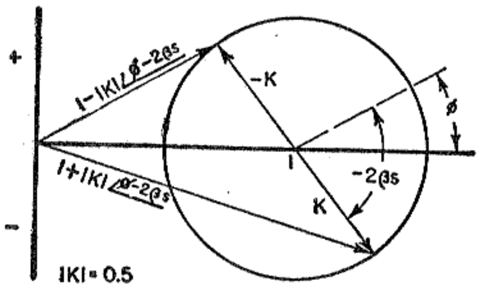

Fig. 7-11. Phasor diagram illustrating Eq. 7-47 with $|K| = 0.5$.

separate phasors, the results of adding unity to plus or minus K. The diagram is arbitrarily drawn for $|K| = 0.5$ or $S = 3$. Several interesting conclusions may be drawn. At values of $s = \phi/2\beta + n\lambda/4$, the numerator and denominator terms of Eq. 7-48 are in phase ($n = 0, 1, 2, 3, \ldots$). At these points the input of the line is purely resistive, with maximum and minimum values occurring every quarter wavelength. In view of the measurement of the angle of the K vector in terms of $2\beta s$, half a revolution represents $\lambda/4$ in terms of wavelength.

The value of the maximum input impedance (resistive), occurring at $s = \phi/2\beta + n\lambda/2$, or with phasors coincident, can be seen to be

$$R_{max} = R_0\left(\frac{1 + |K|}{1 - |K|}\right)$$

$$= SR_0 \tag{7-49}$$

The minimum input impedance, also resistive, occurring at $s = \phi/2\beta + (2n - 1)\lambda/4$, with phasors again coincident, is

$$R_{min} = R_0\left(\frac{1 - |K|}{1 + |K|}\right)$$

$$= \frac{R_0}{S} \tag{7-50}$$

7-8. Input impedance of open- and short-circuited lines

The input impedance of a dissipationless line has been obtained in the section above as

$$Z_s = R_0\left(\frac{Z_R + jR_0 \tan \beta s}{R_0 + jZ_R \tan \beta s}\right) \tag{7-51}$$

For a short-circuited line, $Z_R = 0$, so that

$$Z_{sc} = jR_0 \tan \beta s$$

Since $\beta = 2\pi/\lambda$, this equation becomes

$$Z_{sc} = jR_0 \tan \frac{2\pi s}{\lambda} \tag{7-52}$$

The variation of $Z_{sc}/R_0 = X/R_0$ with length of line s may be plotted as in (a), Fig. 7-12.

Before the input impedance of an open-circuited line is determined, Eq. 7-51 should be rearranged as

$$Z_s = R_0\left(\frac{1 + j\dfrac{R_0}{Z_R}\tan \beta s}{\dfrac{R_0}{Z_R} + j \tan \beta s}\right)$$

For an open-circuited line, $Z_R = \infty$, so that

$$Z_{oc} = \frac{-jR_0}{\tan \beta s} = -jR_0 \cot \frac{2\pi s}{\lambda} \tag{7-53}$$

Figure 7-12(b) is a plot of $Z_{oc}/R_0 = X/R_0$ as a function of the length of line s.

It can be seen that the input impedance of either an open- or short-circuited line is a pure reactance. The value of reactance is a repetitive function of length with a period of $s = \lambda/2$ as a result of

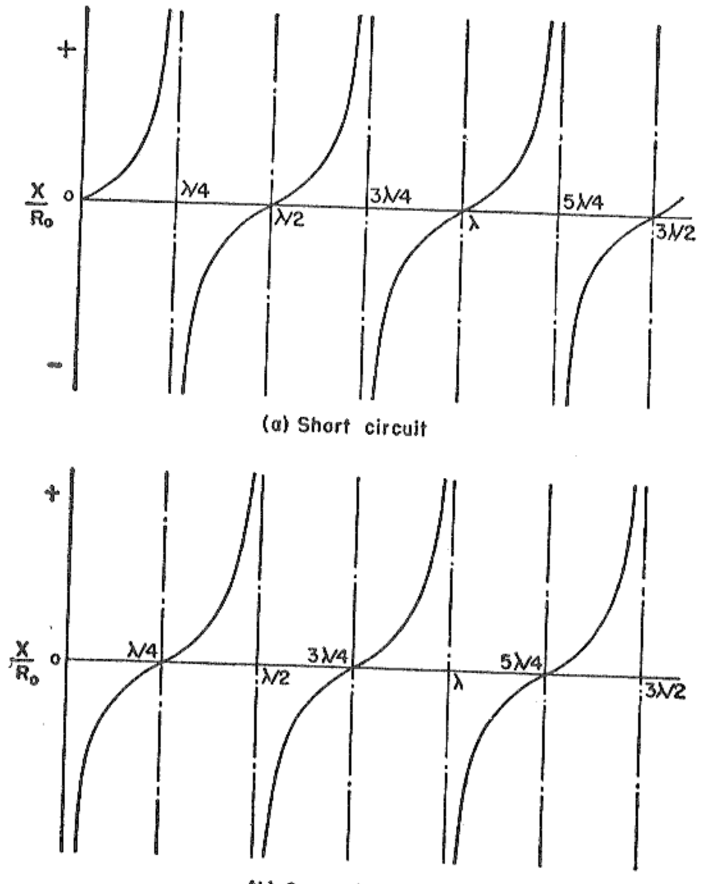

Fig. 7-12. Variation of input impedance of dissipationless line as a function of length: (a) short-circuited line; (b) open-circuited line.

the presence of the tangent function. For the first quarter wavelength, a short-circuited line acts as an inductance, whereas an open-circuited line appears as a capacitance. These reactances reverse each quarter wavelength.

The similarity of performance of open- or short-circuited lines to that of series-resonant or antiresonant circuits may be readily noted by comparison of the curves of Fig. 7-12 with the reactance curves of the resonant circuits of Chapter 2. This similarity suggests the

use of lines as reactive circuit elements or as tuned circuits. As can be seen, a line one-quarter wavelength long or less offers all possible values of reactance, either inductive or capacitive. The input of the quarter-wave short-circuited line appears similar to that of a parallel resonant circuit, and the input of the quarter-wave open line as that of a series resonant circuit. It should also be observed that the input impedance of a $\lambda/4$ short-circuited line appears as an infinite reactance, whereas the input impedance of a $\lambda/4$ open-circuited line appears as a zero reactance or a short circuit. However, the curves of Fig. 7-12 are for the ideal dissipationless line. In a practical line there will always be a small resistance component of the input impedance, indicating some power loss; and zero or infinite impedances are never achieved, the actual values tending to minima and maxima. Further quantitative consideration is given to the subject later in this chapter.

7-9. Power and impedance measurement on lines

The methods of Section 7-7 may be used to rewrite the expressions for the voltage and current on the dissipationless line as

$$E = \frac{I_R(Z_R + R_0)}{2}\,(1 + |K|\underline{/\phi - 2\beta s}) \tag{7-54}$$

$$I = \frac{I_R(Z_R + R_0)}{2R_0}\,(1 - |K|\underline{/\phi - 2\beta s}) \tag{7-55}$$

It can then be seen that Fig. 7-11 may be redrawn for the voltage and current phasors as in Fig. 7-13, phasors A and B being proportional to E and I, respectively.

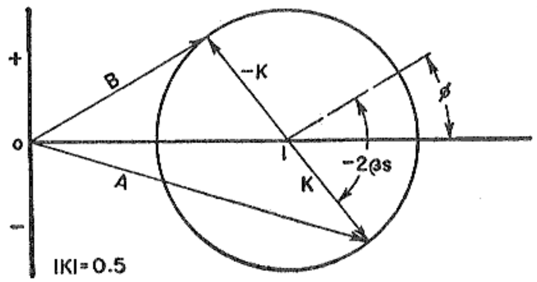

Fig. 7-13. Diagram illustrating Eqs. 7-54 and 7-55.

It has already been reasoned that at a voltage maximum the incident and reflected voltage waves are in phase. This conclusion is confirmed if in Fig. 7-13 it is noted that the line $1\underline{/0°}$ is proportional to the incident voltage and $|K|\underline{/\phi - 2\beta s}$ is proportional to the reflected voltage. Obviously, the in-phase condition is required for E_{max} in Eq. 7-54, so that

$$E_{max} = \frac{I_R|Z_R + R_0|}{2}(1 + |K|) \qquad (7\text{-}56)$$

Similar reasoning shows that at a current maximum the incident and reflected current waves must also be in phase, so that

$$I_{max} = \frac{I_R|Z_R + R_0|}{2R_0}(1 + |K|) \qquad (7\text{-}57)$$

Hence it can be seen that

$$\frac{E_{max}}{I_{max}} = R_0 \qquad (7\text{-}58)$$

Since a change to the values at voltage and current minima requires only the reversal of phase of the reflected waves or a minus sign in front of $|K|$, it can be seen that a similar value for R_0 can be derived as

$$\frac{E_{min}}{I_{min}} = R_0 \qquad (7\text{-}59)$$

These relations furnish an easy method of measuring R_0 if it cannot be readily computed.

From Eqs. 7-54 and 7-55, as from Fig. 7-13, it may be seen that a voltage maximum and a current minimum occur at the same point on the line. This phenomenon has also been noted in the discussion of standing-wave phenomena. Figure 7-13 also shows that at such a point the current and voltage are in phase; or if the impedance of the line is measured looking toward the load, the impedance seen will be resistive. By writing I_{min} as

$$I_{min} = \frac{I_R|Z_R + R_0|}{2R_0}(1 - |K|)$$

the resistive impedance seen at a voltage loop is

$$\frac{E_{max}}{I_{min}} = R_0\left(\frac{1 + |K|}{1 - |K|}\right) \qquad (7\text{-}60)$$

$$= SR_0$$

But this is identifiable as R_{max}, by Eq. 7-49. Then

$$\frac{E_{max}}{I_{min}} = R_{max} \qquad (7\text{-}61)$$

Since the voltage and current are again in phase at a current loop, the resistive impedance seen there may be identified as R_{min}, or

$$\frac{E_{min}}{I_{max}} = \frac{R_0}{S} = R_{min} \qquad (7\text{-}62)$$

The power passing a voltage loop is the power effectively flowing into a resistance R_{max} at voltage E_{max}, so that

$$P = \frac{E^2_{max}}{R_{max}}$$

The same value of power must also pass the current loop, effectively flowing into a resistance R_{min} at voltage E_{min}, since there is no line dissipation, so that

$$P = \frac{E^2_{min}}{R_{min}}$$

Then

$$P^2 = \frac{E^2_{max}E^2_{min}}{R_{max}R_{min}}$$

and by Eqs. 7-60, 7-61, and 7-62, the expression for power passing along the line becomes

$$P = \frac{|E_{max}| \cdot |E_{min}|}{R_0} \qquad (7\text{-}63)$$

The power may also be expressed as

$$P = (|I_{max}| \cdot |I_{min}|)R_0 \qquad (7\text{-}64)$$

These expressions permit easy measurements of power flow on a line of negligible losses.

In many lines, especially those of coaxial construction, the dielectric strength or the voltage breakdown of the line dielectric limits the voltage on the line and thus fixes the maximum power that may be transmitted. For a given maximum line voltage, Eq. 7-63 shows that the greatest amount of power will be transmitted if $|E_{max}| = |E_{min}|$, or the line is operated as a smooth line without standing waves and with an R_0 termination. As will be shown in the next section, it is also advisable for the greatest transfer of power from line to load to operate the line with an R_0 termination, provided the line is sufficiently long that the output image impedance is substantially R_0.

The unknown value of a load impedance Z_R connected to a transmission line may be determined by standing-wave measurements on the open-wire or slotted line. A line may thus be used in lieu of some form of bridge circuit for measuring an unknown impedance. The characteristic impedance R_0 of the line must be known or calculated, and measurements must be made of the standing-wave ratio S and the distance s' from the load to the nearest point of voltage minimum.

At the point of voltage minimum it has been shown that

$$Z_s = R_{min} = \frac{R_0}{S}$$

At any point on a line,

$$Z_s = R_0\left[\frac{Z_R + jR_0 \tan (2\pi s/\lambda)}{R_0 + jZ_R \tan (2\pi s/\lambda)}\right]$$

so that at the point of voltage minimum, distant s' from the load,

$$\frac{R_0}{S} = R_0\left[\frac{Z_R + jR_0 \tan (2\pi s'/\lambda)}{R_0 + jZ_R \tan (2\pi s'/\lambda)}\right]$$

Solution for Z_R gives

$$Z_R = R_0\left[\frac{1 - jS \tan (2\pi s'/\lambda)}{S - j \tan (2\pi s'/\lambda)}\right] \qquad (7\text{-}65)$$

as the value of the connected load impedance.

The point of a voltage minimum is measured rather than a voltage maximum because it is usually possible to determine the exact point of minimum voltage with greater accuracy.

7-10. Reflection losses on the unmatched line

As has been discussed in Section 6-15, if a line is not matched to its load, the energy delivered by the line to the load is less than if the impedances are properly adjusted. This effect is considered as due to reflection at the junction and makes its presence known by

establishment of a reflected wave and a standing-wave system. The voltage at a maximum voltage point is due to the in-phase sum of the incident and reflected waves, so that from Eq. 7-56:

$$|E_{max}| = |E_i| + |E_r| = \frac{|I_R(Z_R + R_0)|}{2}(1 + |K|)$$

The minimum voltage is due to the difference of the incident and reflected waves and is

$$|E_{min}| = |E_i| - |E_r| = \frac{|I_R(Z_R + R_0)|}{2}(1 - |K|)$$

Hence the standing wave ratio is

$$S = \frac{|E_{max}|}{|E_{min}|} = \frac{|E_i| + |E_r|}{|E_i| - |E_r|} \qquad (7\text{-}66)$$

Use of these results and Eq. 7-63 gives for the total power transmitted along the line and delivered to the load

$$P = \frac{|E_{max}| \cdot |E_{min}|}{R_0}$$
$$= \frac{(|E_i| + |E_r|)(|E_i| - |E_r|)}{R_0} = \frac{|E_i|^2 - |E_r|^2}{R_0} \qquad (7\text{-}67)$$

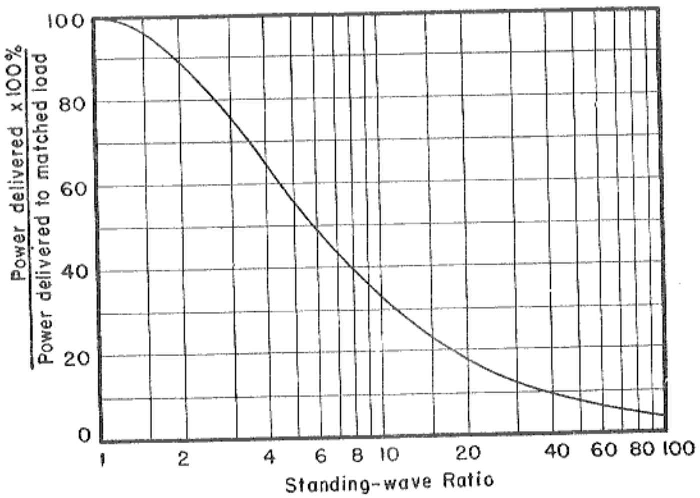

Fig. 7-14. Reflection losses as a function of standing-wave ratio.

But this expression is easily recognized as the difference of two power flows, one being the power P_i transmitted in the incident wave, the other being the power P_r traveling back in the reflected wave.

The ratio of the power P delivered to the load to the power transmitted by the incident wave is

$$\frac{P}{P_i} = \frac{P_i - P_r}{P_i} = \frac{|E_i|^2 - |E_r|^2}{|E_i|^2} = 1 - \frac{|E_r|^2}{|E_i|^2} = 1 - |\boldsymbol{K}|^2$$

$$= 1 - \left(\frac{S-1}{S+1}\right)^2 = \frac{4S}{(S+1)^2} \qquad (7\text{-}68)$$

in view of Eq. 7-42. The ratio of power absorbed by the load to the power transmitted is plotted as a function of S in Fig. 7-14, further illustrating the desirability of operating a line with a Z_0 termination to eliminate reflection, since with proper termination a line has its greatest power capacity.

7-11. The eighth-wave line

The input impedance of a line of length $s = \lambda/8$ is

$$\boldsymbol{Z}_s = R_0 \left[\frac{\boldsymbol{Z}_R + jR_0 \tan(\pi/4)}{R_0 + j\boldsymbol{Z}_R \tan(\pi/4)}\right]$$

$$= R_0 \left(\frac{\boldsymbol{Z}_R + jR_0}{R_0 + j\boldsymbol{Z}_R}\right) \qquad (7\text{-}69)$$

If such a line is terminated in a pure resistance R_R,

$$\boldsymbol{Z}_s = R_0 \left(\frac{R_R + jR_0}{R_0 + jR_R}\right)$$

and the numerator and denominator have identical *magnitudes*, so that

$$|\boldsymbol{Z}_s| = R_0 \qquad (7\text{-}70)$$

Thus an eighth-wave line may be used to transform any resistance to an impedance with a magnitude equal to R_0 of the line, or to obtain a magnitude match between a resistance of any value and a source of R_0 internal resistance.

