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Chapter 5

TRANSMISSION-LINE PARAMETERS

The networks which have so far been discussed are called circuits of *lumped parameters*, wherein the resistance, inductance, and capacitance are individually concentrated or lumped at discrete points in the circuit, and can be identified definitely as representing a particular parameter. The electric line used for transmission of telephone messages or for the transmission of power is a common example of an electric circuit with *distributed parameters*. This term implies that the resistance, inductance, and capacitance are distributed along the circuit, each elemental length of the circuit having its own values, and concentration of the individual parameters is not possible.

The first few sections of this chapter will treat the inductance and capacitance of two special forms of line, namely, the open-wire and the coaxial line. The later sections will develop methods for calculation of the parameters of more general forms of multiconductor lines.

5-1. Line parameters

A common form of transmission line is known as the *open-wire line* because of its construction. The ordinary telephone line, strung on cross arms on poles, or the power transmission line on towers, are examples of the open line. The conductors of such lines may be considered parallel and separated by air dielectric.

Another form of line construction is the *cable*. For telephone use this consists of hundreds of individually paper-insulated conductors, twisted in pairs, and combined inside a protective lead or plastic sheath. Power transmission cables will employ only two or three large conductors, insulated with oil impregnated paper or other solid dielectric, inside the protective sheath. In either case the conductors may be considered again as parallel, with a solid dielectric.

A different form of construction is employed with the *coaxial* *line*, in which one conductor is a hollow tube, the second conductor being located inside and coaxial with the tube. The dielectric may be solid or gaseous, but if the cable is gas filled, occasional solid dielectric disks are employed to maintain accurate spacing and location of the central wire.

Knowledge of the values of electric circuit parameters associated with these forms of line is necessary to understanding and design. These parameters include *resistance,* which is uniformly distributed along the length of the conductors. Since current will be present, the conductors will be surrounded and linked by magnetic flux, and this phenomena will demonstrate its effect in distributed *inductance* along the line. The conductors are separated by insulating dielectric, so that *capacitance* will be distributed along the conductor length. This dielectric, or the insulators of the open-wire line, may not be perfect, and a leakage current will flow and *leakage conductance* will exist between the conductors.

These four parameters, all distributed along the line, are known by the symbols R, L, C, and G, where usually quantities *per unit length of line* are meant. Thus for resistance, both wires are included in the value of R for a unit of line.

Methods of calculating resistance will be discussed later. Calculations for inductance and capacitance will be presented here, starting from fundamental relations.

5-2. Inductance of a line of two parallel round conductors

Self-inductance was defined in Chapter 3 as

$$L = N\frac{d\phi}{di} \tag{5-1}$$

Permeability μ is defined as

$$\mu = \mu_r\mu_v \tag{5-2}$$

where μ_v is the magnetic permeability of space and has the value $4\pi \times 10^{-7}$ in MKS units, and μ_r is the relative permeability of the particular material.

In a material or region where μ_r is independent of flux density, as in air and most dielectrics,

$$\frac{d\phi}{di} = \frac{\phi}{i}$$

and

$$L = \frac{N\phi}{i} = \frac{\lambda}{i} \text{ henrys} \tag{5-3}$$

where $\lambda = N\phi$, or is the *flux linkage* of the current.

The open-wire telephone line, or the cable pair, may be considered as two parallel, round conductors immersed in air or a solid dielectric, and if the flux linkages $N\phi = \lambda$ can be calculated, Eq. 5-3 will compute the inductance. For this purpose a unit flux linkage is defined as a surrounding or linking of one weber of flux with the

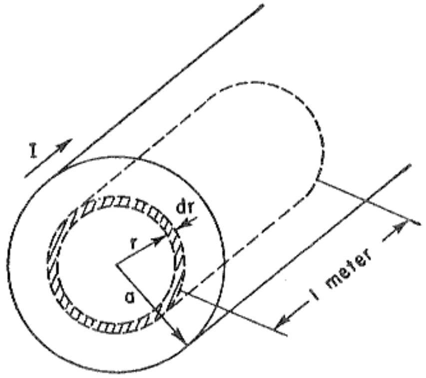

Fig. 5-1. A section of a long round wire with a hypothetical cylinder of length 1 meter and wall thickness dr shown.

entire conductor current. Thus one weber of flux linking one-half the current in the conductor produces one-half of a flux linkage, whereas one weber of flux entirely linking the same current in n turns of a conductor gives rise to n flux linkages. Inductance of a conductor in a region of constant permeability is then stated as flux linkages per ampere of current flowing in the circuit. This will also apply for conductors of magnetic material if μ is assumed constant or independent of flux density.

The flux paths surrounding a current in a long, straight, round wire are concentric circles, as in Fig. 5-1. The magnetic field intensity at any distance r from the center of the conductor is, by definition

$$H_r = \frac{NI_r}{l} \tag{5-4}$$

where I_r is the current enclosed by the flux path around which H is measured. In the figure, the length of the path at any distance r is the circumference of the dotted circle, or $l = 2\pi r$. The magnetic field intensity at any point inside or outside the conductor is

$$H_r = \frac{I_r}{2\pi r} \tag{5-5}$$

since only one turn or current path is linked.

For a flux path *inside* the conductor, as shown by the dashed circle at radius r in Fig. 5-1, the current enclosed is

$$I_r = \frac{\pi r^2}{\pi a^2} I \tag{5-6}$$

where I is the total conductor current, assumed to flow uniformly distributed over the whole cross section of the conductor. Then

$$H_r = \frac{rI}{2\pi a^2}$$

and since $B = \mu H$,

$$B_r = \frac{\mu r I}{2\pi a^2} \tag{5-7}$$

This is the flux density *at any point inside the conductor.* The conductor may be magnetic, that is μ_r may have a value other than unity, the only requirement on later use of this expression being that μ_r is independent of flux density.

The flux present in the wall of the hollow cylinder of elemental thickness dr, and 1 meter in length as shown in the figure, is

$$d\phi = B_r\, dA = \frac{\mu r I}{2\pi a^2}\, dr \tag{5-8}$$

This flux in the elemental cylinder wall links the fraction r^2/a^2 of the total current I, so that each weber of flux produces the fraction r^2/a^2 of a full flux linkage. Thus the linkages $d\lambda$ due to the flux passing in the wall of the cylinder of radius r, and elemental thickness dr, in length 1 meter, is

$$d\lambda = \frac{\mu r I}{2\pi a^2}\frac{r^2}{a^2}\, dr \tag{5-9}$$

The total internal linkages λ_{int} due to all the flux inside the conductor are obtained by integrating from the center of the conductor to the

radius a:

$$\lambda_{int} = \frac{\mu I}{2\pi a^4}\int_0^a r^3\, dr = \frac{\mu I}{8\pi} \text{ linkages/m} \tag{5-10}$$

where μ is that of the conductor material.