7-12. The quarter-wave line; impedance matching

The expression for the input impedance of a dissipationless line may be rearranged as

$$\boldsymbol{Z}_s = R_0 \left[\frac{\dfrac{\boldsymbol{Z}_R}{\tan(2\pi s/\lambda)} + jR_0}{\dfrac{R_0}{\tan(2\pi s/\lambda)} + j\boldsymbol{Z}_R}\right]$$

If the line is made a quarter-wave long, or $s = \lambda/4$,

$$\boldsymbol{Z}_s = \frac{R_0^2}{\boldsymbol{Z}_R} \qquad (7\text{-}71)$$

That is, the input impedance of the line is equal to the square of R_0 of the line divided by the load impedance. A quarter-wave section of line may be thought of as a transformer to match a load of $\boldsymbol{Z}_R$ ohms to a source of $\boldsymbol{Z}_s$ ohms. Such a match can be obtained if the characteristic impedance R_0' of the matching quarter-wave section of line is properly chosen according to

$$R_0' = |\sqrt{\boldsymbol{Z}_s \boldsymbol{Z}_R}| \qquad (7\text{-}72)$$

The R_0' of the matching section should thus be equal to the geometric mean of the source and load impedances.

A quarter-wave line may be considered as an impedance inverter in that it can transform a low impedance into a high impedance and vice versa. This effect is illustrated by the action of the $\lambda/4$ short-circuited line in transforming the zero impedance short-circuit termination to an apparent open circuit, and of the open-circuited $\lambda/4$ line in transforming the open circuit termination to a low value or an apparent short circuit.

An important application of the quarter-wave matching section is to couple a transmission line to a resistive load such as an antenna. The quarter-wave matching section then must be designed to have a characteristic impedance R_0' so chosen that the antenna resistance R_A is transformed to a value equal to the characteristic impedance R_0 of the transmission line. The line then is terminated in its R_0 and is operated under conditions of no reflection. The characteristic impedance R_0' of the matching section then should be

$$R_0' = \sqrt{R_A R_0} \qquad (7\text{-}73)$$

It is interesting to note that the value of R_0', the characteristic impedance of the matching section, is just the value required to achieve critical coupling and maximum power transfer from the transmission line to the load, as indicated by the coupled-circuit theory of Chapter 3.

In cases where physical spacing is greater than can be reached with a line a quarter wave in length the same transformation can be produced by a line three quarter waves long or a line any odd number of quarter waves in length. Greater lengths reduce the efficiency slightly due to increased losses in a line of practical construction.

As a practical matter, the range of values of R_A and R_0 that can be matched satisfactorily is limited roughly to about 10 to 1. The transformer is also a single-frequency or narrow-band device. The band width may be increased by using two or more quarter-wave sections in series, each accomplishing part of the total transformation.

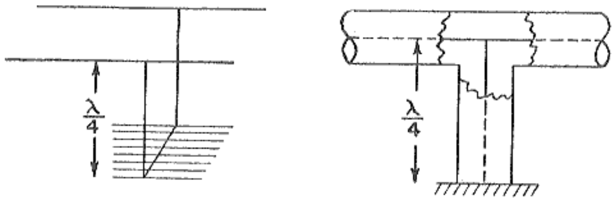

Fig. 7-15. Quarter-wave lines as insulators.

A quarter-wave transformer may also be used if the load is not pure resistance. It should then be connected between points corresponding to I_{max} or E_{min}, at which places the transmission line has resistive impedances given by R_0/S or SR_0. For step down in impedance from the line value of R_0 the matching transformer characteristic impedance should be

$$R_0' = \sqrt{R_0 \frac{R_0}{S}} = R_0 \sqrt{\frac{1}{S}} \qquad (7\text{-}74)$$

Another application of the short-circuited quarter-wave line is as an insulator to support an open-wire line or the center conductor of a coaxial line. This application, illustrated in Fig. 7-15, makes use of the fact that the input impedance of a quarter-wave shorted line is very high. Such lines are sometimes referred to as *copper insulators*. They will be given further study later in this chapter.

7-13. The half-wave line

When a length of line having $s = \lambda/2$ is used, the input impedance is

$$Z_s = R_0 \left(\frac{Z_R + jR_0 \tan \pi}{R_0 + jZ_R \tan \pi} \right)$$

$$Z_s = Z_R \qquad (7\text{-}75)$$

This result is obvious, since the conditions on the line have a period of π, or one-half wavelength.

A half wavelength of line may then be considered as a one-to-one transformer. It has its greatest utility in connecting a load to a source in cases where the load and source cannot be made adjacent. A group of capacitors may be placed in parallel by connecting them with sections of line n half waves in length. As a result, insulators on a high-frequency line should not be spaced at half-wave intervals, since their effect would then be cumulative, lowering the insulation resistance of the line.

7-14. The exponential line for impedance transformation

The characteristic impedance of a line is a function of the spacing and size of the conductors among other factors. If the spacing of a dissipationless line is made to vary in a uniform manner along the length of a line, R_0 will likewise vary along the line. Using the image-impedance concept, such a tapered transmission line would not be symmetrical, the image impedance at the sending end differing from that at the load end, for reasonable electrical line lengths. This variation indicates the possibility of lines of tapered spacing being used as matching sections between a line and a load. If such a dissipationless line were matched on an image basis at both ends, it could serve as a magnitude matching transformer between some impedance Z_1 and some other load impedance Z_R.

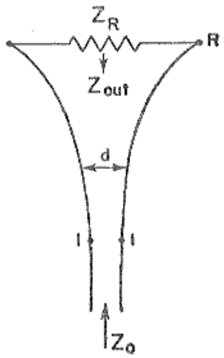

Fig. 7-16. The exponential-line impedance transformer.

While any arbitrary taper may be employed, an exponential variation of parameters may be most readily handled mathematically. Assume that the line parameters are made to vary in such a way that

$$L = L_1\epsilon^{2\theta s} \tag{7-76}$$

where L is the inductance per meter of line at any distance s from the source at point 1, Fig. 7-16, L_1 is the inductance per meter at the sending end, and θ is a *transformation function* to be defined. From the properties of lines it is apparent that

$$C = C_1\epsilon^{-2\theta s} \tag{7-77}$$

where C and C_1 are line capacitances per meter at distance s and the sending end, respectively. The line spacing is small at the sending end and large at the load for positive taper or positive θ.

It is desired that this line operate without internal reflections. It was mentioned in Section 6-10 that, for a dissipationless line operating without reflection, the energy in each of the electric and magnetic fields was maintained constant, but that if a variation of line constants (Z_0) caused an exchange of energy between the fields, a reflection was set up. It is obvious in the present case that a variation of line constants does exist. If, however, the line parameters are varied in such a way that the energy in the fields can be forced to remain constant (dissipationless line) and a wave propagated, no reflection will be produced.

If the energy in the magnetic field at any point is equated to that at the sending end as a requirement of zero reflection,

$$\frac{LI^2}{2} = \frac{L_1I_1^2}{2}$$

Likewise, for electric field energy,

$$\frac{CE^2}{2} = \frac{C_1E_1^2}{2}$$

Use of these statements and Eqs. 7-76 and 7-77 gives the following as the required manner of current and voltage variation on the exponentially tapered line:

$$I = I_1\epsilon^{-\theta s}, \qquad E = E_1\epsilon^{\theta s}$$

Introducing a phase constant β for a moving wave gives

$$E = E_1\epsilon^{\theta s}\epsilon^{j\beta s} = E_1\epsilon^{(\theta+j\beta)s} \tag{7-78}$$

$$I = I_1^{-\theta s}\epsilon^{j\beta s} = I_1\epsilon^{-(\theta-j\beta)s} \tag{7-79}$$

These voltage and current relations indicate inverse variation of current and voltage with length of line s. For positive θ the voltage increases and the current decreases along the line, transformed in an exponential manner from element to element.

The variation of voltage and current along the line with the above distributions is

$$\frac{dE}{ds} = (\theta + j\beta)E_1\epsilon^{(\theta+j\beta)s} \tag{7-80}$$

$$\frac{dI}{ds} = -(\theta - j\beta)I_1\epsilon^{-(\theta-j\beta)s} \tag{7-81}$$

If a propagating wave exists on the lossless tapered line, the line differential equations must be satisfied or

$$\frac{dE}{ds} = -ZI = -j\omega LI$$

$$\frac{dI}{ds} = -YE = -j\omega CE$$

s being measured from the sending end, and causing the negative signs. Forcing the line current and voltage to satisfy these traveling wave relations, thereby ensuring the presence of the desired propagating wave, gives

$$(\theta + j\beta)E_1\epsilon^{(\theta+j\beta)s} = -j\omega LI = -j\omega L_1\epsilon^{2\theta s}I_1\epsilon^{-(\theta-j\beta)s}$$

$$-(\theta - j\beta)I_1\epsilon^{-(\theta-j\beta)s} = -j\omega CE = -j\omega C_1\epsilon^{-2\theta s}E_1\epsilon^{(\theta+j\beta)s}$$

and cancellation of the exponentials leads to

$$(\theta + j\beta)E_1 = -j\omega L_1I_1 \tag{7-82}$$

$$-(\theta - j\beta)I_1 = -j\omega C_1E_1 \tag{7-83}$$

Eliminating E_1 and I_1 from the above gives

$$\theta^2 + \beta^2 = \omega^2L_1C_1$$

from which it is found that the phase constant β for the propagating wave must have the value

$$\beta = \pm\sqrt{\omega^2L_1C_1 - \theta^2} \tag{7-84}$$

The phase constant β may then be either real or imaginary. For large values of ω or high frequencies, β is real, and the voltage and current propagate without attenuation although they may be transformed in magnitude because of the presence of θ, the transformation function. The positive sign for the incident wave will be chosen, the negative sign merely indicating the possibility of a reflected or reverse-propagating wave.

The above expression can be recognized as the usual form for propagation on a lossless line, modified by the function θ^2. The value of β reduces to that of the lossless line if θ be made small or the rate of taper very gradual.

For large rates of taper, where $\theta^2 > \omega^2 L_1 C_1$, the phase constant β becomes imaginary or

$$\beta = j\sqrt{\theta^2 - \omega^2 L_1 C_1}$$

and the voltage and current expressions are

$$E = E_1 \epsilon^{\theta s} \epsilon^{-\beta s}, \qquad I = I_1 \epsilon^{-\theta s} \epsilon^{-\beta s}$$

indicating attenuation of both voltage and current due to the $\epsilon^{-\beta s}$ factors. The exponentially tapered line then has properties similar to those of a high-pass filter, with the cutoff frequency f_c occurring for β equal to zero or

$$f_c = \frac{\theta}{2\pi\sqrt{L_1 C_1}} \tag{7-85}$$

This indicates that there is a limiting rate of taper for a given frequency, beyond which the tapered line will not propagate a wave, or acts as a discontinuity. Physically another limit is imposed by the fact that the greatest rate of taper is reached when the two wires are directed oppositely.

The impedance at any point along the line is the ratio of E to I at that point or

$$Z = \frac{E}{I} = \frac{E_1 \epsilon^{\theta s}}{I_1 \epsilon^{-\theta s}} = Z_1 \epsilon^{2\theta s} \tag{7-86}$$

If Z_R is the load impedance to be matched, then

$$\left|\frac{Z_R}{Z_1}\right| = \epsilon^{2\theta s}$$

from which it is possible to develop a definition for θ as

$$\theta = \frac{1}{2s} \ln \left|\frac{Z_R}{Z_1}\right| \tag{7-87}$$

where Z_R is the load and Z_1 the source or uniform line impedance to which the load is to be matched. Thus the desired rate of taper, for a given length of line, may be determined.

It is also possible to show that

$$\left|\frac{E_R}{E_1}\right| = \sqrt{\left|\frac{Z_R}{Z_1}\right|} \tag{7-88}$$

which indicates that the exponential line is analogous to the ideal transformer.

The impedance at any point on the line may be found from Eq. 7-86 and the fact that

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{L_1}{C_1}}\,\epsilon^{2\theta s}$$

so

$$Z = \sqrt{\frac{L}{C}}\left(\sqrt{1 - \frac{f_c^2}{f^2}} + \frac{jf_c}{f}\right) \tag{7-89}$$

where L and C are per meter values of a small section of the line at the point under consideration. For pass-band frequency values which make $f \gg f_c$, this reduces to $Z = \sqrt{L/C} = Z_0$ of a dissipationless line, so that best operation will occur at frequencies well above cutoff value.

It is preferable that the tapered section be one-half to one wavelength long at the operating frequency and it is found that with this length a linear rate of taper may be employed without appreciable reflection. This simplifies the construction problem. A linear taper implies that Eq. 7-87 has been written by use of the logarithmic series and the first term taken. For small ratios of impedance transformation this is a good approximation.

The exponential line may be designed by use of Eq. 7-76 and the relation for parallel wire lines, as

$$\frac{L}{L_1} = \frac{\ln(d/a)}{\ln(d_1/a_1)} = \epsilon^{2\theta s} = \left|\frac{Z_R}{Z_1}\right|$$

$$\frac{d}{a} = \left(\frac{d_1}{a_1}\right)\left|\frac{Z_R}{Z_1}\right| \tag{7-90}$$

for the d/a ratio at the load end of the tapered line.

In construction of open-wire or coaxial lines it is necessary to avoid discontinuities or sudden changes in mechanical dimensions or construction, otherwise distortions will occur in the electric and magnetic fields present, and these will set up reflections. The tapered section provides a method of making transitions between line sizes and configurations without serious reflections.

7-15. Single-stub impedance matching on a line

For greatest efficiency and delivered power, a high-frequency transmission line should be operated as a smooth line or with an R_0 termination. However, the usual loads, such as antennas, do not in

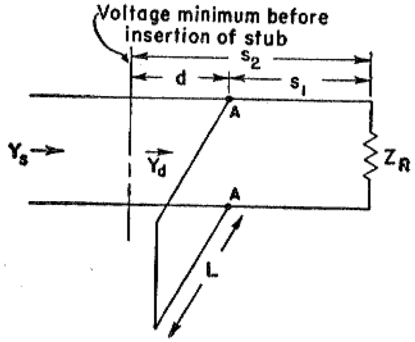

Fig. 7-17. Location of the single stub for impedance matching.

general have resistances of value equal to R_0, so that in many cases it is necessary to introduce some form of an impedance-transforming section between line and load to make the load appear to the line as a resistance of value R_0. The quarter-wave line or transformer and the tapered line are such impedance-matching devices. Another means of accomplishing the desired result is the use of an open or closed stub line of suitable length as a reactance shunted across the transmission line at a designated distance from the load, to tune the length of line and the load to resonance with an antiresonant resistance equal to R_0. The arrangement is as in Fig. 7-17.

The theory of the method may be easily stated in general terms. Since the input conductance of a line is $1/SR_0$ at a voltage maximum and S/R_0 at a voltage minimum, then at some intermediate point A the real part of the input admittance may have an intermediate value of $1/R_0$, or the input admittance at A has a value

$$Y_s = \frac{1}{R_0} \pm jB \tag{7-91}$$

The susceptance B is the shunt value at the point in question. After the point having a conductance equal to $1/R_0$ is located, a short stub line having input susceptance of $\mp B$ may be connected across the transmission line. The input admittance at this point then is

$$Y_s = \frac{1}{R_0} \pm jB \mp jB = \frac{1}{R_0}$$

or the input impedance of the transmission line at point A looking toward the load is

$$Z_s = R_0$$

The line from the source to A is then terminated in R_0 and is a smooth line. From A to the load, reflection and standing waves occur; but since this distance can always be made less than a wavelength, the losses are not severe.

Since both the location and length of the stub must be determined, two independent measurements must be made on the original line and load to secure sufficient data. The most easily obtained measurements are the standing-wave ratio S and the position of a voltage minimum, usually the minimum nearest to the load. A voltage minimum is chosen rather than a maximum, since its position usually can be determined more accurately. If the location of the stub is fixed with respect to an original voltage minimum, no knowledge of the load impedance is needed.