It is then necessary to determine the external flux linkages.

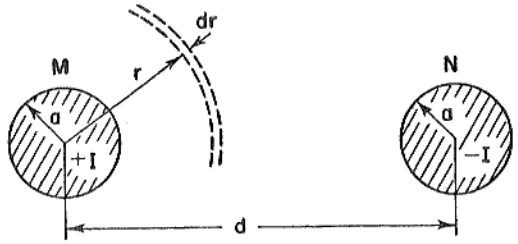

Fig. 5-2. Cross section of a parallel-wire line.

Referring to Fig. 5-2, the flux density in the *space outside* the conductor is

$$B_r = \mu_v H_r = \frac{\mu_v I}{2\pi r} \text{ webers/m}^2 \tag{5-11}$$

The flux present in the dashed hollow cylinder, drawn at radius r about conductor M with thickness dr and length 1 meter, is

$$d\phi = \frac{\mu_v I}{2\pi}\frac{dr}{r}$$

Flux passing in the space between the two conductors, and around the current in M, or having a radius between a and $d - a$, therefore contributes full linkages of the current in M, and these linkages are given by

$$\lambda_1 = \frac{\mu_v I}{2\pi}\int_a^{d-a}\frac{dr}{r} \tag{5-12}$$

Flux due to the current in M, that passes through wire N, or has a radius between $d - a$ and $d + a$, links values of current varying between $+I$ for a flux element having a radius just greater than $d - a$, and zero current for a flux element having a radius of $d + a$. An average value may be obtained by integrating to the center of wire N, thus counting as full linkages that flux passing through the left half of N, and as zero linkages the flux passing through the right half of N. The contribution by the flux cutting wire N is

$$\lambda_2 = \frac{\mu_v I}{2\pi} \int_{d-a}^{d} \frac{dr}{r} \tag{5-13}$$

Flux due to the current in M that has a radius greater than $d + a$, or that surrounds conductor N, links both $+I$ and $-I$ currents or zero net current, and contributes zero flux linkages.

The total *external* flux linkages about wire M are then

$$\lambda_{ext} = \lambda_1 + \lambda_2 = \frac{\mu_v I}{2\pi}\left[\int_a^{b-a} \frac{dr}{r} + \int_{d-a}^{d} \frac{dr}{r}\right]$$

$$\lambda_{ext} = \frac{\mu_v I}{2\pi}\int_a^d \frac{dr}{r} = \frac{\mu_v I}{2\pi} \ln \frac{d}{a} \text{ linkages/m}$$

The total inductance of both wires or of the circuit is then obtained from the definition:

$$L = \frac{\lambda}{I} = 2\left(\frac{\mu}{8\pi} + \frac{\mu_v}{2\pi} \ln \frac{d}{a}\right) \tag{5-14}$$

Rearranging and using the value of μ_v:

$$L = 10^{-7}\left(\frac{\mu}{\mu_v} + 4 \ln \frac{d}{a}\right) \text{ henrys/m} \tag{5-15}$$

$$= 10^{-7}\left(\mu_r + 9.210 \log_{10} \frac{d}{a}\right) \text{ henrys/m} \tag{5-16}$$

$$= 0.1609\mu_r + 1.482 \log_{10} \frac{d}{a} \text{ mh/mile} \tag{5-17}$$

The first term on the right of each of these four expressions is called the *internal inductance* of the line, since it is due to the internal flux linkages in the conductors. The second term on the right is the *external inductance* due to linkages with flux external to the wires. It should be emphasized that this relation was derived under the assumption of uniform current distribution over the conductors. This can be true only if $d \gg a$, or the wires are widely spaced with respect to the radius of a wire; otherwise the magnetic fields force a nonuniform distribution known as *proximity effect*. With alternating current, the current also crowds toward the surface, and at radio

frequencies the current flows so near the surface that the internal inductance becomes negligible.

5-3. Inductance of the coaxial line

Figure 5-3 represents a cross section of a coaxial line, in which there is a central conductor enclosed by an outer conducting sheath, the two separated by a dielectric. It is assumed that the current is uniformly distributed over the conductor cross section, or that the frequency is low.

A current of $+I$ amperes will be carried on the inner conductor and a current of $-I$ amperes on the outer conductor. A flux path surrounding the outer conductor encloses zero net current, and therefore there is no flux external to the outer conductor. The flux linkages of the inner and outer conductors may be added to obtain the total inductance.

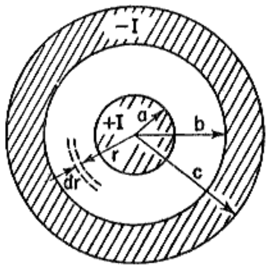

Fig. 5-3. Cross section of a coaxial line with flux paths at radius r.

The flux linkages internal to the central conductor are obtained from Eq. 5-10 as

$$\lambda_1 = \frac{\mu_1 I}{8\pi} \text{ linkages/m} \tag{5-18}$$

The linkages of the central conductor current $+I$, due to flux in the dielectric between conductors, is given by Eq. 5-11, as

$$\lambda_2 = \frac{\mu_v I}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_v I}{2\pi} \ln \frac{b}{a} \text{ linkages/m} \tag{5-19}$$

Flux surrounding the central conductor and passing within the outer conductor material is produced by a net current that varies from $+I$ at the inner surface to zero at the outer surface of the outer conductor. The current enclosed by flux at radius r, where $b < r < c$, is

$$I + \frac{\pi r^2 - \pi b^2}{\pi c^2 - \pi b^2}(-I) = I\frac{(c^2 - r^2)}{(c^2 - b^2)}$$

and the flux density B_r at radius r within the outer conductor material, due to the above current, is

$$B_r = \frac{\mu_3 I}{2\pi r}\frac{(c^2 - r^2)}{(c^2 - b^2)} \tag{5-20}$$

A flux line at the inner surface of the outer conductor at radius b makes a complete linkage of current I, whereas a flux line at the radius c links zero current. The fractional part of a linkage contributed by a flux line at radius r between b and c is

$$1 - \frac{\pi r^2 - \pi b^2}{\pi c^2 - \pi b^2} = \frac{c^2 - r^2}{c^2 - b^2}$$

The linkages due to the flux passing through the outer conductor, between b and c, are

$$\lambda_3 = \frac{\mu_3}{2\pi} \int_b^c \frac{(c^2 - r^2)}{(c^2 - b^2)} \frac{(c^2 - r^2)}{(c^2 - b^2)} I \frac{dr}{r}$$

$$= \frac{\mu_3 I}{2\pi(c^2 - b^2)^2} \left[c^4 \ln \frac{c}{b} - c^2(c^2 - b^2) + \frac{1}{4}(c^4 - b^4) \right] \text{linkages/m} \tag{5-21}$$

The inductance of the line may be found as the sum of λ_1, λ_2, and λ_3, divided by the current I. Ordinarily the materials of a line are nonmagnetic, so that $\mu_1 = \mu_3 = \mu_v$. The inductance of the coaxial line can then be found as

$$L = \frac{\lambda_1 + \lambda_2 + \lambda_3}{I}$$

$$L = \frac{\mu_v}{8\pi} \left\{ 1 + 4 \ln \frac{b}{a} + \frac{4}{(c^2 - b^2)^2} \left[c^4 \ln \frac{c}{b} - c^2(c^2 - b^2) + \frac{1}{4}(c^4 - b^4) \right] \right\}$$

Inserting the value of μ_v and rearranging gives

$$L = 10^{-7} \left[2 \ln \frac{b}{a} + \frac{2c^4 \ln \frac{c}{b}}{(c^2 - b^2)^2} - \frac{c^2}{c^2 - b^2} \right] \text{henrys/m} \tag{5-22}$$

as the value of the inductance of the coaxial line.

At high radio frequencies, the current crowds to the outer surface of the inner conductor and the inner surface of the outer conductor, as explained qualitatively in the next section. There is then no flux inside the inner or outer conductors, eliminating λ_1 and λ_3 from the above. All flux then exists in the dielectric between radii a and b, and only the linkages λ_2 contribute to the inductance, so

that

$$L = \frac{\mu_v}{2\pi} \ln \frac{b}{a} = 2 \times 10^{-7} \ln \frac{b}{a} \text{ henrys/m} \tag{5-23}$$

$$= 4.60 \times 10^{-7} \log_{10} \frac{b}{a} \quad \text{henrys/m} \tag{5-24}$$

$$= 0.741 \log_{10} \frac{b}{a} \quad \text{mh/mile} \tag{5-25}$$

In order to produce a flexible coaxial cable the outer conductor is frequently braided from small wires. The assumption of uniform current flow in the outer conductor is then upset, and because of the discontinuities contributed by the braiding, some flux will leak out and appear external to the cable. This may be undesirable in some applications where extreme shielding is desired. Doubly shielded cables are available to reduce this leakage of flux, if solid outer conductors cannot be used.

5-4. Qualitative discussion of skin effect

Consider a conductor made up of a large number of fine strands of wire, all strands carrying the same current. A strand at the center is linked by all the internal flux in the conductor, whereas a strand on the surface is not linked by the internal flux. The inductance and reactance of the strand at the center is greater than that of the strand at the surface. If the current is permitted to vary with time and all strands still carry the same current, then the voltage drop along the center strand will be greater than that along an outside strand. This effect, however, is a direct violation of Kirchhoff's law. Therefore the currents carried by the strands cannot be equal in magnitude, since the impedances are unequal. The interior strand thus carries less current than the outer so as to produce equal impedance drops along the strands. This phenomenon is known as *skin effect*. Though causing some deviation from uniform current distribution even at 60 cycles, it is so effective at radio frequencies that essentially the whole current flows in a layer a few thousandths of an inch thick on the conductor surface.

The decrease in effective conductor cross section at the higher frequencies increases the conductor resistance. The increase in resistance can be shown to be proportional to the square root of the frequency.

A more fundamental treatment of the subject is given in Chapter 8.

5-5. Capacitance of two parallel round conductors

In order to arrive at a means of calculation of the capacitance between the two conductors of a transmission line, it is desirable to start with consideration of the very long (infinite) round conductor of Fig. 5-4. This conductor is given an electric charge of q coulombs

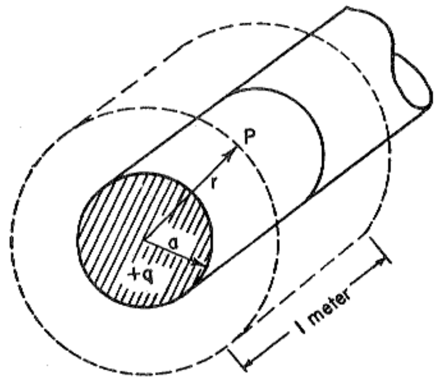

Fig. 5-4. A section of a long round conductor of radius a. Inscribed about a 1-meter length is a hypothetical cylinder of radius r.

per meter of length. According to Gauss's law, the electric flux passing outward from the wire is equal to q lines per meter of wire length. By reason of the specification that the wire be long, no *net* flux emerges from the two drumhead surfaces of any section of the wire. If a section of wire one meter in length be considered, the area of an enclosing cylinder of radius r meters is

$$A = 2\pi r \cdot 1 \quad (\text{m}^2)$$

The electric flux density D at P, a point on the enclosing cylinder, is

$$D = \frac{q}{2\pi r} \text{ coulombs/m}^2$$

TABLE 1

PROPERTIES OF DIELECTRICS

Material	Dielectric constant	Dielectric strength v/meter	Volume resistivity, ohm-meters
Vacuum	1.00000	....	....
Air	1.00059	3×10^6	....
Bakelite, pure resin	4.5	20×10^6	2×10^{14}
Celluloid	7	12×10^6	2×10^8
Ceresin wax	2.2	....	5×10^{16}
Fiber, vulcanized	2.5	7×10^6	2×10^8
Glass, Pyrex	4.8	13×10^6	1×10^{12}
Mica	7	90×10^6	2×10^{15}
Mycalex	8.0	14×10^6	5×10^{13}
Paper, kraft	3.5	3×10^6	....
Polyethylene	2.3	30×10^6	....
Polystyrene	2.5	20×10^6	....
Quartz, fused	4.2	80×10^6	5×10^{16}
Rubber, hard	3	17×10^6	1×10^{16}
Steatite	5.9	....	2.5×10^{12}
Sulphur	4	....	1×10^{15}

from which the electric field intensity at radius r is

$$\mathcal{E} = \frac{D}{\epsilon} = \frac{q}{2\pi r\epsilon} \text{ (v/m)} \tag{5-26}$$

The *permittivity* ϵ is defined as

$$\epsilon = \epsilon_r \epsilon_v \tag{5-27}$$

where ϵ_r is the *dielectric constant* or *relative permittivity*, and ϵ_v is the permittivity of space, which in MKS units is given the value $10^{-9}/36\pi = 8.85 \times 10^{-12}$. Properties of some common dielectrics are summarized in Table 1.