Because of the paralleling of elements, it is most convenient to work with admittances. From Eq. 7-48 the input admittance Y_s, looking toward the load from any point on the line, may be written as

$$Y_s = \frac{1}{R_0}\left(\frac{1 - |K|/\phi - 2\beta s}{1 + |K|/\phi - 2\beta s}\right) \tag{7-92}$$

Writing $G_0 = 1/R_0$ and changing to rectangular coordinates gives

$$Y_s = G_0\left[\frac{1 - |K| \cos(\phi - 2\beta s) - j|K| \sin(\phi - 2\beta s)}{1 + |K| \cos(\phi - 2\beta s) + j|K| \sin(\phi - 2\beta s)}\right]$$

and upon rationalizing,

$$Y_s = G_0\left[\frac{1 - |K|^2 - 2j|K| \sin(\phi - 2\beta s)}{1 + |K|^2 + 2|K| \cos(\phi - 2\beta s)}\right]$$

Expressing the shunt conductance as a dimensionless ratio G_s/G_0, or on a *per unit* basis,

$$\frac{G_s}{G_0} = \left[\frac{1 - |K|^2}{1 + |K|^2 + 2|K| \cos (\phi - 2\beta s)}\right] \tag{7-93}$$

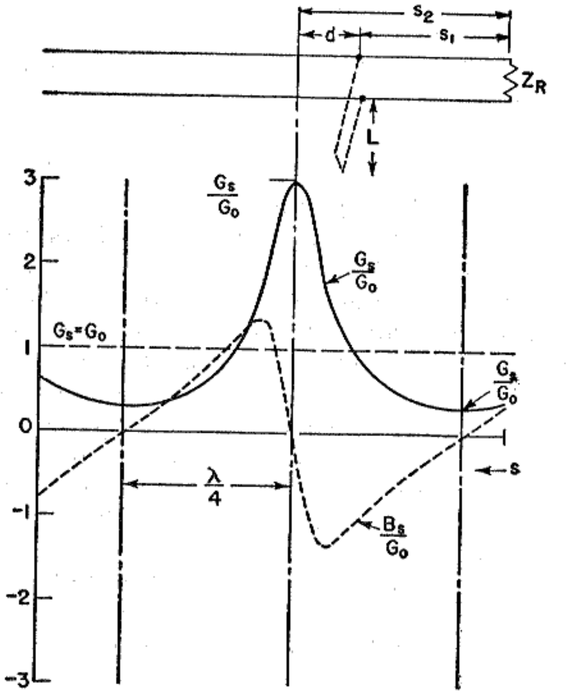

Fig. 7-18. Admittance conditions on a line indicating the proper location of the stub for $|K| = 0.5$.

and the shunt susceptance on a *per unit* basis is

$$\frac{B_s}{G_0} = \left[\frac{-2|K| \sin (\phi - 2\beta s)}{1 + |K|^2 + 2|K| \cos (\phi - 2\beta s)}\right] \tag{7-94}$$

Equations 7-93 and 7-94 are plotted in Fig. 7-18 for a value of $|K|$ arbitrarily chosen as 0.5. The value of G_s/G_0 can be seen as having a maximum. Inspection of Eq. 7-93 shows that this maximum

occurs for the value s_2 at which the cosine term is -1, or

$$\phi - 2\beta s_2 = -\pi$$

$$s_2 = \frac{\phi + \pi}{2\beta} \tag{7-95}$$

The distance s_2 is identified in Fig. 7-18. At this distance from the load,

$$\frac{G_s}{G_0} = \frac{1 - |K|^2}{1 + |K|^2 - 2|K|} = \frac{1 + |K|}{1 - |K|} = S \tag{7-96}$$

Since this equation states that $R_s = R_0/S$, the point of maximum G_s/G_0 is recognized as a point of *minimum voltage*, at a distance s_2 from the load.

At a distance s_1 from the load it can be seen in Fig. 7-18 that $G_s = G_0$. This is the point at which the stub is to be connected. The value of G_s/G_0 is unity there, so that from Eq. 7-93,

$$1 = \frac{1 - |K|^2}{1 + |K|^2 + 2|K| \cos (\phi - 2\beta s_1)}$$

from which $\quad \cos (\phi - 2\beta s_1) = -|K|$

Since $\quad \cos^{-1} (-|K|) = -\pi + \cos^{-1} |K|$

$$s_1 = \frac{\phi + \pi - \cos^{-1} |K|}{2\beta} \tag{7-97}$$

Hence the distance d from the voltage minimum to the point of stub connection is

$$d = s_2 - s_1$$

which, from Eqs. 7-95 and 7-97, is

$$d = \frac{\cos^{-1} |K|}{2\beta} = \frac{\cos^{-1}\left(\frac{S - 1}{S + 1}\right)\frac{\lambda}{4}}{\pi} \tag{7-98}$$

The stub should then be located at this distance d measured in either direction from a voltage minimum. Ordinarily the stub is placed on the load side of that minimum which is nearest to the load.

The input susceptance of the line at the stub location nearest the load can be obtained from Eqs. 7-94 and 7-97 as

$$B_s = G_0\left[\frac{-2|K|\sin(\pi + \cos^{-1}|K|)}{1 + |K|^2 + 2|K|\cos(\pi + \cos^{-1}|K|)}\right]$$

For an angle whose cosine is $|K|$, the sine is $\sqrt{1 - |K|^2}$, so that

$$B_s = G_0\left(\frac{2|K|\sqrt{1 - |K|^2}}{1 - |K|^2}\right) \tag{7-99}$$

The susceptance of the stub required to cancel the line susceptance must be the negative of B_s. The susceptance of a short-circuited stub is

$$B_{sc} = -G_0 \cot \beta L$$

where L is the length of the short-circuited stub. If stub and line have equal G_0, then

$$\frac{G_0}{\tan \beta L} = G_0\left(\frac{2|K|\sqrt{1 - |K|^2}}{1 - |K|^2}\right)$$

Then

$$L = \frac{\lambda}{2\pi}\tan^{-1}\frac{\sqrt{1 - |K|^2}}{2|K|} \tag{7-100}$$

By use of the standing-wave ratio existing *before connection of the stub,* this equation may be conveniently expressed as

$$L = \frac{\lambda}{2\pi}\tan^{-1}\frac{\sqrt{S}}{S - 1} \tag{7-101}$$

This is the length of the short-circuited stub to be placed d meters toward the load from a point at which a voltage minimum existed before attachment of the stub. The susceptance of the line at d is then canceled, and the line appears to be terminated in a resistance of value R_0 at that point; it will be a smooth line between the generator and the point of connection of the stub.

It is also possible to place the stub d meters toward the source from the voltage minimum. The sign of the reactance is reversed on that side with respect to the sign for the location nearer the load, as shown by Fig. 7-18. The stub length L' then should be

$$L' = \frac{\lambda}{2} - L \tag{7-102}$$

for a short-circuited stub.

A short-circuited stub is ordinarily preferred to an open-circuited stub because of greater ease in construction and because of the inability to maintain high enough insulation resistance at the open-circuit point to ensure that the stub is really open-circuited. A shorted stub also has a lower loss of energy due to radiation, since the short circuit can be definitely established with a large metal plate, effectively stopping all field propagation. This is especially true for a coaxial line where the short-circuiting plate can be made to seal the line completely.

7-16. The circle diagram for the dissipationless line

The diagram of Fig. 7-11 has certain inherent properties valuable in a qualitative discussion of line phenomena, but it does not present a quantitative impedance answer. A somewhat similar circle diagram may be obtained, however, that solves the impedance equation and simplifies the design of dissipationless lines considerably. The input-impedance equation for a dissipationless line may be written on a per unit basis as

$$\frac{Z_s}{R_0} = \frac{1 + |K|/\phi - 2\beta s}{1 - |K|/\phi - 2\beta s} \tag{7-103}$$

In terms of Z_s/R_0 the equation is applicable to all lines, regardless of their characteristic impedance values. Since Z_s/R_0 is complex, it is possible to write

$$\frac{Z_s}{R_0} = r_a + jx_a \tag{7-104}$$

where r_a and x_a are values of resistance or reactance *per unit* of R_0. Then

$$r_a + jx_a = \frac{1 + |K|/\phi - 2\beta s}{1 - |K|/\phi - 2\beta s} \tag{7-105}$$

The most easily measured line quantity is the standing-wave ratio S, so that it is convenient to replace $|K|$ by its equivalent in terms of S, or

$$(r_a + 1 + jx_a)\left(\frac{S - 1}{S + 1}\right)/\phi - 2\beta s = r_a - 1 + jx_a \tag{7-106}$$

Equating the squares of the magnitudes and after clearing of fractions, this becomes

$$r_a^2 - r_a \frac{(S^2+1)}{S} + x_a^2 = -1$$

By adding a term to complete the square, there results

$$r_a^2 - \frac{2r_a(S^2+1)}{2S} + \left(\frac{S^2+1}{2S}\right)^2 + x_a^2 = \left(\frac{S^2+1}{2S}\right)^2 - 1$$

from which can be obtained

$$\left[r_a - \left(\frac{S^2+1}{2S}\right)\right]^2 + x_a^2 = \left(\frac{S^2-1}{2S}\right)^2 \tag{7-107}$$

This is an equation of the form

$$(x-c)^2 + y^2 = r^2$$

which is recognizable as that of a family of circles of radius r and with centers shifted c units from the origin on the positive x axis.

An actual circle will then have a radius

$$r = \frac{S^2-1}{2S} = \frac{S - \dfrac{1}{S}}{2} \tag{7-108}$$

and a shift of the center of the circle on the positive r_a axis (abscissa)

$$c = \frac{S^2+1}{2S} = \frac{S + \dfrac{1}{S}}{2} \tag{7-109}$$

A family of circles may be drawn for successive values of S as in Fig. 7-19. In drawing particular circles it is interesting to note that for any circle the intercept near the origin is at $1/S$, and that far removed from the origin is at S units on the r_a axis.

Since the minimum value for S is unity, Eq. 7-109 shows that all S circles must surround the 1,0 point. In fact, the circle for $S = 1$ is represented by the 1,0 point. The maximum value of S is infinity, for the case of open-circuit or short-circuit line termination. As S increases, the radius of the S circle increases, and the center moves to the right; for the limiting case of S = infinity, the circle becomes the x_a axis.

Hence a given constant-S circle represents all possible values of

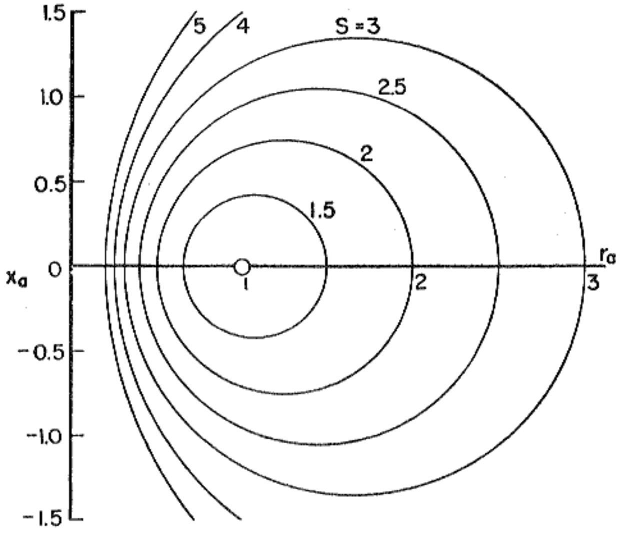

Fig. 7-19. A family of constant-S circles.

r_a and x_a for a given value of Z/R_0. The line from the origin to a given point on the circle represents Z_s/R_0 in both magnitude and angle, with real and reactive components r_a and jx_a, respectively. When Z_s/R_0 lies on the abscissa with magnitude S, the line impedance has a maximum value, and

$$\frac{Z_s}{R_0} = S = \frac{1+|K|}{1-|K|} \tag{7-110}$$

so that by comparison with Eq. 7-103 it is seen that

$$\phi - 2\beta s = 0$$

Thus the point $r_a = S$, $x_a = 0$ on the S circle represents a resistive line impedance at a voltage maximum. This point is chosen as the $\beta s = 0$ condition by convention.

When Z_s/R_0 terminates at the circle intercept $1/S$, the line impedance has a minimum value, and

$$\frac{Z_s}{R_0} = \frac{1}{S} = \frac{1-|K|}{1+|K|} \tag{7-111}$$

from which it can be reasoned that for a resistive load ($\phi = 0$),

$$\beta s = -\frac{\pi}{2}$$

Thus, moving through $\beta s = \pi/2$ radians back along the line has caused the tip of the impedance vector to travel over a *distance* on the circle of $-\pi$ radians. Hence it is seen advisable to place a βs scale on the S circles. The βs scale increases *clockwise*, or in the direction of increasing negative angles.

Rewriting Eq. 7-106 as

$$\left(\frac{S-1}{S+1}\right)\underline{/\phi - 2\beta s} = \frac{r_a - 1 + jx_a}{r_a + 1 + jx_a}$$

and rationalizing the right side gives

$$\left(\frac{S-1}{S+1}\right)\underline{/\phi - 2\beta s} = \frac{r_a^2 - 1 + x_a^2 + j2x_a}{(r_a+1)^2 + x_a^2} \qquad (7\text{-}112)$$

The angle ϕ may be made zero in order that the βs scale may start at 0° on the abscissa. Then, equating the tangents of the angles in Eq. 7-112,

$$\tan(-2\beta s) = \frac{2x_a}{r_a^2 - 1 + x_a^2} \qquad (7\text{-}113)$$

$$r_a^2 - 1 + x_a^2 + \frac{2x_a}{\tan 2\beta s} = 0$$

After the square has been completed, this may be written

$$r_a^2 + \left(x_a + \frac{1}{\tan 2\beta s}\right)^2 = 1 + \frac{1}{\tan^2 2\beta s}$$

$$= \frac{1}{\sin^2 2\beta s} \qquad (7\text{-}114)$$

Lines of equal βs value are then seen to be circles of radius

$$= \frac{1}{\sin 2\beta s} \qquad (7\text{-}115)$$

with a shift of center downward on the x_a axis (ordinate)

$$= \frac{1}{\tan 2\beta s} \qquad (7\text{-}116)$$

A family of such circles is drawn in Fig. 7-20. All the βs circles pass through the point $r_a = 1$, $x_a = 0$.

Superposition of the βs circles on the S circles provides a scale of βs angles and results in the circle diagram of Fig. 7-21. Although the constant-βs circles are computed from Eqs. 7-116 and 7-115 in

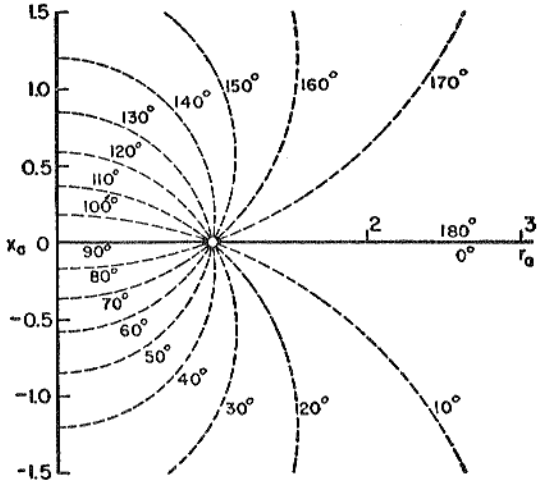

Fig. 7-20. A family of constant-βs circles.

terms of $2\beta s$, they are labeled in terms of βs for ease of use of the diagram.

Per unit admittance may be written

$$\frac{Y_s}{G_0} = \frac{1}{r_a + jx_a} = \frac{r_a}{r_a^2 + x_a^2} - \frac{jx_a}{r_a^2 + x_a^2}$$

$$= g_a - jb_a \qquad (7\text{-}117)$$

Thus a positive inductive reactance becomes a negative susceptance. From Eq. 7-103, it is possible to write

$$\frac{Y_s}{G_0} = g_a - jb_a = \frac{1 - |K|\underline{/\phi - 2\beta s}}{1 + |K|\underline{/\phi - 2\beta s}} \qquad (7\text{-}118)$$

It can be seen that this appears as if formed from Eq. 7-103 by replacing r_a by g_a, x_a by $-b_a$, and $+|K|$ by $-|K|$. Making the same

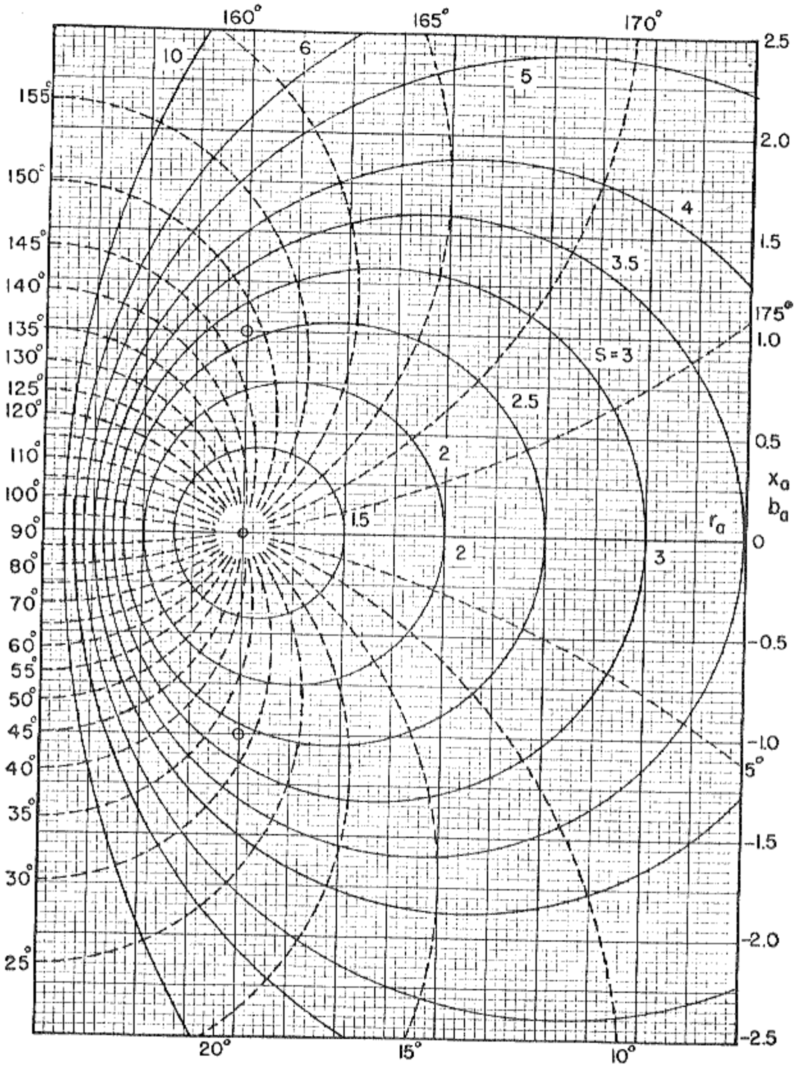

Fig. 7-21. The transmission-line circle diagram.

substitutions in Eq. 7-107 gives

$$\left[g_a - \left(\frac{S^2+1}{2S}\right)\right]^2 + b_a{}^2 = \left(\frac{S^2-1}{2S}\right)^2 \qquad (7\text{-}119)$$

which is an equation of exactly similar form to Eq. 7-107. The circle diagram obtained for impedance, resistance, and reactance can be used for admittance, conductance, and susceptance merely by changing the r_a scale to g_a. The substitution of $-b_a$ for x_a need not be made, since Eq. 7-119 is independent of the sign of the b_a term. The important point to be observed in the use of the chart for susceptance is that inductive reactance is negative susceptance and thus plots downward, whereas positive (capacitive) susceptance plots upward from the axis of reals, with the same scale used as for x_a.

7-17. Application of the circle diagram

The circle diagram may be used to find the input impedance of a line of any chosen length. Compute first the *per unit* value of the load impedance as

$$\frac{Z_R}{R_0} = r_a + jx_a$$

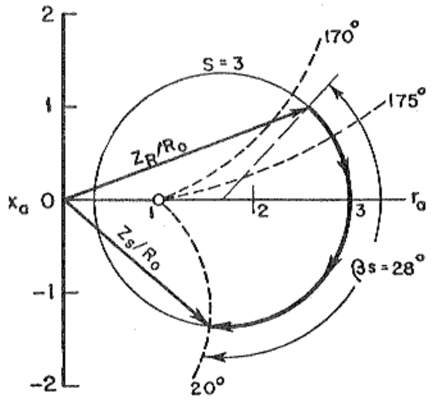

Fig. 7-22. Application of the circle diagram to obtain the input impedance of a 28 deg line terminated in $Z_R/R_0 = 2.6 + j1.0$.

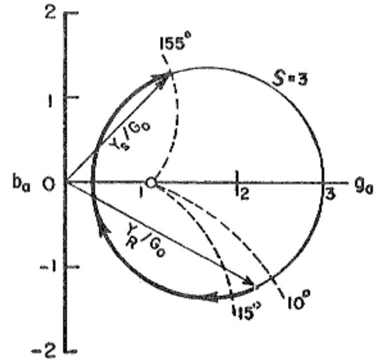

Fig. 7-23. Illustration of the application of the circle diagram to a problem employing admittances.

Locate the point on the chart having coordinates $r_a + jx_a$. This point is the end of a line from the origin that represents the per unit input impedance of the transmission line of zero length (the load only). Follow the constant-S circle passing through the point so located, in a clockwise or negative angle direction through a total number of degrees corresponding to the value of βs or the electrical length of the line, and read the coordinates r_a' and jx_a' of the point reached. It may be necessary to interpolate between βs or S circles. The input impedance of the length of line is then

$$Z_s = R_0(r_a' + jx_a') \tag{7-120}$$

The method is illustrated in Fig. 7-22. For the example chosen, the load is inductive, of value $Z_R/R_0 = 2.6 + j1$; the line is 28° long; the standing-wave ratio is found to be 3; and the per unit input impedance of the line is determined as $Z_s/R_0 = 1.58 - j1.35$.

The input admittance of a length of line may be found by a similar method. An example is given in Fig. 7-23 for a line having per unit load admittance $Y_R/G_0 = 2.25 - j1.20$, representing an inductive load component. This value is plotted below the axis of reals. The standing-wave ratio is found to be 3; and with the line 143° long, the per unit input admittance is $Y_s/G_0 = 1.21 + j1.28$, showing a capacitive component.

An open-circuited line has $S = \infty$, the corresponding S circle appearing as the vertical axis. The input impedance is then pure reactance, with the value for various electrical lengths determined by the intersections of the corresponding βs circles with the vertical axis.

A short-circuited line may be solved by determining its admittance. The S circle is again the vertical axis, and susceptance values may be read off at appropriate intersections of the βs circles with the vertical axis.

7-18. The Smith circle diagram

A modified form of circle diagram for the dissipationless line has been developed by P. H. Smith (Reference 5). The Smith diagram is obtained from a transformation of Eq. 7-112

$$\left(\frac{S-1}{S+1}\right)\underline{/\phi - 2\beta s} = |K|\underline{/\phi - 2\beta s} = \frac{r_a^2 - 1 + x_a^2 + j2x_a}{(r_a + 1)^2 + x_a^2}$$

by introducing new variables, $U + jV$. Thus

$$U + jV = \frac{r_a^2 - 1 + x_a^2}{(r_a + 1)^2 + x_a^2} + \frac{j2x_a}{(r_a + 1)^2 + x_a^2}$$

Equating reals and imaginaries,

$$U = \frac{r_a^2 - 1 + x_a^2}{(r_a + 1)^2 + x_a^2} \tag{7-121}$$

$$V = \frac{2x_a}{(r_a + 1)^2 + x_a^2} \tag{7-122}$$

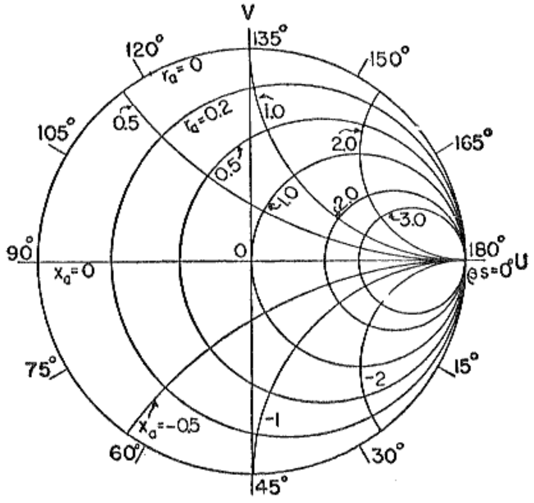

Fig. 7-24. Basis of the Smith circle diagram.

Elimination first of x_a and then of r_a from Eqs. 7-121 and 7-122 results in two equations:

$$\left[U - \left(\frac{r_a}{r_a + 1}\right)\right]^2 + V^2 = \frac{1}{(r_a + 1)^2} \tag{7-123}$$

$$(U - 1)^2 + \left(V - \frac{1}{x_a}\right)^2 = \frac{1}{x_a^2} \tag{7-124}$$

The first of these equations represents a family of constant-r_a circles having centers on the U axis at $r_a/(r_a + 1)$ and radii of $1/(r_a + 1)$. The second equation is that of a family of constant-x_a circles with

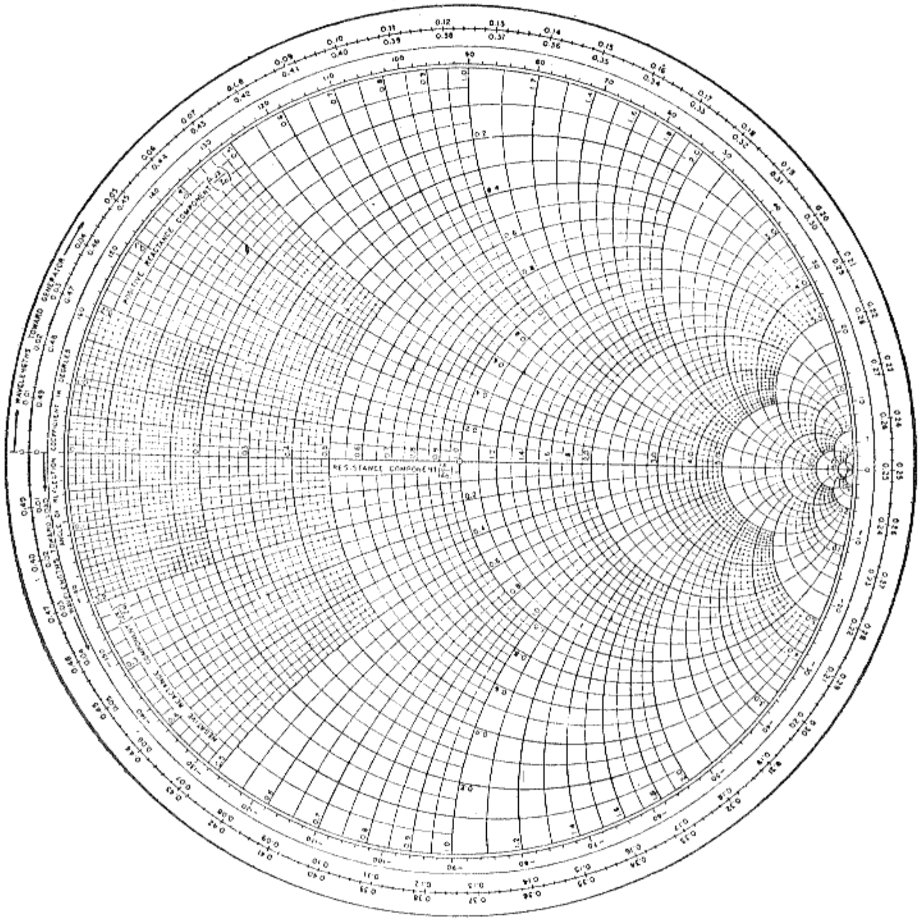

Fig. 7-25. The Smith transmission-line chart. (Reproduced by permission of the Emeloid Co., Inc.)

centers at $1 + j/x_a$ and radii equal to $1/x_a$. The two circle families are drawn in Fig. 7-24.

The diagram was obtained under the assumption that

$$|K|\underline{/\phi - 2\beta s} = U + jV$$

Hence the maximum magnitude of $U + jV$ is fixed at unity by the maximum value of K. Thus *all possible values of impedance are*

contained inside the outer circle of unit radius. The same relation between polar and rectangular coordinates fixes the βs angles or electrical-line-length coordinates at equal increments around the $K = 1$ circle ($S = \infty$) as indicated in the figure.

The circle diagram of Section 7-16 was drawn in terms of S with values from zero to infinity, all lying outside the unit circle, which appeared as a point at 1,0. The Smith diagram is essentially an inversion of the other diagram, mapping all points or S values inside a unit circle. This diagram has achieved considerable popularity.

In a commercially available form of the Smith chart, the βs increments are indicated around the outer edge of the chart in terms of wavelengths, as in Fig. 7-25. A transparent straightedge is pivoted at the center to serve as a distance coordinate to any point on the chart. This straightedge is marked in terms of K or S, giving the effect of adding constant-S circles to the chart without complicating the figure with additional lines.

The impedance of a line may be read at any point on the appropriate standing-wave ratio or S circle. The point at the center of the chart represents the impedance of a line terminated in its characteristic impedance, where $Z/R_0 = 1$ for all distances. The point at the extreme left of the resistance or r_a axis represents zero impedance or short circuit, the point at the extreme right represents infinite impedance or open circuit, and the outer circle represents $S = \infty$.

The chart may be used for admittance as well as for impedance, the r_a and x_a axes simply becoming g_a and b_a axes, with the usual implication that capacitive susceptance is positive or above, and inductive susceptance below, the U or real axis. The point at the left of the conductance or g_a axis then represents zero conductance or an open circuit, while the point at the extreme right represents infinite conductance or a short circuit.

7-19. Application of the Smith chart

The use of the Smith chart is identical with that of the previously discussed circle diagram, and a number of applications will be illustrated.

Consider a line with load of $Z_R/R_0 = 2.6 + j1$, which is 28° long, as was discussed in Section 7-17. The input impedance may be found by entering the chart of Fig. 7-26 at point A, having coordinates $2.6 + j1$, the resistance component scale being along the

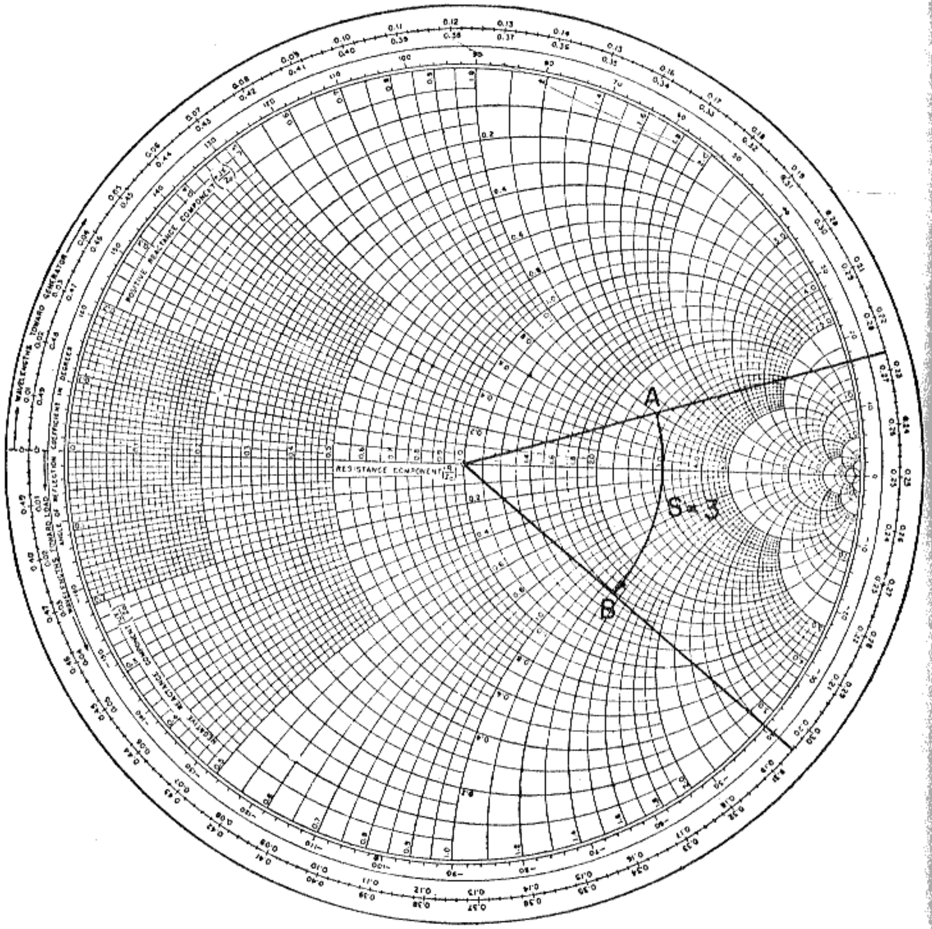

Fig. 7-26. Input impedance of a line.

horizontal axis, the reactive component having a scale at the inner edge of the bounding circle. A constant βs line through A intersects the outer distance scale at 0.227λ.