The potential at point P, in Fig. 5-4, with respect to the conductor, is then obtained as the negative of the integral of the field from a to r:

$$V_r = -\int_a^r \frac{q}{2\pi r\epsilon} dr = \frac{-q}{2\pi\epsilon} \ln \frac{r}{a} \text{ volts} \tag{5-28}$$

A parallel-wire transmission line is ordinarily constructed so that the spacing d between wires, Fig. 5-5, is large with respect to the radius a of a conductor. With that condition established it is permissible to consider the charge as uniformly distributed around

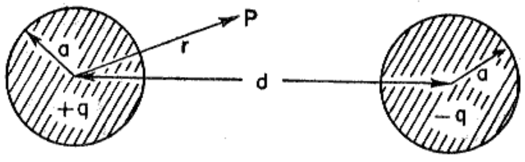

Fig. 5-5. Section of parallel-wire transmission line.

the periphery of each conductor; and then Eq. 5-28 obtained from the single isolated wire of Fig. 5-4, becomes valid for the parallel-wire case.

The potential difference between the two conductors of Fig. 5-5, when the left wire has a charge of $+q$ and the right wire a charge of $-q$ coulombs per unit length, and with the negatively charged wire as reference, is

$$V = \frac{q}{2\pi\epsilon} \ln \frac{d-a}{a} - \frac{(-q)}{2\pi\epsilon} \ln \frac{d-a}{a}$$

$$= \frac{q}{2\pi\epsilon} \ln \frac{d-a}{a} \text{ volts} \qquad (5\text{-}29)$$

The capacitance C is defined as the charge which may be held on the electrodes per volt of potential difference, so

$$C = \frac{q}{V} = \frac{q}{\dfrac{q}{\pi\epsilon} \ln \dfrac{d-a}{a}} = \frac{\pi\epsilon}{\ln \dfrac{d-a}{a}} \text{ farads/m}$$

It has already been assumed that d is large with respect to a so that the term $d - a$ in the logarithm may be reduced to d, giving

$$C = \frac{\pi\epsilon}{\ln \dfrac{d}{a}} \text{ farads/m} \qquad (5\text{-}30)$$

$$= \frac{12.07\epsilon_r}{\log_{10} \dfrac{d}{a}} \text{ farads/m} \qquad (5\text{-}31)$$

For the telephone line, capacitance per mile is more useful, and since 1 mile = 1609.4 meters,

$$C = \frac{1.943 \times 10^4 \epsilon_r}{\log_{10} \dfrac{d}{a}} \ \mu\mu\text{f/mile} \qquad (5\text{-}32)$$

If the wires are located such that d is not large with respect to a, then the charge is not uniformly distributed around the periphery of the wire, because of the attraction of unlike charges. A more detailed analysis in which this effect is considered gives, for the capacity of two round wires,

$$C = \frac{\pi\epsilon}{\cosh^{-1} \dfrac{d}{2a}} \text{ farads/m} \qquad (5\text{-}33)$$

Equations 5-30 and 5-33 give equivalent results within better than 2 per cent for values of d/a above 5.

5-6. Capacitance of the coaxial line

A cross section of a coaxial line is shown in Fig. 5-6. If the inner conductor is given a charge of $+q$ coulombs per meter of length, and the outer conductor an equal negative charge, the charges will distribute uniformly on the outer surface of the inner conductor and on the inner surface of the outer conductor. The field at point P will then be identical with that of the isolated round conductor of Fig. 5-4, because of symmetry and because no field contribution

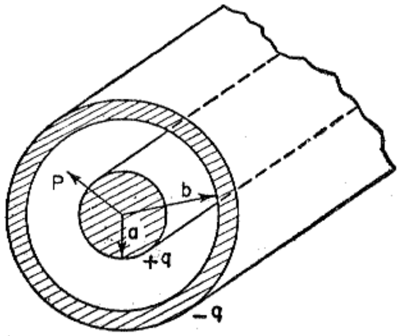

Fig. 5-6. Cross section of a coaxial transmission line.

comes from the outer charge. The field intensity at P is, from Eq. 5-26,

$$\mathcal{E} = \frac{q}{2\pi r\epsilon} \text{ (v/m)}$$

and the potential difference from outer to inner conductor, with the negative outer conductor as reference, is

$$V = -\int_b^a \frac{q}{2\pi r\epsilon}\,dr = \frac{q}{2\pi\epsilon}\ln\frac{b}{a} \text{ volts} \qquad (5\text{-}34)$$

The capacitance of the coaxial line then is

$$C = \frac{q}{V} = \frac{2\pi\epsilon}{\ln\dfrac{b}{a}} \text{ farads/m} \qquad (5\text{-}35)$$

$$= \frac{24.14\,\epsilon_r}{\log_{10}\dfrac{b}{a}} \ \mu\mu\text{f/m} \qquad (5\text{-}36)$$

$$= \frac{3.886 \times 10^4\epsilon_r}{\log_{10}\dfrac{b}{a}} \ \mu\mu\text{f/mile} \qquad (5\text{-}37)$$

The dielectric of the coaxial line may be a solid or a gas. In the use of air or nitrogen it is necessary to maintain the position of the central conductor with frequent solid dielectric spacers. These spacers raise the average value of the dielectric constant or permittivity above that of space. The effective value of the permittivity can be computed as

$$\epsilon_r' = 1 + (\epsilon_r - 1)\frac{t}{S} \qquad (5\text{-}38)$$

where ϵ_r is the relative permittivity of the dielectric spacers, t the thickness, and S the center-to-center distance of the spacers. This expression applies where S is small with respect to a wavelength.