Then draw a constant-S circle as through point A, using the chart center at the point 1,0 as center. This circle is found to intersect the resistance axis at 3.0, indicating a standing-wave-ratio of 3.0. The line length is $28°/360° = 0.078\lambda$, so that the $S = 3$ circle is followed in a clockwise direction (toward the generator) to a βs

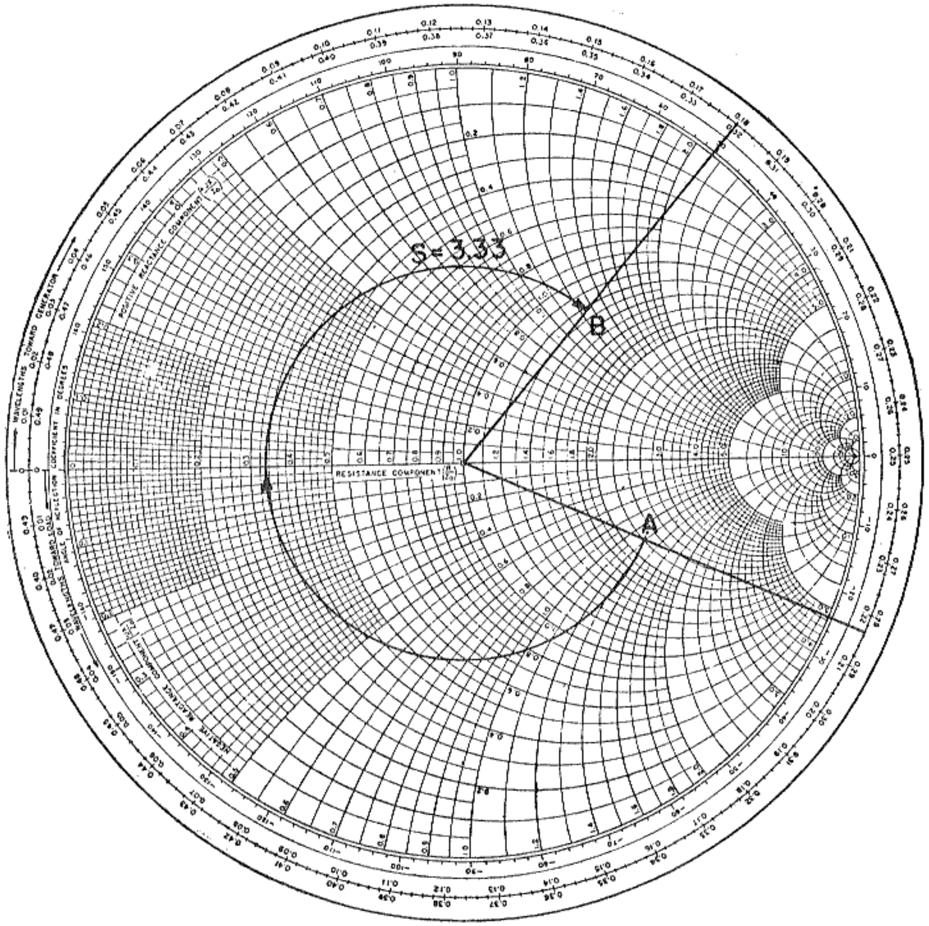

Fig. 7-27. Sending-end input admittance.

line drawn through $(0.227 + 0.078)\lambda = 0.305\lambda$. Point B at the source is thus found. The input impedance can then be read as $Z_s/R_0 = 1.58 - j1.35$, indicating a capacitive reactance for the line input, a result identical to that obtained in Section 7-17.

A similar method yields the input admittance of a loaded line. Assume a line with a load of admittance $Y_R/G_0 = 2.25 - j1.20$, representing an inductive load component. The line length is 143° $= 0.396\lambda$. Using Fig. 7-27, the load admittance is plotted at A, the

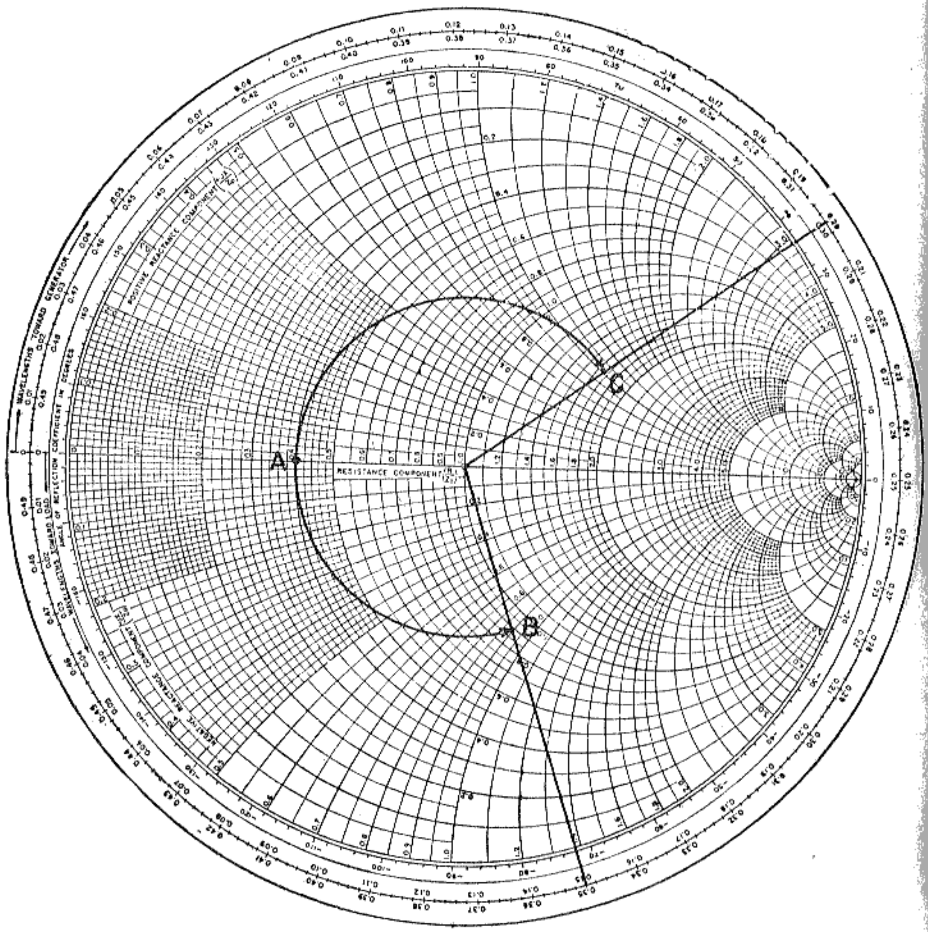

Fig. 7-28. Load and input impedances.

βs line being located at 0.284λ. Using 1,0 as a center the constant-S circle is drawn, showing a standing-wave ratio of 3.0. The βs line for the generator end of the transmission line is drawn at $(0.284 + 0.396)\lambda = 0.680\lambda$, and point B is found. The value of Y_s/G_0 is then read as $1.21 + j1.23$, which checks the value from the previous chart. Since a positive admittance is capacitive, this line will have a capacitive input admittance.

As another example, assume that a line has $S = 2.5$, and it is

found that a voltage minimum exists 0.15λ from the load. Find the load and input impedances for a line of 0.35λ length. A voltage minimum occurs where the S circle crosses the left half of the resistance axis; for $S = 2.5$ this will be at point A at 0.4 in Fig. 7-28. Drawing the circle for $S = 2.5$ and measuring 0.15λ counterclockwise, in the direction away from the generator, will locate point B as the load impedance. The inner wavelength scale might also have been used in terms of distance toward the load. The load impedance at B is read as $Z_R/R_0 = 0.89 - j0.89$.

Measurement on the S circle in the clockwise direction for a distance of 0.35λ from B carries to point C which is the location of the source or generator. The input impedance of the line can here be read as $Z_s/R_0 = 1.68 + j1.03$.

The reflection factor $\mathbf{K}$ can be calculated from $(S - 1)/(S + 1) = 0.428$. The angle of $\mathbf{K}$ can be read from the βs line at the load, giving $-72°$ from the innermost scale of the chart.

It may be noted that the open-circuited line has $\mathbf{K} = 1\underline{/0°}$ or $S = \infty$, and the appropriate S circle is the outer bound of the chart. For this circle $r_a = 0$, and the input impedance of an open line will be pure reactance, as determined by the electrical length of the line.

A short-circuited line may be considered by use of its admittance, its S circle is then the outer bound of the chart, and the input susceptance is determined by the length or βs angle around the chart.

The Smith chart may also be used for lossy lines, and the locus of points on a line then follows a spiral path toward the chart center, due to attenuation.

7-20. Single-stub matching with the Smith chart

The solution of the stub-matching problem of Section 7-15 may be easily carried out on the Smith chart, and an example is shown in Fig. 7-29. Working with admittances, since the tuning stub is to be connected in parallel with the line, enter the chart at A, Fig. 7-29, for a load having a capacitive component and given as $Y_R/G_0 = 2.75 + j1.75$. Drawing the constant-S circle shows the standing-wave ratio before use of the stub to be 4.0, by reason of the intersection of the S circle with the right half of the real axis.

Note that the chart circle for $Y/G_0 = 1$ is the locus of all points for which the real part of the line conductance is unity. This is the

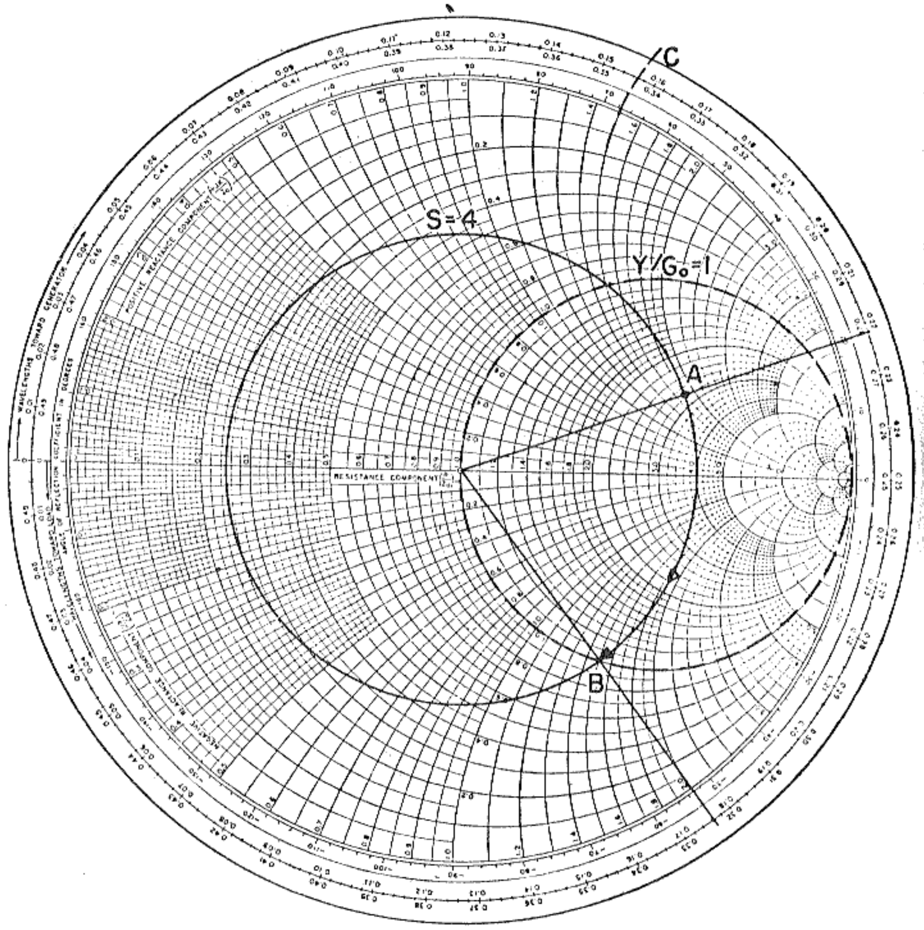

Fig. 7-29. Single stub matching.

desired condition at the point of stub connection, so that the intersection of the S circle and the chart circle for $Y/G_0 = 1$, as indicated at B, will locate the stub. Use of the βs lines through A and B shows that the stub should be located $0.102\lambda = 37°$ from the load.

The b_a value at B represents the susceptance of the line at the stub connection, and this is read as -1.5, indicating inductive susceptance. This value of line susceptance must be resonated by a stub line having an input susceptance of $+1.5$. The electrical

length for a capacitive (short-circuited) stub may be computed from

$$\beta s = \cot^{-1}(-b_a)$$

or may be readily found on the chart.

The input admittance of a short-circuited stub line having capacitive susceptance of $b_a/G_0 = +1.5$ would plot at the intersection of the $+1.5$ susceptance circle and the $K = 1$ ($S = \infty$ circle) or outer bound of the chart at point C. This intersection occurs at 0.156λ on the βs scale or 0.406λ from a short circuit at the right end of the real axis or infinite admittance point (measuring toward the load or counterclockwise). A short-circuited stub line 0.406λ or 146° in length would then have the required capacitive susceptance.

It is also apparent that an open stub line 0.156λ or 56° long might be used, the length of this line in wavelengths being determined from $\beta s = 0.156\lambda$ counterclockwise to zero, or on the $K = 1$ circle to the left end of the real axis which represents an open circuit or zero admittance load.

In Section 7-15 similar information was developed; there only standing-wave ratio before stub connection and the location of the voltage minimum nearest the load were the available data. This is the usual information, and the Smith chart can be readily applied. From the standing-wave ratio the proper S circle is immediately determined by reason of its intercept on the real or U axis. This intersection on the g_a axis at 4.0 for the above example represents maximum conductance where $Y/G_0 = S$, and this is also a point of minimum voltage. Following the S circle to its intersection with the $Y/G_0 = 1$ circle shows that the stub should be connected $(0.324 - 0.25)\lambda = 0.074\lambda = 27°$ toward the source as measured from the voltage minimum nearest the load. The length of stub would be as previously determined. Thus no knowledge of the load admittance is required.

7-21. Double-stub impedance matching on a line

Single-stub impedance matching requires that the stub be located at a definite point on the line. This requirement frequently calls for placement of the stub at an undesirable place from a mechanical viewpoint. For a coaxial line, it is not possible to determine the location of a voltage minimum without a slotted line section, so that placement of a stub at the exact required point is difficult. In the case of the single stub it was mentioned that two adjustments were required, these being location and length of the stub. Another possible method is to use two stubs in which the locations of the stubs are arbitrary, the two stub lengths furnishing the required adjustments. The spacing is frequently made $\lambda/4$. Half-wave spacing should be avoided because it places the two stubs in parallel, resulting in only one effective adjustment being available. The same difficulty arises if the stubs are closely spaced. An arrangement of two short-circuited stubs is illustrated for an open-wire line in Fig. 7-30, $\lambda/4$ spacing being indicated.

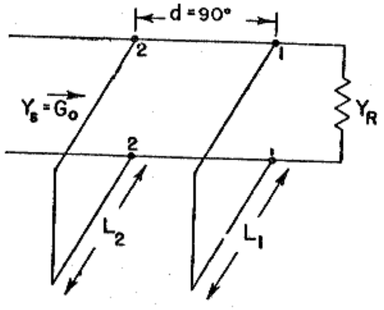

Fig. 7-30. Double-stub impedance matching.

For smooth line operation, the input admittance of the line looking toward the load at the 2,2 location of Fig. 7-30 should be

$$Y_s = G_0$$

or the line should appear terminated in its characteristic impedance at that point. Thus the 2,2 point should be at a location on the line having a per unit admittance of

$$\frac{Y_s}{G_0} = 1 \pm jb_a$$

The Y/G_0 circle passing through $g_a = 1$ will be the locus of all such admittances, and this locus is shown as circle A in short dashes in Fig. 7-31. All points on this circle can be resonated by a stub of susceptance $\mp b_a$ to give the desired per unit admittance of $Y/G_0 = 1$ at 2,2 on the line.

The transformer formed by the quarter-wave of line between 2,2 and 1,1 will transform all admittances on the locus circle A into admittances which will lie on a second locus circle B, found by displacing each point of circle A counterclockwise by a quarter wavelength on the chart (180° rotation on the chart). This second locus circle B is indicated in the figure.

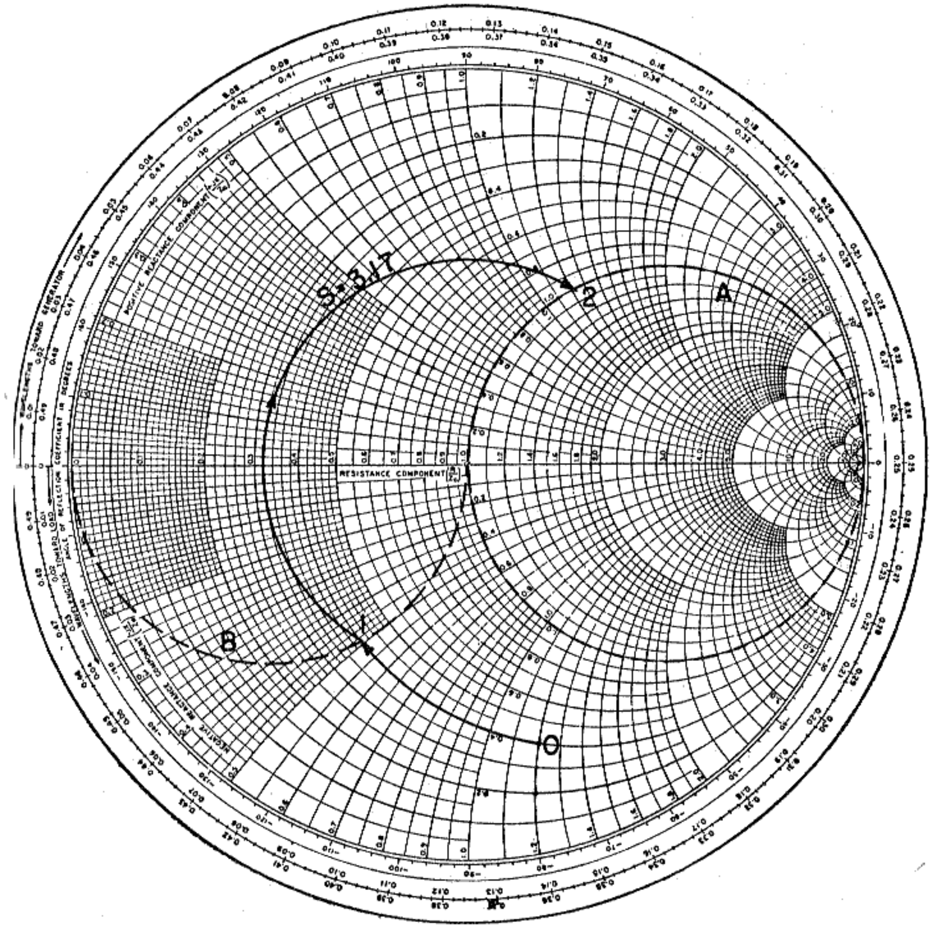

Fig. 7-31. Solution of the double-stub problem.

Therefore if stub 1 succeeds in transforming the input admittance of the line and load to the right of 1,1 into an admittance which will plot on the circle B locus, the quarter-wavelength line will further transform the admittance into a value at 2,2 which will plot on the locus circle A and will have $Y/G_0 = 1 + jb_a$. Stub 2 can then resonate the admittance at 2,2 to the desired value $Y/G_0 = 1 + j0$ for properly terminating the main line to the left of 2,2.

It is desirable that stub 1 be connected at or near the load. It is not always possible to make such a connection at the load, since the largest conductance component at 1,1 that can be transformed is $g_a = 1$, or the input admittance to the left of 1,1 when plotted on the chart cannot lie inside the $g_a = 1$ circle. If this happens, a point on the line toward the source having per unit conductance less than 1.0 must be selected for connection of the stub. Such a point will be available, since the per unit conductance varies between S and $1/S$ in a $\lambda/4$ distance along the line.

An example is illustrated by Fig. 7-31. Before connection of the stubs, the per unit admittance of the line (or of line and load) at the desired location for connection of stub 1 is plotted at 0 on the chart as

$$\frac{Y}{G_0} = 0.4 - j1.2$$

Stub 1 adds a susceptance in parallel, and this must change Y/G_0 to a value Y_1/G_0 which lies on locus circle B. Stub 1 cannot alter the conductance, so following a constant-conductance circle from point 0 locates point 1 on circle B, where

$$\frac{Y_1}{G_0} = 0.4 - j0.5$$

Thus stub 1 must contribute a susceptance of $b_a = +0.7$ at 1,1 on the line.

If Y_s/G_0 had a conductance greater than 1.0, a constant-conductance circle from the Y_s/G_0 plot would miss circle B and point 1 could not be established, confirming the limitation on conductance given above.

The section of line between points 1,1 and 2,2 changes the admittance at 1,1 to that of 2,2 along a constant-S circle, and S along this circle is found to have a value of 3.15. The admittance at 2,2 on the line, without stub 2 connected, can be read from the chart at point 2 on locus circle A, giving

$$\frac{Y_2}{G_0} = 1.0 + j1.2$$

Stub 2 should then be adjusted to provide an inductive susceptance of $b_a = -1.2$, after which the admittance at 2,2 will be $Y_2/G_0 = 1 + j0$, and the main line to the left of 2,2 will be properly terminated.

The required stub lengths may be readily found from the chart. Since short-circuited stubs are most desirable, because of ease of construction and adjustment, their lengths will be

$$L_1 = 0.348\lambda = 125°, \qquad L_2 = 0.11\lambda = 40°$$

If $\lambda/4$ spacing is not suitable, the design method may still be applied by drawing a different locus circle B. The desired circle B should be drawn by rotating circle A through the wavelength spacing chosen for the two stubs. That is, if $\frac{3}{8}\lambda$ spacing is desired, circle B will have its center lying on an axis rotated 270 space degrees counterclockwise from the axis of circle A, or the diameter of B will project vertically downward from the chart center.

7-22. Constants for the line of "small" dissipation

At very high frequencies, of the order of 100 megacycles or more, open- or short-circuited lines are frequently employed as reactances or as resonant circuits. The assumption of zero dissipation would indicate that all such elements have infinite Q values, an obvious fallacy. Consideration of the small losses present in all such elements is necessary to give a true quantitative understanding of the situation. At the frequencies employed, the value of ω is so large that R is always small with respect to ωL, and G is assumed zero. Under such an assumption an analysis for the line constants may readily be made. The line parameters are

$$Z = R + j\omega L, \qquad Y = j\omega C$$

so that the characteristic impedance Z_0 is

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}\left(1 - j\frac{R}{\omega L}\right)}$$

Expansion by the binomial theorem gives

$$Z_0 = \sqrt{\frac{L}{C}}\left(1 - j\frac{R}{2\omega L}\right) \tag{7-125}$$

after neglecting high order terms in $R/\omega L$. This equation shows the impedance Z_0 to have an angle

$$\theta = \tan^{-1}\left(-\frac{R}{2\omega L}\right)$$

and since $R/\omega L$ is small with respect to unity, the characteristic impedance may be written as

$$Z_0 = R_0 \cong \sqrt{\frac{L}{C}} \tag{7-126}$$

and is essentially resistive, in which case it is again given the symbol R_0.

The propagation constant γ for the line of small dissipation is

$$\gamma = \sqrt{ZY} = \sqrt{(R + j\omega L)j\omega C} \tag{7-127}$$
$$= (-\omega^2 LC + j\omega CR)^{1/2}$$

Since $R \ll \omega L$, a more convenient value for γ can be obtained by expansion by the binomial theorem as

$$\gamma = (-\omega^2 LC)^{1/2} + \frac{1}{2}(-\omega^2 LC)^{-1/2} j\omega CR + \frac{1}{8}(-\omega^2 LC)^{-3/2}\omega^2 C^2 R^2$$
$$- \frac{1}{16}(-\omega^2 LC)^{-5/2} j\omega^3 C^3 R^3 + \cdots$$
$$= j\omega\sqrt{LC} + \frac{R}{2}\sqrt{\frac{C}{L}} + j\frac{R^2}{8\omega L}\sqrt{\frac{C}{L}} - \frac{1}{16}\frac{R^3}{\omega^2 L^2}\sqrt{\frac{C}{L}} + \cdots$$
$$= \frac{R}{2}\sqrt{\frac{C}{L}}\left(1 - \frac{R^2}{8\omega^2 L^2}\right) + j\omega\sqrt{LC}\left(1 + \frac{R^2}{8\omega^2 L^2}\right) + \cdots \tag{7-128}$$

The values of α and β are then seen to be approximately

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}}\left(1 - \frac{R^2}{8\omega^2 L^2}\right) \tag{7-129}$$

$$\beta = \omega\sqrt{LC}\left(1 + \frac{R^2}{8\omega^2 L^2}\right) \tag{7-130}$$

Usually the value of $R/\omega L$ will be such that the expressions for α and β may be reduced further to

$$\alpha \cong \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{R}{2R_0} \tag{7-131}$$

$$\beta \cong \omega\sqrt{LC} \tag{7-132}$$

where all values of parameters are usually per meter.

The velocity of propagation may be found by use of Equation 7-130 as

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}\,(1 + R^2/8\omega^2 L^2)} \tag{7-133}$$

In Section 6-4 it was found that for the ideal conditions of zero losses and zero internal inductance of a line, $v = 1/\sqrt{LC}$, and the velocity of propagation is equal to that of light in the medium surrounding the conductors. Equation 7-133 shows that when dissipation is considered, the velocity is less than that of light by reason of a factor dependent on line resistance. In Section 6-4 it was shown that the internal inductance of the line is responsible for slowing of propagation, and here it is seen that the resistance of the line is also a factor in reducing the velocity below the theoretical value, that of light in the medium. Thus it has been found that internal inductance, line resistance, and line insulation with a value of ϵ_r other than unity all contribute to slowing the velocity of propagation below that of light in free space. In actual lines, since $R/\omega L$ is small, the reduction in velocity due to resistance is usually only a fraction of 1 per cent.

The wavelength on the line is

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}\,(1 + R^2/8\omega^2 L^2)} \tag{7-134}$$

The wavelength in free space is

$$\lambda = \frac{1}{f\sqrt{LC}}$$

so that the wavelength on the line is reduced slightly below the free-space value by reason of the line resistance.

7-23. Voltages and currents on the line of small dissipation

Voltage and current expressions on the long line were originally developed as

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}(\epsilon^{\gamma s} + K\epsilon^{-\gamma s})$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}(\epsilon^{\gamma s} - K\epsilon^{-\gamma s})$$

Since it has been found that

$$\alpha = \frac{R}{2R_0}$$

where R is the line resistance per meter of length, then if R is small with respect to $2R_0$ and the line is short, the $\epsilon^{\gamma s}$ terms may be written

$$\epsilon^{\gamma s} = \epsilon^{\alpha s}\epsilon^{j\beta s} = (1 + \alpha s)(\cos \beta s + j \sin \beta s)$$

$$\epsilon^{-\gamma s} = \epsilon^{-\alpha s}\epsilon^{-j\beta s} = (1 - \alpha s)(\cos \beta s - j \sin \beta s)$$

the higher-order terms of the exponential series for $\epsilon^{\alpha s}$ being neglected as small. After insertion of the value of K, the voltage and current equations reduce to

$$E = E_R\left[\left(1 + \alpha s\frac{Z_0}{Z_R}\right)\cos \beta s + j\left(\alpha s + \frac{Z_0}{Z_R}\right)\sin \beta s\right] \quad (7\text{-}135)$$

$$I = \frac{E_R}{Z_0}\left[\left(\alpha s + \frac{Z_0}{Z_R}\right)\cos \beta s + j\left(1 + \alpha s\frac{Z_0}{Z_R}\right)\sin \beta s\right] \quad (7\text{-}136)$$

If it is desired to start with the conventional hyperbolic relations

$$E = E_R \cosh (\alpha + j\beta)s + I_R Z_0 \sinh (\alpha + j\beta)s$$

$$I = I_R \cosh (\alpha + j\beta)s + \frac{E_R}{Z_0} \sinh (\alpha + j\beta)s$$

the cosh and sinh may be expanded as the sum of two angles, and trigonometric functions substituted. If αs is sufficiently small, cosh αs may be replaced by unity and sinh αs by αs, giving the same results as obtained above from the exponential expressions.

7-24. Open- and short-circuit impedances when considering dissipation

The input impedance of a line of length $s = l$ in which the dissipation is small may be written from Eqs. 7-135 and 7-136 as

$$Z_s = \frac{E_s}{I_s} = Z_0\left[\frac{(Z_R + \alpha l Z_0)\cos \beta l + j(Z_R\alpha l + Z_0)\sin \beta l}{(Z_R\alpha l + Z_0)\cos \beta l + j(Z_R + \alpha l Z_0)\sin \beta l}\right] \quad (7\text{-}137)$$

For a line terminated in a short circuit, $Z_R = 0$, and the input impedance becomes

$$Z_{sc} = Z_0\left(\frac{\alpha l \cos \beta l + j \sin \beta l}{\cos \beta l + j\alpha l \sin \beta l}\right) \quad (7\text{-}138)$$

which, after rationalizing and writing $Z_0 = R_0$, becomes

$$Z_{sc} = R_0\left[\frac{\alpha l + j(1 - \alpha^2 l^2)\sin \beta l \cos \beta l}{1 - (1 - \alpha^2 l^2)\sin^2 \beta l}\right]$$

It will be found more convenient if β is eliminated by writing

$$\beta l = \frac{2\pi l}{\lambda}$$

so that

$$Z_{sc} = R_0\left[\frac{\alpha l + j(1 - \alpha^2 l^2)\sin (2\pi l/\lambda)\cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2)\sin^2 (2\pi l/\lambda)}\right] \quad (7\text{-}139)$$

Such a short-circuited line may be used as a circuit element having components as follows:

$$R_{sc} = \frac{R_0 \alpha l}{1 - (1 - \alpha^2 l^2)\sin^2 (2\pi l/\lambda)} \quad (7\text{-}140)$$

$$X_{sc} = \frac{R_0(1 - \alpha^2 l^2)\sin (2\pi l/\lambda)\cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2)\sin^2 (2\pi l/\lambda)} \quad (7\text{-}141)$$

For values of l between 0 and $\lambda/4$ this reactance is inductive; for values of l between $\lambda/4$ and $\lambda/2$ it is capacitive and is seen to oscillate in sign every quarter wavelength. This performance is similar to that of the dissipationless line studied in Section 7-8 except that the above expression for reactance reaches a maximum but does not become infinite.

For certain applications it is convenient to have the admittance of the short-circuited line of small dissipation. It can be obtained by rationalizing the reciprocal of Eq. 7-138, giving

$$Y_{sc} = G_0\left[\frac{\alpha l - j(1 - \alpha^2 l^2)\sin (2\pi l/\lambda)\cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2)\cos^2 (2\pi l/\lambda)}\right] \quad (7\text{-}142)$$

After the numerator and denominator of Eq. 7-137 are divided by Z_R and after Z_R has been set equal to ∞, an expression for the input impedance of an open-circuited line is obtained as

$$Z_{oc} = Z_0\left(\frac{\cos \beta l + j\alpha l \sin \beta l}{\alpha l \cos \beta l + j \sin \beta l}\right) \quad (7\text{-}143)$$

Upon rationalizing and substituting $R_0 = Z_0$ and $\beta = 2\pi/\lambda$, this becomes

$$Z_{oc} = R_0 \left[\frac{\alpha l - j(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \cos^2 (2\pi l/\lambda)} \right] \quad (7\text{-}144)$$

The open-circuit impedance has components

$$R_{oc} = \frac{R_0 \alpha l}{1 - (1 - \alpha^2 l^2) \cos^2 (2\pi l/\lambda)} \quad (7\text{-}145)$$

$$X_{oc} = \frac{-R_0(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \cos^2 (2\pi l/\lambda)} \quad (7\text{-}146)$$

For values of l between 0 and $\lambda/4$ this reactance is capacitive; for values of l between $\lambda/4$ and $\lambda/2$ it is inductive and can be seen to oscillate in sign every quarter wavelength, always being opposite in sign to the reactance of the short-circuited line.

The input admittance of the open-circuited line is

$$Y_{oc} = G_0 \left[\frac{\alpha l + j(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \sin^2 (2\pi l/\lambda)} \right] \quad (7\text{-}147)$$

At very high frequencies the distributed inductive reactance of the leads of a capacitor may be greater than its capacitive reactance,

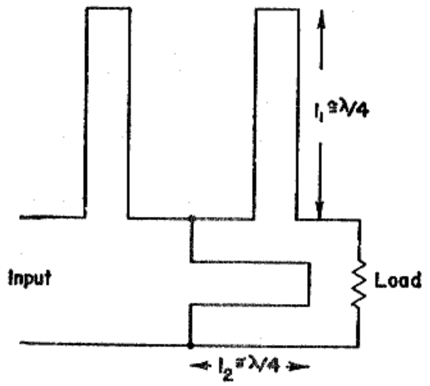

Fig. 7-32. Use of lines as reactive elements in a filter section. Lengths adjusted to give desired capacitive or inductive reactance.

so that the capacitor may appear as an inductor. An inductor may have so much distributed shunting capacitance that it becomes a parallel-resonant circuit or even behaves as a capacitor. Analysis by the methods of distributed-constant circuits has shown a way to

avoid these difficulties by replacing the conventional low-frequency forms of lumped inductors or capacitors with distributed reactive elements and resonant circuits.

Lines may be used as efficient impedance matching sections and as reactive elements in wave filters. At very high frequencies, lines of appropriate length may be used to replace any reactive element called for by the circuit design. Such an application is shown in Fig. 7-32. A wave filter of this nature will be a multiband-pass type because of the multiple resonances of the elements. As will be shown in Section 7-26, the dissipation is lower, or the Q factor greater, for lines than for equivalent lumped reactances, so that filter performance more closely approaches the ideal.

7-25. Quarter- and half-wave lines of small dissipation

In the analysis of the dissipationless line it was indicated that the input reactance of a short-circuited line was infinite at $l = \lambda/4$ and zero at $l = \lambda/2$. Both such values are obviously impossible, and it is interesting to discover the more nearly true state of affairs by analysis of the line of small dissipation.