5-7. Flux linkages in a system of multiple parallel conductors

For systems involving more than two conductors, a more general method than that employed in Section 5-2 is required for computation of flux linkages and inductance.

Assume that Fig. 5-7 shows a cross section of a group of N round conductors which constitute a complete circuit, that is, the net current perpendicular to the page is zero, or

$$I_A + I_B + I_C + \cdots + I_N = 0 \qquad (5\text{-}39)$$

The conductors of the system will be labeled from A to N, and these letters as subscripts will designate conductor radii and currents.

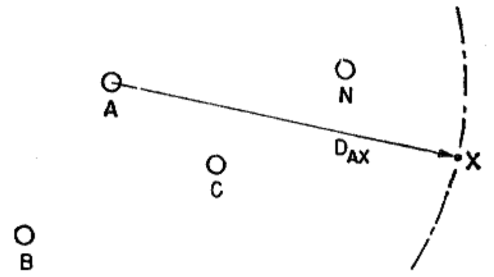

Fig. 5-7. Arbitrary arrangement of conductors $A, \ldots N$, showing point X.

The letter D with appropriate subscripts will indicate center-to-center distances for the various elements of the system.

Let wire A be the conductor under immediate consideration, and whose flux linkages are to be computed. Select a point in the plane of the page of Fig. 5-7, much more distant from A than any other wire of the group of conductors, and label this point X as in the figure. By use of Eqs. 5-10 and 5-12, with appropriate limits, the total linkages about wire A due to its own current I_A, produced by flux surrounding A and having a radius less than the distance D_{AX} is

$$\lambda = I_A\left(\frac{\mu}{8\pi} + \frac{\mu_v}{2\pi}\int_r^{D_{AX}}\frac{dr}{r}\right)$$

and after using the value of μ_v,

$$\lambda = 10^{-7}I_A\left(\frac{\mu}{2\mu_v} + 2\ln\frac{D_{AX}}{r_A}\right) \text{ linkages/m} \qquad (5\text{-}40)$$

Now consider the flux due to a current I_J in any other conductor J. Flux produced by this current and which links A, but does not have a radius greater than D_{JX}, (X chosen so $D_{JX} \gg D_{JA}$), is given by

$$\lambda = \frac{\mu_v}{2\pi}\int_{D_{JA}}^{D_{JX}} \frac{dr}{r} = 2 \times 10^{-7} \ln \frac{D_{JX}}{D_{JA}}$$

All the other conductors likewise produce similar linkage terms, so that the total linkage around conductor A, due to flux not having a radius greater than D_{AX} is

$$\lambda_A = 10^{-7}\left(\frac{I_A}{2}\frac{\mu}{\mu_v} + 2I_A \ln \frac{D_{AX}}{r_A} + 2I_B \ln \frac{D_{BX}}{D_{AB}} + 2I_C \ln \frac{D_{CX}}{D_{AC}} + \cdots + 2I_N \ln \frac{D_{NX}}{D_{AN}}\right) \text{ linkages/m} \quad (5\text{-}41)$$

Because of Eq. 5-39 it is possible to write

$$I_N = -I_A - I_B - I_C - \cdots - I_{N-1}$$

This expression may be used to replace the last term of Eq. 5-41, giving a number of negative logarithmic terms. These may be combined with their companion positive terms as ratios, so that

$$\lambda_A = 10^{-7}\left(\frac{I_A}{2}\frac{\mu}{\mu_v} + 2I_A \ln \frac{D_{AX}}{r_A}\frac{D_{AN}}{D_{NX}} + 2I_B \ln \frac{D_{BX}}{D_{AB}}\frac{D_{AN}}{D_{NX}} + 2I_C \ln \frac{D_{CX}}{D_{AC}}\frac{D_{AN}}{D_{NX}} + \cdots + 2I_{N-1} \ln \frac{D_{(N-1)X}}{D_{A(N-1)}}\frac{D_{AN}}{D_{NX}}\right) \quad (5\text{-}42)$$

If the distances from all the wires to point X be made very large by allowing X to approach infinity, all the ratios such as D_{AX}/D_{NX}, involving distances to X, will approach unity as a limit. In this manner all flux from each wire to infinity is considered in calculating the linkages. Because of the way point X was chosen, and since the result is finite and independent of any distance to the point X, it is proved that flux which links all the conductors, or zero net current, does not contribute to the linkages or to the inductance.

Then rewriting Eq. 5-42 as

$$\lambda_A = 10^{-7}\left(\frac{I_A}{2}\frac{\mu}{\mu_v} + 2I_A \ln D_{AN} - 2I_A \ln r_A + 2I_B \ln D_{AN} - 2I_B \ln D_{AB} + \cdots + 2I_{N-1} \ln D_{AN} - 2I_{N-1} \ln D_{A(N-1)}\right)$$

The terms involving $\ln D_{AN}$ may be collected and Eq. 5-39 may again be applied, finally leading to a single term replacement as $-2I_N \ln D_{AN}$.

If *nonmagnetic conductors are assumed*, then $\mu/\mu_v = 1$, and the linkages about wire A can be written

$$\lambda_A = 2 \times 10^{-7}\left(\frac{I_A}{4} + I_A \ln \frac{1}{r_A} + I_B \ln \frac{1}{D_{AB}} + I_C \ln \frac{1}{D_{AC}} + \cdots + I_N \ln \frac{1}{D_{AN}}\right) \quad (5\text{-}43)$$

The first two terms inside the parentheses involving I_A, may be combined as

$$I_A\left(\frac{1}{4} + \ln \frac{1}{r_A}\right) = I_A\left(\ln \frac{\epsilon^{1/4}}{r_A}\right) = I_A \ln \frac{1}{r_A\epsilon^{-1/4}}$$

so that

$$\lambda_A = 2 \times 10^{-7}\left(I_A \ln \frac{1}{r_A\epsilon^{-1/4}} + I_B \ln \frac{1}{D_{AB}} + I_C \ln \frac{1}{D_{AC}} + \cdots + I_N \ln \frac{1}{D_{AN}}\right) \quad (5\text{-}44)$$

The linkages about each of the other wires of the group could then be written by inspection, through following the same pattern with respect to subscripts. The inductance of each of the conductors of the group could then be calculated by use of the known currents and the relation $L = \lambda/I$.