The input impedance of a *quarter-wave short-circuited line* may be found by insertion of $l = \lambda/4$ in the trigonometric functions of Eq. 7-139, giving

$$Z_{sc} = \frac{R_0}{\alpha l} \quad (7\text{-}148)$$

If in general the line is made any odd number of quarter wavelengths long by setting

$$l = (2n - 1)\frac{\lambda}{4} \quad (n = 1, 2, 3, \ldots)$$

then

$$Z_{sc} = \frac{4R_0}{\alpha(2n - 1)\lambda}$$

and upon substitution of the value of α,

$$Z_{sc} = \frac{8R_0^2}{R(2n - 1)\lambda} \quad (7\text{-}149)$$

is obtained as the input impedance of a *shorted line an odd number of quarter waves long.*

The result is obviously resistive and represents an impedance maximum. Because of the factor n present, the highest impedance value will be obtained with the shortest line or with $n = 1$. It has been shown (Reference 2) that since both R_0 and R are functions of spacing and wire diameter for open-wire lines, or of b/a for coaxial lines, the greatest value of impedance will be obtained if $d/a = 8.0$ for open-wire lines and $b/a = 9.2$ for coaxial lines. The performance of the $\lambda/4$ short-circuited line with varying frequency approximates closely that of an antiresonant circuit *near resonance.*

For a *short-circuited line any number n of half wavelengths long,* the input impedance may be computed as

$$Z_{sc} = R_0 \alpha l = \frac{Rn\lambda}{4} \tag{7-150}$$

This is again resistive and represents an impedance minimum. The $n\lambda/2$ short-circuited line behaves in approximately the same manner as a series resonant circuit when the frequency is varied *near resonance.*

For the open-circuited line the analysis starts with Eq. 7-144. Minima in the impedance characteristics occur with $l = (2n - 1)$ $\lambda/4$, or odd numbers of quarter wavelengths. The impedance then is

$$Z_{oc} = R_0 \alpha l = \frac{R(2n-1)\lambda}{8} \tag{7-151}$$

for the *open-circuited, odd quarter-wavelength line.* The performance of the odd $\lambda/4$ open line is comparable to a series resonant circuit *near resonance.*

The maximum resistive impedance of an *open-circuited line* occurs at $l = n\lambda/2$ and is

$$Z_{oc} = \frac{R_0}{\alpha l} = \frac{4R_0^2}{Rn\lambda} \tag{7-152}$$

Such a line is comparable to the parallel resonant circuit.

It should be noted that the short-circuited $\lambda/4$ line gives a higher input impedance than the open-circuit $\lambda/2$ line for any n. The open-circuited $\lambda/4$ line gives a lower input impedance than the short-circuited $\lambda/2$ line. That is, that line which is physically shortest will usually be the most desirable. The factor n appearing in the equations has the value 1, 2, 3, ... It definitely influences the resistive impedances, lowering the maxima and raising the minima obtainable as n increases. Physically this is reasonable, since if the line were very long, its input impedance would be R_0, regardless of the termination.

A common application of the shorted $\lambda/4$ line is as an insulator, as mentioned in Section 7-12. Equation 7-149 permits calculation of the *insulation resistance* of such a line. Though not infinite, it will ordinarily be found to have a value of some hundreds of thousands of ohms, which usually is sufficiently high to be neglected in comparison with an R_0 of only a few hundred ohms or less. The copper insulator is mechanically rugged and maintains the line at low potential to ground for all frequencies except the one desired, or its odd harmonics.

7-26. The tapped quarter-wave line as an impedance transformer

The resistive impedance seen at the open end of a *short-circuited* line $\lambda/4$ long is given by Eq. 7-149 as

$$Z_{sc} = \frac{8R_0^2}{R\lambda} = \frac{4R_0}{\alpha\lambda}$$

The impedance seen at the short-circuited end of the line is obviously zero. Intermediate points along the line will present impedances

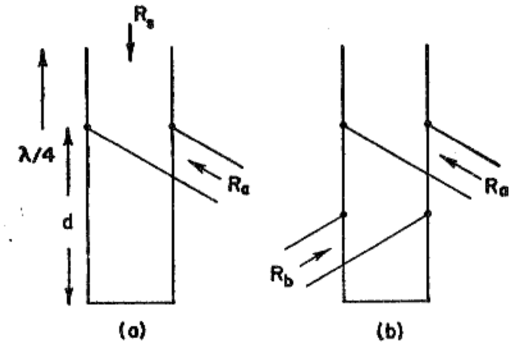

Fig. 7-33. The tapped quarter-wave line.

intermediate to the two values. Some line having a resistance R_a can then be matched to some load having resistance R_s by use of a $\lambda/4$ line as in (a), Fig. 7-33. The line having R_a impedance is connected at a point distant d meters from the short circuit. The load having resistance R_s is connected at the open end of the $\lambda/4$ line. The line then appears as an autotransformer.

In some cases neither of the devices to be coupled has a resistance equal to R_s. The two devices having resistances R_a and R_b may be connected at appropriate points on the line at which the line impedance matches that of the connected sources or loads, as in (b), Fig. 7-33. Loads or sources having impedances above the value of R_s, the input resistance at the open end of the line, cannot be matched by this system.

An expression by which the impedance of an unloaded line at any distance d from the short circuit can be found, can be obtained by writing Y_{sc} and Y_{oc} from Eqs. 7-138 and 7-143 as

$$Y_{sc} = G_0\left[\frac{1 + j\alpha l \tan(2\pi l/\lambda)}{\alpha l + j\tan(2\pi l/\lambda)}\right] \tag{7-153}$$

$$Y_{oc} = G_0\left[\frac{\alpha l + j\tan 2\pi l/\lambda}{1 + j\alpha l \tan 2\pi l/\lambda}\right] \tag{7-154}$$

The admittance Y_a at distance d from the short circuit is the sum of two admittances in parallel; one a short-circuited line d meters long, the second an open-circuited line $(\lambda/4 - d)$ meters long. Then

$$Y_a = G_0\left[\frac{1 + j\alpha d\tan(2\pi d/\lambda)}{\alpha d + j\tan(2\pi d/\lambda)} + \frac{\alpha(\lambda/4 - d) + j\tan(2\pi/\lambda)(\lambda/4 - d)}{1 + j\alpha(\lambda/4 - d)\tan(2\pi/\lambda)(\lambda/4 - d)}\right]$$

By use of the trigonometric identities

$$\tan\frac{2\pi}{\lambda}\left(\frac{\lambda}{4} - d\right) = \cot\frac{2\pi d}{\lambda}$$

$$\tan\frac{2\pi d}{\lambda} + \cot\frac{2\pi d}{\lambda} = \frac{2}{\sin(4\pi d/\lambda)}$$

the above may be reduced to

$$Y_a = G_0\frac{j\alpha\lambda/[2\sin(4\pi d/\lambda)]}{\alpha(2d - \lambda/4) + j[\tan(2\pi d/\lambda) + \alpha^2 d(\lambda/4 - d)\cot(2\pi d/\lambda)]} \tag{7-155}$$

Since α will be small, the denominator terms involving α may be considered negligible in magnitude and dropped. In the case of the first term in the denominator, this is equivalent to neglecting a small susceptance component. Inversion and rationalization then

produces

$$Z_a = \frac{2R_0}{\alpha\lambda}\sin\frac{4\pi d}{\lambda}\tan\frac{2\pi d}{\lambda} = \frac{4R_0}{\alpha\lambda}\sin^2\frac{2\pi d}{\lambda} \tag{7-156}$$

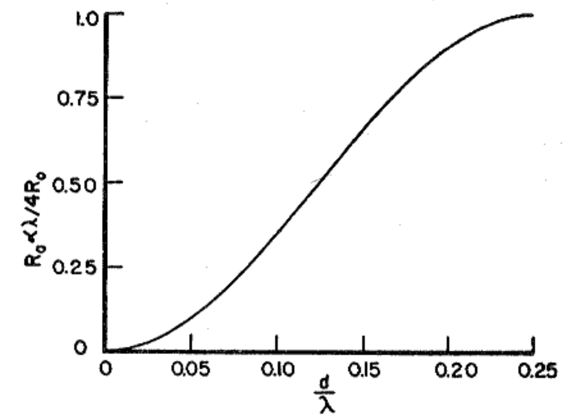

Fig. 7-34. Variation of resistance with distance d from short-circuit point.

The coefficient is recognizable as the resistive impedance at the open end of the $\lambda/4$ line. Thus the resistance at any point varies as the square of the sine of the distance function. The result is plotted in Fig. 7-34.

A load connected at the open end of the line will parallel the line impedance, and if this load has a resistance R_L, the resultant resistive impedance seen at the tap at any distance d will vary as

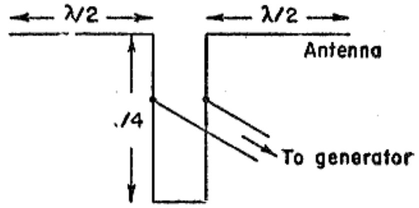

Fig. 7-35. Application of tapped quarter-wave line as an impedance transformer in matching a line to an antenna of length λ

$$Z_a = \frac{4R_0R_L}{\alpha\lambda(R_L + 4R_0/\alpha\lambda)}\sin^2\frac{2\pi d}{\lambda} = \frac{4R_0}{\alpha\lambda}\left(\frac{1}{1 + 4R_0/\alpha\lambda R_L}\right)\sin^2\frac{2\pi d}{\lambda} \tag{7-157}$$

The tapped quarter-wave line is frequently employed for impedance matching between an antenna, or several antennas, and a transmission line, as shown in Fig. 7-35. While an open-line has theoretical impedance-matching properties it is rarely used because of radiation losses from the open end, end capacitance effects, and the difficulty of a smooth adjustment of length.

7-27. Voltage step-up on the resonant line

A quarter-wave open-circuited line has the property of producing a voltage step-up between its ends. By writing the voltage relation of Eq. 7-135 as

$$\frac{E_s}{E_R} = \left(1 + \alpha l \frac{Z_0}{Z_R}\right) \cos \frac{2\pi l}{\lambda} + j\left(\alpha l + \frac{Z_0}{Z_R}\right) \sin \frac{2\pi l}{\lambda}$$

and setting $Z_R = \infty$ and $l = \lambda/4$, there is obtained

$$\frac{E_R}{E_s} = \frac{1}{\alpha l} = \frac{4}{\alpha\lambda} \tag{7-158}$$

as the voltage ratio between the ends of the line. Since α may be quite small, the voltage step-up can be large.

7-28. *Q* of a line as a circuit element; band width

As has been pointed out in the preceding sections, lines may be employed as resonant circuits. When they are so used, it is convenient to be able to analyze and proportion them to achieve desired frequency selection properties. This may be done conveniently in terms of *band width,* which was defined in Chapter 2 as the width of the resonant curve, in cycles, at the point at which the power in the circuit is one-half the maximum power, expressed as a percentage of the resonant frequency.

It was discovered in Chapter 2 that at the frequency f_2 at which the power was one-half of the value at the resonant frequency f_r, the reactance of the circuit was equal to the resistance. In an effort to find the frequency at which the power is one-half the maximum value (current = 0.707 maximum), the reactance and resistance of an open-circuited line as given by Eqs. 7-145 and 7-146 may be

equated. That is, at frequency f_2 or phase-constant value β_2,

$$R_0\alpha l = -R_0(1 - \alpha^2 l^2) \sin \beta_2 l \cos \beta_2 l$$

$$\frac{\sin 2\beta_2 l}{2} = \frac{-\alpha l}{1 - \alpha^2 l^2}$$

If αl is small, its square may be neglected with respect to unity in the denominator, so that

$$2\beta_2 l = \sin^{-1}(-2\alpha l)$$

Again, since αl is small, the sine may be replaced by the angle, so that

$$\beta_2 l = \frac{\pi}{2} + \alpha l \tag{7-159}$$

The frequency f_2 at which reactance and resistance are equal is then available by substitution of the value of β, giving

$$f_2 = \frac{v}{2\pi l}\left(\frac{\pi}{2} + \alpha l\right) \tag{7-160}$$

The chosen open-circuited line behaves like a series resonant circuit when its length is equal to $\lambda/4$. The value of the resonant frequency f_r is then available from the fact that, for $\lambda/4$ length,

$$\beta l = \frac{\pi}{2} = \frac{2\pi f_r l}{v}$$

$$f_r = \frac{v}{4l} \tag{7-161}$$

There will be a second frequency f_1, below f_r, at which the resistance equals the reactance. This frequency is

$$f_1 = \frac{v}{2\pi l}\left(\frac{\pi}{2} - \alpha l\right) \tag{7-162}$$

The band width is then given by

$$\frac{\Delta f}{f_r} = \frac{f_2 - f_1}{f_r} = \frac{(v/2\pi l)(\pi/2 + \alpha l - \pi/2 + \alpha l)}{f_r} \tag{7-163}$$

Since $\beta = 2\pi f/v$,

$$\frac{\Delta f}{f_r} = \frac{2\alpha}{\beta} \tag{7-164}$$

It has also been shown that the circuit Q is equal to the reciprocal of the band width, so that the merit factor of the line is given by

$$Q = \frac{\beta}{2\alpha} \tag{7-165}$$

Since α is ordinarily small, the value of Q for such a line may be very high. For the usual line designs it may approximate 1000 to 4000, which is considerably higher than the Q possible for lower-frequency lumped-constant circuits. The selectivity possible with lines as resonant circuits is correspondingly greater.

Since Q is increased if α is made small, the design of lines for minimum attenuation is of interest and will be considered in the next section.

7-29. Optimum design of the open-wire resonant line

In previous sections it has been found that the Q of a resonant line, the voltage step-up ratio, and the magnitudes of impedances or resistances available when lines are used as reactive or resonant elements all depend on the value of α. It is usually desirable that α be small. Since for a line of small losses

$$\alpha = \frac{R}{2R_0}$$

and both R and R_0 depend on the radius of the wires used, it is desirable to determine that value of the radius of the wire in an open-wire line which will make α a minimum.

From Eqs. 7-10 and 7-15 for a *copper line*, the value of α is

$$\alpha = \frac{(8.33 \times 10^{-8} \sqrt{f})/a}{2 \times 120 \ln (d/a)} \tag{7-166}$$

where d is the center-to-center spacing and a is the radius of the conductors of the open-wire line, in meters. Then

$$\alpha = \frac{3.47 \times 10^{-10} \sqrt{f}}{a \ln (d/a)} \text{ nepers/m} \tag{7-167}$$

Considering d constant and minimizing α with respect to a gives $d/a = \epsilon = 2.718$ as the value of the ratio for minimum attenuation on an open-wire line. However, this value of d/a is so small that the assumption of large d/a ratio used to derive Eq. 5-30 for the

capacity of an open line does not hold. Consequently Eq. 7-167 should be rewritten, giving consideration to the fact that the charges will not be uniformly distributed around the conductor periphery, thereby altering the capacitance value to that of Eq. 5-33

$$C = \frac{\pi\epsilon}{\cosh^{-1}(d/2a)}$$

and adding a proximity factor to the resistance term. With these considerations the value of d/a for minimum attenuation is approximately 3.6 for an open-wire air-insulated line.

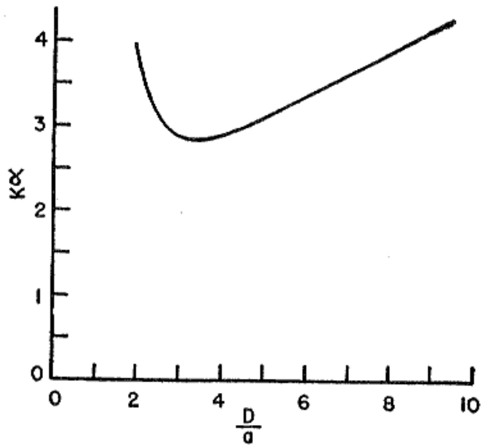

Fig. 7-36. Variation of α of an open-wire line with d/a ratio of the line.

The variation of a dimensionless factor, derivable by this metho and proportional to α, as a function of d/a is shown in Fig. 7-36. This curve displays the expected minimum, but it is apparent that the minimum is quite broad, thereby permitting considerable latitude in the choice of element sizes for resonant lines.

Qualitatively, the minimum value provides a balance between R_0 and the size of conductor. If the conductor is made smaller in radius, then R_0 increases, thereby reducing the value of current flowing in a given line and reducing the losses. For the same change the conductor resistance rises, increasing the losses. The value $d/a = 3.6$ achieves a balance of these opposing actions.

With a resonant line designed for a minimum value of α, the highest value of Q and other desirable properties of resonant lines are obtained.

7-30. Design considerations for the coaxial line

For optimum performance, the design of a high-frequency coaxial line is dependent on desirable values of the ratio b/a, where b is the inner radius of the outer conductor and a the radius of the inner conductor. If α is written as

$$\alpha = \frac{R}{2R_0}$$

it is found, by use of Eqs. 7-15 and 7-21, that α for a *copper coaxial line* may be written as

$$\alpha = \frac{4.16 \times 10^{-8}\sqrt{f}(1/b + 1/a)}{[2 \times 60 \ln (b/a)]/\sqrt{\epsilon_r}}$$

$$= 3.47 \times 10^{-10}\sqrt{f\epsilon_r}\,\frac{[(1/b) + (1/a)]}{\ln (b/a)} \text{ nepers/m} \qquad (7\text{-}168)$$

all dimensions being in meters. The term in parentheses may be

$$= \frac{1}{b}\frac{[1 + (b/a)]}{\ln (b/a)} \qquad (7\text{-}169)$$

and if the outer conductor radius b is considered constant, the expression for α may be minimized, giving

$$\frac{dx}{d\left(\dfrac{b}{a}\right)} = \ln\frac{b}{a} - \frac{a}{b}\left(1 + \frac{b}{a}\right) = 0 \qquad (7\text{-}170)$$

$$\ln\frac{b}{a} = \frac{a}{b} + 1$$

This may be solved graphically, giving a value of b/a for minimum attenuation:

$$\frac{b}{a} = 3.6$$

The variation in attenuation for the coaxial line with the ratio b/a is shown in Fig. 7-37. The minimum occurring at $b/a = 3.6$ is broad,

so that the dimensions to produce minimum attenuation are not critical. The characteristic impedance for an air-spaced coaxial line with dimensions for minimum attenuation is 77 ohms.