Now consider the two groups of conductors $a \ldots n$, and $1 \ldots N$, of Fig. 5-8. It will be assumed that these conductors in

Fig. 5-8. Cross section of parallel conductors.

each group share the current equally, each conductor of the group $a \ldots n$ carrying $+I/n$ amperes, and each conductor of the group $1 \ldots N$ carrying $-I/N$ amperes, where n does not necessarily equal N.

Following the pattern of Eq. 5-44 it is possible to write the flux linkages about any conductor, a for example, as:

$$\lambda_a = 2 \times 10^{-7} \frac{I}{n} \left(\ln \frac{1}{r_a \epsilon^{-1/4}} + \ln \frac{1}{D_{ab}} + \ln \frac{1}{D_{ac}} + \cdots + \ln \frac{1}{D_{an}} \right)$$
$$- 2 \times 10^{-7} \frac{I}{N} \left(\ln \frac{1}{D_{a1}} + \ln \frac{1}{D_{a2}} + \ln \frac{1}{D_{a3}} + \cdots + \ln \frac{1}{D_{aN}} \right) \tag{5-45}$$

This is the linkage about one conductor carrying $1/n$ of the current $+I$, and therefore each such linkage represents $1/n$ of a linkage about the total current I, or about the group $a \ldots n$. Similar expressions may be written for the linkages about any conductor of the group $a \ldots n$. The sum of these linkages, when divided by n to weight properly the fractional linkages, will give the *total* conductor linkages about current $+I$, from which the inductance of the $a \ldots n$ system may be determined. This inductance, when multiplied by dI/dt, will give the inductive voltage drop in the $a \ldots n$ group of conductors.

A similar method would lead to the inductance of the $1 \ldots N$ return group of conductors. In the case of polyphase circuits each of the other phase conductors may be considered as return circuits, with appropriate current values.

The total linkages about the conductors carrying the total current $+I$ amperes is

$$\lambda = 2 \times 10^{-7} \frac{I}{n^2} \left(\ln \frac{1}{r_a \epsilon^{-1/4}} + \ln \frac{1}{D_{ab}} + \ln \frac{1}{D_{ac}} + \cdots + \ln \frac{1}{D_{an}} \right.$$
$$+ \ln \frac{1}{D_{ba}} + \ln \frac{1}{r_b \epsilon^{-1/4}} + \ln \frac{1}{D_{bc}} + \cdots + \ln \frac{1}{D_{bn}}$$
$$+ \ln \frac{1}{D_{ca}} + \ln \frac{1}{D_{cb}} + \ln \frac{1}{r_c \epsilon^{-1/4}} + \ln \frac{1}{D_{cd}} + \cdots + \ln \frac{1}{D_{cn}}$$
$$\left. + \cdots + \ln \frac{1}{D_{na}} + \ln \frac{1}{D_{nb}} + \cdots + \ln \frac{1}{r_n \epsilon^{-1/4}} \right)$$
$$- 2 \times 10^{-7} \frac{I}{nN} \left(\ln \frac{1}{D_{a1}} + \ln \frac{1}{D_{a2}} + \ln \frac{1}{D_{a3}} + \cdots + \ln \frac{1}{D_{aN}} \right.$$

$$+ \ln \frac{1}{D_{b1}} + \ln \frac{1}{D_{b2}} + \ln \frac{1}{D_{b3}} + \cdots + \ln \frac{1}{D_{bN}}$$
$$+ \ln \frac{1}{D_{c1}} + \ln \frac{1}{D_{c2}} + \ln \frac{1}{D_{c3}} + \cdots + \ln \frac{1}{D_{cN}}$$
$$\left. + \cdots + \ln \frac{1}{D_{n1}} + \ln \frac{1}{D_{n2}} + \ln \frac{1}{D_{n3}} + \cdots + \ln \frac{1}{D_{nN}} \right) \tag{5-46}$$

It should be noted that the first parenthesis involves distances from conductors in the $a \ldots n$ group to other conductors in the same group, and these might be referred to as *self* distances. The second parenthesis involves distances from all the conductors in the $a \ldots n$ group to all the conductors in the $1 \ldots N$ group.

By combining the various logarithms, it is possible to write

$$\lambda = 2 \times 10^{-7} I \left[\ln \frac{\sqrt[nN]{D_{a1} D_{a2} \cdots D_{aN} D_{b1} D_{b2} \cdots D_{bN} \cdots D_{n1} D_{n2} \cdots D_{nN}}}{\sqrt[n^2]{r_a \epsilon^{-1/4} D_{ab} D_{ac} \cdots D_{an} D_{ba} r_b \epsilon^{-1/4} D_{bc} \cdots D_{bn} D_{ca} D_{cb} r_c \epsilon^{-1/4} \cdots D_{cn} \cdots r_n \epsilon^{-1/4}}} \right] \tag{5-47}$$

There are nN terms under the numerator radical, for which the nNth root is to be taken, so that a geometric mean is involved. This is the geometric mean of all the distances between the conductors or elements of the system carrying the total current $+I$, and all the return current conductors. The numerator is called the *geometric mean distance* of one conductor set to all other conductors, and is abbreviated GMD and given the symbol D_M. This is a mathematically derived concept, which has no relation to the electrical problem at hand except as a useful tool.

It is then possible to write

$$\lambda = 2 \times 10^{-7} I \ln \frac{D_M}{\sqrt[n^2]{r_a \epsilon^{-1/4} \cdots r_n \epsilon^{-1/4} D_{ab} \cdots D_{an} D_{ba} \cdots D_{bn} D_{ca} \cdots D_{cn} \cdots D_{na} \cdots D_{n(n-1)}}} \tag{5-48}$$

There are n^2 terms under the denominator radical, for which the n^2 root is to be taken, so that this is again a geometric mean.

This may be further explained. Mathematically, the geometric mean of the distances between all possible pairs of points in an area is called the *geometric mean radius* of the area, abbreviated GMR. For a circular area of radius r the geometric mean radius can be shown equal to $r\epsilon^{-1/4}$. Thus it is recognized that the radical above contains terms having the GMR of each of the $a \ldots n$ conductors, times the $n(n-1)$ distances of each wire of the group to every other wire of the group. The denominator radical may be considered as the *self-geometric mean distance* among the elements of the conductor group carrying current $+I$, and is symbolized by D_S.