A second factor of importance in the design of coaxial lines

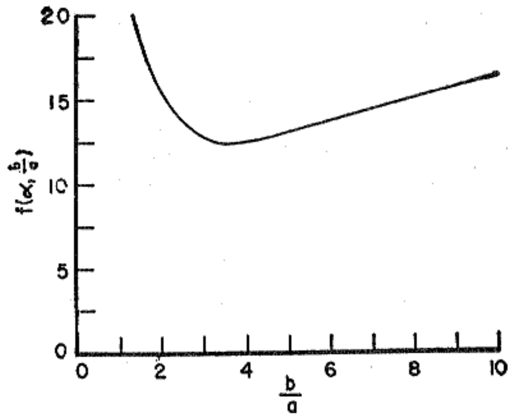

Fig. 7-37. Variation of α of a coaxial line with b/a ratio of the line.

required to handle considerable amounts of power is the design for minimum voltage gradient in the dielectric, or maximum voltage breakdown of the line. It may be argued that if a nearly equals b, the full line voltage appears across a thin layer of dielectric and the voltage gradient is large. If a is very small, the field at the central conductor surface is also very large. Between these extremes should be some value of a that will produce a minimum voltage gradient.

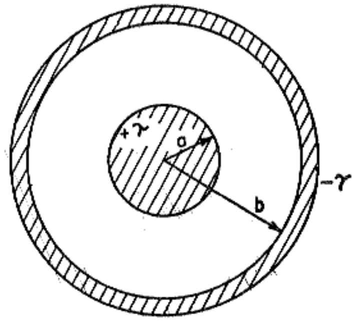

Fig. 7-38. Cross section of a coaxial line.

In a practical case the voltage between conductors will be held constant, with line charges $+\tau$ and $-\tau$ appearing on the conductors as shown in Fig. 7-38. The charges in coulombs per meter of length are related to the voltage between the cylinders according to

$$\tau = CV$$

where C is the line capacitance per meter of length. Insertion of Eq. 5-35 for the capacitance of the coaxial line gives

$$\tau = \frac{2\pi\epsilon V}{\ln(b/a)} \text{ coulombs/m} \qquad (7\text{-}171)$$

According to Eq. 5-26, the electric field intensity in the dielectric is

$$\mathcal{E} = \frac{\tau}{2\pi r\epsilon} \text{ v/m}$$

The maximum value of field intensity will obviously occur at the surface of the inner conductor at $r = a$, so that

$$\max \mathcal{E} = \frac{\tau}{2\pi a\epsilon} \text{ v/m} \qquad (7\text{-}172)$$

From these equations, the line charge τ may be eliminated, giving

$$\max \mathcal{E} = \frac{V}{a \ln(b/a)} \text{ v/m} \qquad (7\text{-}173)$$

To obtain the dimensions that result in minimum field, the above expression may be minimized, holding the voltage V and dimension b

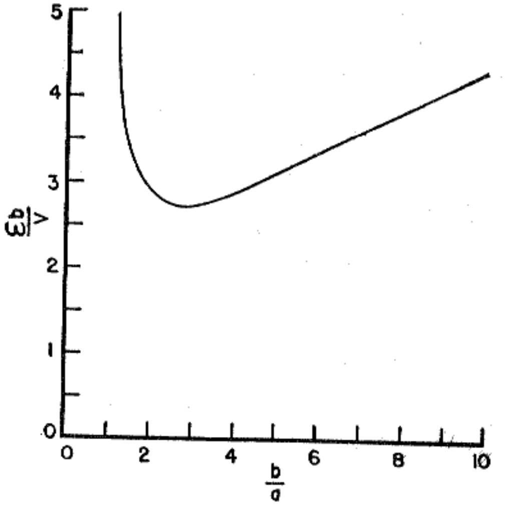

Fig. 7-39. Variation of maximum field intensity with b/a ratio for a coaxial line.

constant. Then

$$\frac{d\mathcal{E}}{da} = -V(\ln b - \ln a - 1) = 0$$

$$\ln\frac{b}{a} = 1 \qquad (7\text{-}174)$$

For a minimum value of voltage gradient in the region between the conductors, the ratio of conductor radii should be

$$\frac{b}{a} = \epsilon = 2.718$$

The variation of field intensity at the inner conductor surface is expressed in a dimensionless plot as a function of b/a in Fig. 7-39. It can be seen from Figs. 7-37 and 7-39 that the difference between the values of b/a for minimum attenuation and minimum field intensity can be readily compromised.

Optimum design conditions for lines are summarized in Table 6.

TABLE 6

OPTIMUM DESIGN CONDITIONS FOR HIGH-FREQUENCY LINES

$\epsilon_r = 1$	Coaxial		Open-wire	
	$\frac{b}{a}$	Z_0	$\frac{d}{a}$	Z_0
Minimum attenuation	3.6	77	3.6	154
Maximum resonant impedance	9.2	133	8	250
Maximum Q	3.6	77	3.6	154
Maximum voltage breakdown	2.7	60	5.4	202
Maximum power transfer	1.6	30		

PROBLEMS

7-1. A 50-megacycle open-wire line is to be built of No. 8 copper wire and to have $Z_0 = 425$ ohms.

(a) Find the desired spacing D.

(b) Calculate the total L and C of 5 meters of this line.

7-2. An air-filled coaxial line of copper is to have a capacitance of 22 $\mu\mu$f per meter. The inner conductor has a diameter of 0.1 cm.

(a) Calculate the inductance of the line.

(b) Find the inner radius of the outer conductor.

(c) What is the characteristic impedance of the line?

(d) Find the phase constant and wavelength at a frequency of 25 megacycles, neglecting dissipation.

7-3. (a) Find Z_0, L, and C of an air-spaced coaxial line having a No. 12 inner copper conductor and $b/a = 10$, at a frequency of 40 megacycles.

(b) Repeat (a) if the line is filled with polyethylene as a dielectric.

(c) Find the velocity of propagation and λ for (a) and (b), neglecting resistance.

7-4. A line and load have the following values: $R_0 = 150$ ohms, $Z_L = 25 + j40$ ohms. Plot curves of the magnitude and phase angle of the input impedance as the value of βl varies from 0 to 2π.

7-5. (a) Calculate the reflection coefficient and standing-wave ratio for a line of open wire having No. 10 conductors spaced 3 cm, if the line is terminated in $50 + j75$ ohms at 7.3 megacycles. Neglect line resistance.

(b) If the line is supplied by a generator of 100 v, 200 ohms internal resistance, find the power delivered to the load at 7.3 megacycles if $l = 30$ m.

(c) Calculate the power that could be delivered to the load if it were matched to the line.

7-6. A transmission line is terminated in Z_L. Measurements indicate that the standing-wave minima are 102 cm apart and that the last minimum is 35 cm from the load end of the line. The value of S is 2.4 and $R_0 = 250$ ohms.

(a) Find Z_L in terms of real and reactive components.

(b) What frequency is being transmitted if line dissipation is neglected?

7-7. A certain line of $R_0 = 400$ ohms is $\frac{7}{16}\lambda$ long and open at both ends. Find the impedance seen by a generator connected at a point $\lambda/4$ from one end.

7-8. Find the input impedance of a coaxial line having $R_0 =$ 95 ohms. The line is 20 m long, short-circuited at the far end, and operated at 10 megacycles. Neglect the line dissipation.

7-9. Given a dissipationless line of $R_0 = 300$ ohms, what lengths are needed to obtain an inductance of 7 μh at 75 megacycles, with both open- and short-circuit terminations.

7-10. (a) Find the b/a ratio for a coaxial line used to match a load of 300 ohms resistance to a line having $R_0 = 95$ ohms.

(b) If a coaxial matching section is used as above and made with a 0.2-cm-diameter internal conductor, find the internal diameter of the outer conductor.

7-11. A 200-m length of line of No. 8 copper wire is to be used to supply a load of 100 ohms and is to be operated at 30×10^6 c. Under short-circuit conditions the maximum voltage on the line is 310 v, and the maximum current is 0.584 amp. Find the spacing of 1-cm-diameter conductors needed for a $\lambda/4$ matching section to match this line to its load, neglecting conductor proximity.

7-12. An air-insulated coaxial line has an inside conductor of 3.6 mm copper (No. 7) and a copper outside conductor of 10 mm inside diameter. Find the input impedance of a quarter-wave short-circuited line of this material operating at 1500 megacycles.

7-13. An antenna, as load on a transmission line, produces a standing-wave ratio of 2.8, with a voltage minimum 0.12λ from the antenna terminals. Find the antenna impedance and the reflection factor at the antenna, if $R_0 = 300$ ohms for the line.

7-14. A lossless line $\frac{7}{16}\lambda$ long has an input impedance $Z_s/R_0 =$ $1.2 + j0.95$. Find the load impedance and the standing-wave ratio.

7-15. A prototype T-section filter is designed to operate at 10^8 c. One inductance of 5.2 μh and two capacitances of 7.8 $\mu\mu$f, are needed. The completed filter is to transmit direct current as well as the 10^8-c signal. Calculate the line lengths required and state the terminations to be used. Draw the completed circuit, if R_0 of the line is 200 ohms.

7-16. In Fig. 7-40(a), $R = 50$ ohms and $R_0 = 100$ ohms. Find the length L of an open stub needed to make the input impedance at A,B purely resistive.

7-17. An impedance $Z_R = 41 + j15$ ohms is connected as in Fig. 7-40(b) to a coaxial line of $R_0 = 100$ ohms. Find the lengths d_1 and d_2 to match the line to the load, by use of the circle diagram.

7-18. Considering dissipation, derive an expression for the impedance Z_{in} seen at distance d in Fig. 7-40(c). Plot the variation of Z_{in} against distance.

7-19. A coaxial line has $b = 0.55$ cm, $a = 0.10$ cm, and polystyrene dielectric.

(a) With this line terminated in 100 ohms resistance at 144 megacycles, find the shortest length that will have a reactive component of input impedance equal to $+j25$ ohms.

(b) Find the standing-wave ratio.

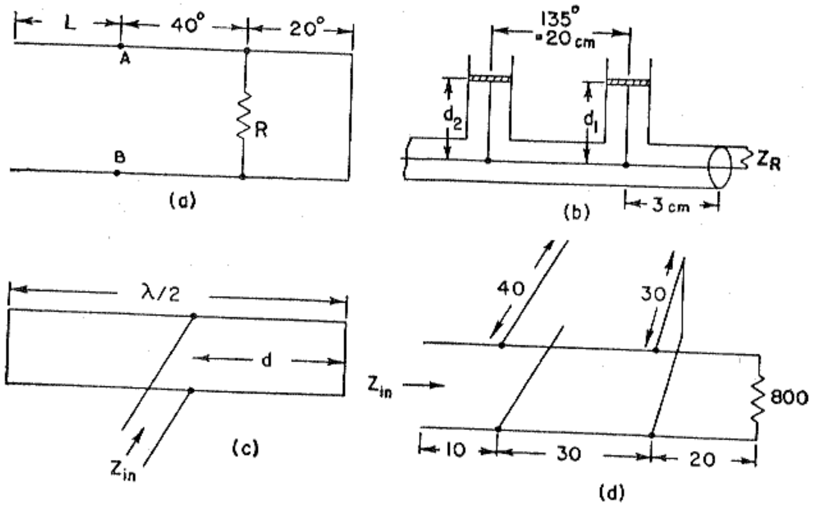

Fig. 7-40.

7-20. A standard coaxial line is designed as follows: central conductor No. 22 copper, polyethylene dielectric ($\epsilon_r = 2.25$), $b = 0.073$ in., design $Z_0 = 73$ ohms.

(a) Check the Z_0 by calculation from the dimensions.

(b) If the line is terminated in a reactance of $j150$ ohms, find the input impedance of a section 25 cm long at a frequency of 250 megacycles.

7-21. By use of the circle diagram find the input impedance of the line in Fig. 7-40(d), if the line and stubs are constructed of No. 4 copper wire spaced 10 cm. The frequency of operation is 220 megacycles. All dimensions are in centimeters.

7-22. A load having impedance of $Z_R = 140$ ohms is to be connected to a line of $R_0 = 100$ ohms by a quarter-wave matching transformer.

(a) Find the Z_0 of the matching transformer.

(b) What is S for the transformer?

(c) If the input voltage to the line is 100 v, find the load voltage, (dissipationless line).

(d) Find the location and magnitude of the maximum voltage on the quarter-wave transformer.

7-23. A line has a standing-wave-ratio of 4. The R_0 is 150 ohms and the maximum voltage measured on the line is 135. Find the power being delivered to the load.

7-24. A line of $R_0 = 300$ ohms is connected to a load of 73 ohms resistance. For a frequency of 45 megacycles, find the length, termination, and location nearest the load of a single stub to produce an impedance match.

7-25. The line of Prob. 7-24 is to be matched with double stubs, one located as near the load as possible. Specify length of both stubs, termination, and location of the second stub, using $\lambda/4$ spacing.

7-26. For a load of $Z_R/Z_0 = 0.8 + j1.2$, design a double-stub tuner, making the distance between stubs $\frac{3}{8}\lambda$. Specify the stub lengths and the distance from load to first stub. Stubs are short-circuited.

7-27. A load has admittance $Y_R/G_0 = 1.25 + j0.25$. Find the length and location for a single-stub tuner, short-circuited.

7-28. A line is made up to have $R = 0.008$ ohm, $L = 2.0$ μh, and $C = 5.0$ $\mu\mu$f, all per meter of line length. Frequency $= 8$ megacycles.

(a) Find Z_0.

(b) Determine the true velocity and wavelength.

(c) Calculate the Q and band width of a quarter wave of this line open-circuited.

7-29. An air-spaced coaxial line is shorted and used as a quarter-wave resonant circuit. If the line is 1.2 m long and has an outer copper conductor with inner radius of 4 cm and inner conductor radius of 0.8 cm, calculate the resonant frequency. Determine also the impedance, Q, and band width of the line.

7-30. The line of Prob. 7-29 is tapped at a point 0.3 m up from the short circuit. Find the impedance seen at the tap.

7-31. What is the maximum value of conductance that can be matched by a double-stub tuner with one stub at the load and the other stub at $\frac{3}{8}\lambda$ back from the load? at $\lambda/2$ stub spacing? at $\frac{5}{16}\lambda$?

7-32. A line of $\frac{1}{4}$ in. diameter copper rods is spaced 2 in. At 250 megacycles, what short-circuited length of this line would serve as an insulator? What would be the value of the insulation resistance presented to the circuit being isolated?

7-33. A lossless line terminated in a resistance is found to have a standing-wave ratio of 4. The R_0 is calculated as 100 ohms. A short-circuited stub that matches the line to the load is placed less than $\lambda/8$ from the load.

(a) What is the value of the load resistance?

(b) What is the stub length in wavelengths?

7-34. An open-wire dissipationless line has $R_0 = 200$ ohms. When it is short-circuited, the first voltage minimum is 20 cm from the short-circuited end. The short circuit is then removed and replaced with a load. The first voltage minimum is found to have moved to a point 11 cm from the load, and S is measured as 3.33. What is the impedance of the load?

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Chapter 8

THE LINE AT POWER FREQUENCIES

While ruled by the same fundamental physical theory, the design and operation of a line for transmission of electric power at low frequencies involve considerable differences in operational techniques and viewpoint from those employed in the use and understanding of the telephone or radio-frequency line.

These differences are created by the following factors:

Power transmission lines are electrically short, none exceeding $\lambda/10$ in length at the present time.

They must have high power efficiency, since the price obtained for the energy output of the line must be competitive with other energy sources.

Operation is at a fixed frequency, thereby simplifying design.

They are operated at constant output voltage, because of our wide use of the parallel form of distribution to voltage-rated loads.

The lines are operated with variable-impedance termination, determined by the power demand at the fixed output voltage. Good regulation of the output voltage will also be required as the load fluctuates.

The latter two paragraphs show a dissimilarity with the operation of the telephone or radio-frequency line at fixed Z_0 termination and variable output voltage. As a point of similarity with the radio-frequency line, the line insulation is usually excellent and the shunt conductance may be considered zero.

For the study of a single specific transmission line, the engineer will either employ calculation with the transmission line equations by use of the equivalent circuit, or will use a graphical method with a circle diagram especially adapted to power usage. In the analysis of a complete *power system* involving many lines, some calculation may be used, but for very complex systems an analogue computer known as a *network analyzer* is ordinarily employed. In either case an individual line is again represented by its equivalent circuit.