A simplified expression for the inductance of a conductor group carrying a total current I, with the return current $-I$ in one or more other conductors, may be written as

$$L = 2 \times 10^{-7} \ln \frac{D_M}{D_S} \text{ henrys/m} \qquad (5\text{-}49)$$

where D_M and D_S have the meanings given in Eqs. 5-47 and 5-48.

It should be noted that the denominator of Eq. 5-49 relates to a conductor and to its size and shape, using the word conductor in a general sense as a group of wires carrying equal portions of a common current. The numerator is determined solely by the distances between the various wires of the several conductors, again using the latter term in a general sense.

The method may be extended to lines having individual conductors of irregular cross section as in Fig. 5-9. Each conductor

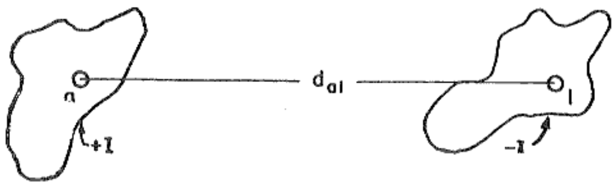

Fig. 5-9. Parallel conductors of irregular cross section.

may be divided into many elementary areas, each area carrying the same fractional current $+I/n$ or $-I/N$. To compute the inductance of the left conductor of the figure, its elementary filaments may be considered as identical to the group of wires $a \ldots n$ of Fig. 5-8. Likewise the elementary filaments of the right conductor are identical to the conductor group $1 \ldots N$ of Fig. 5-8. An expression for the flux linkages about the left conductor of Fig. 5-9 can be written in accordance with Eq. 5-47. The number of ele-

mentary conducting areas or filaments in each conductor may then be allowed to increase without limit. The radical of the numerator then approaches the GMD of the two conductor areas as a limit, and the denominator radical approaches the GMR of the conductor whose inductance is being calculated. Equation 5-49 will then apply to the case of the irregular conductors.

As an example, the two wire line of Fig. 5-2 may be used. The GMD of the two conductors is obviously the center-to-center distance $d = D_M$. The GMR of a round conductor of radius a is $a\epsilon^{-1/4}$. Therefore for *both* wires

$$L = 2 \times 2 \times 10^{-7} \ln \frac{D_M}{D_S}$$

$$= 4 \times 10^{-7} \ln \frac{d}{a\epsilon^{-1/4}} \text{ henrys/m} \qquad (5\text{-}50)$$

However, Eq. 5-15 gave

$$L = 10^{-7} \left(\frac{\mu}{\mu_v} + 4 \ln \frac{d}{a} \right) \text{ henrys/m}$$

This may be rewritten for nonmagnetic conductors as

$$L = 4 \times 10^{-7} \left(\frac{1}{4} + \ln \frac{d}{a} \right)$$

and by combining the constant $\frac{1}{4}$ with the logarithm as $\ln \epsilon^{1/4}$ this becomes

$$L = 4 \times 10^{-7} \ln \frac{d}{a\epsilon^{-1/4}} \text{ henrys/m} \qquad (5\text{-}51)$$

which is identical with the result by GMD methods in Eq. 5-50. Further examples for multiconductor circuits will follow.

5-8. GMR and GMD of various conductor arrangements

It has been stated that the GMR of a circular *area* of radius r is given by $r\epsilon^{-1/4} = 0.7788r$. It may be reasoned that the GMR of a circular *line* is equal to the radius of the circle. Another useful theorem, derived purely on a mathematical basis, is that the GMD between two nonoverlapping circular areas of any size is equal to the distance between their centers, this applying to two wires.

For computation with rectangular bus bars, the GMR of a rectangular area of sides a and b is closely equal to

$$D_S = 0.2235(a + b) \tag{5-52}$$

Other theorems are available in Reference 1.

Stranded cable is most frequently used in the large conductor sizes for reasons of flexibility, and the GMR value is not quite the same as that of a solid conductor of equivalent radius. Three-wire cables are used, but other common types start with a central strand surrounded by a spiraled layer made of six more strands as in Fig.

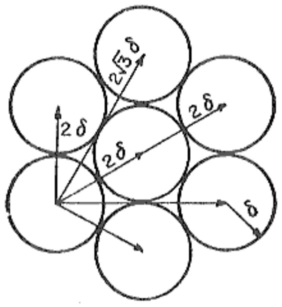

Fig. 5-10. Seven-strand cable.

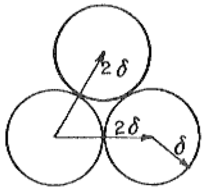

Fig. 5-11. Three-strand cable.

5-10. The next larger size is made by adding another oppositely spiraled layer which will require 12 more strands, the next layer would require 18 to fill the annular space, and so on increasing in each layer by multiples of six. The total number of strands in such cables is then 7, 19, 37, 61, 91, and so on.

Successive layers are spiraled oppositely to prevent untwisting, and also to prevent one layer from settling into the interstices of the layer below. It is desirable to illustrate the GMR calculations for several cables:

1. *Three-strand cable* of Fig. 5-11. If the radius of an individual strand is δ, then the area of each strand is $4\delta^2$, and the cable cross section $A = 3 \cdot 4\delta^2 = 12\delta^2$ *in circular measure.* The GMR value may be found by application of Eq. 5-48. First write the GMR value for a strand as $\delta\epsilon^{-1/4}$, and repeat this three times, once for each strand, giving $(\delta\epsilon^{-1/4})^3$. The distance from the center of one strand to another is 2δ, and this is repeated six times, since there are six such distances, giving $(2\delta)^6$. Collecting these nine terms

gives

$$D_S = \sqrt[9]{(\delta\epsilon^{-1/4})^3(2\delta)^6} = \delta\sqrt[9]{(\epsilon^{-1/4})^3 2^6}$$
$$= 1.46\delta$$

Since the radius R_c of the cable is equal to 2.15δ, then

$$D_S = 0.677R_c$$

2. *Seven-strand cable* of Fig. 5-10. The area A of such a cable is $A = 7 \cdot 4\delta^2 = 28\delta^2$ in circular measure. To find D_S, first write the GMR value for each strand as $\delta\epsilon^{-1/4}$, and repeat this seven times for the seven strands, giving $(\delta\epsilon^{-1/4})^7$. The distance from the center strand to the six outer strands is 2δ, and this will be repeated six times, or $(2\delta)^6$. The distances from one outside strand to the six other strands are shown in the figure, and their product is

$$2\delta \cdot 2\delta \cdot 2\delta \cdot 2\sqrt{3}\,\delta \cdot 2\sqrt{3}\,\delta \cdot 4\delta = 2^7 \cdot 3 \cdot \delta^6$$

and this quantity will be repeated six times, giving $(2^7 \cdot 3 \cdot \delta^6)^6$.

Collecting these 49 distances gives

$$D_S = \sqrt[49]{(\delta\epsilon^{-1/4})^7(2\delta)^6(2^7 \cdot 3 \cdot \delta^6)^6}$$
$$= \delta\sqrt[49]{(\epsilon^{-1/4})^7 2^{48} 3^6} = 2.18\delta$$

in terms of the radius of one strand. Since the radius R_c of the cable is 3δ, then

$$D_S = 0.726R_c$$

for the seven-strand cable.

Similar methods can be followed for cables with larger numbers of strands, leading to Table 2.

Table 2

GMR Values for Stranded Cable

Strands	D_S	Strands	D_S
3	$0.677R_c$	61	$0.772R_c$
7	$0.726R_c$	91	$0.774R_c$
19	$0.758R_c$	solid	$0.778R_c$
37	$0.768R_c$		

Some conductors are essentially tubular in shape, the long line from Hoover Dam to Los Angeles being of this form. This is desirable to provide a large outside diameter to reduce the external electric field and corona loss at the high voltage of operation. The calculation of the D_S value for a tube is complicated, but Table 3 supplies values for various ratios of outside radius c and inside radius b.

TABLE 3

GMR VALUES FOR TUBULAR CONDUCTORS

Ratio b/c	D_S	Ratio b/c	D_S
0.0 (Solid wire)	0.7788	0.6	0.879
0.1	0.781	0.7	0.908
0.2	0.791	0.8	0.938
0.3	0.806	0.9	0.967
0.4	0.826	1.0	1.0
0.5	0.850		

Aluminum is frequently employed in transmission lines as conductor material. When so used, it is customary for the central strand, or the central strand and one or more of the first layers, to be made of steel for increased mechanical strength. Such a cable is known as *aluminum cable steel reinforced* or ACSR. Because of the disparity in conductivities between aluminum and steel, a reasonable assumption is to consider that the aluminum strands carry all the current, and to compute D_S as if the steel were absent. Accurate calculations which include the effect of the steel are difficult because of the necessity for knowledge of the permeability of the steel strands. For accurate work it is best to consult tables of inductance values prepared from experimental data by the cable manufacturers.

5-9. Inductance of a symmetrical three-phase line

The simplest three-phase transmission line arrangement is the equilateral spacing of Fig. 5-12. The conductors are indicated as a, b, c, and have a center-to-center spacing D. If the currents are designated I_a, I_b, I_c, then by Eq. 5-45 the linkages about phase a are

$$\lambda_a = 2 \times 10^{-7}\left(I_a \ln \frac{1}{D_S} + I_b \ln \frac{1}{D} + I_c \ln \frac{1}{D}\right) \tag{5-53}$$

since $D_{a1} = D_{a2} = D$.

Now in a balanced three-phase system the currents are related as

$$I_b = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)I_a \qquad I_c = \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)I_a$$

so that Eq. 5-53 becomes

$$\lambda_a = 2 \times 10^{-7} I_a \left(\ln \frac{1}{D_S} - \ln \frac{1}{D}\right)$$

$$= 2 \times 10^{-7} I_a \ln \frac{D}{D_S} \text{ linkages/m} \tag{5-54}$$

Dividing by I_a gives the inductance of phase a as

$$L_a = 2 \times 10^{-7} \ln \frac{D}{D_S} \text{ henrys/m} \tag{5-55}$$

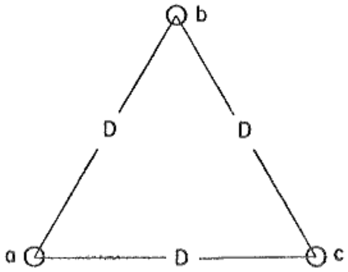

Fig. 5-12. Equilateral spacing of a three-phase line.

where D_S is the GMR of conductor a. It can be noted that the GMD between a and conductors b and c might have been written as $\sqrt[4]{D^4} = D$, and thus use of Eq. 5-49 would have lead directly to the above result.

When used in a series reactance term as ωL_a, and multiplied by the effective line current, the series voltage drop in the phase would be obtained. When this voltage drop is subtracted from the sending end line-to-neutral voltage the result is the receiving end line-to-neutral voltage.

Because of the symmetry of the conductor arrangement of Fig. 5-12, the same result would be obtained for all phases, and balanced operation would result.

5-10. The unsymmetrical three-phase line; transposition

Unsymmetrical conductor arrangements are most commonly employed for three-phase transmission, since they allow cheaper and more convenient tower and line construction. Several arrangements appear in Fig. 5-13.

Use of the linkage relations of Eq. 5-45 for either arrangement of the figure, leads to a linkage for phase a of

$$\lambda_a = 2 \times 10^{-7}\left(I_a \ln \frac{1}{D_S} + I_b \ln \frac{1}{D_{ab}} + I_c \ln \frac{1}{D_{ac}}\right)$$

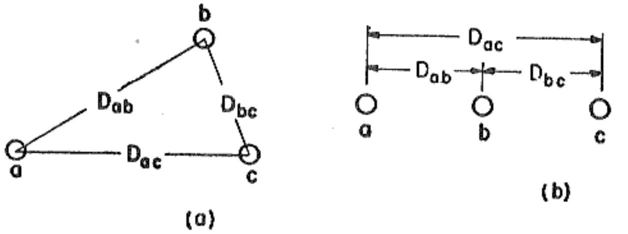

Fig. 5-13. Two unsymmetrical three-phase line arrangements.

and under *assumed balanced current* conditions

$$\lambda_a = 2 \times 10^{-7} I_a\left(\ln \frac{1}{D_S} - \frac{1}{2}\ln \frac{1}{D_{ab}} - \frac{1}{2}\ln \frac{1}{D_{ac}}\right)$$

$$= 2 \times 10^{-7} I_a \ln \frac{\sqrt{D_{ab}D_{ac}}}{D_S} \text{ linkages/m} \tag{5-56}$$

A similar expression could be obtained for phase b as

$$\lambda_b = 2 \times 10^{-7} I_b\left(\ln \frac{1}{D_S} - \frac{1}{2}\ln \frac{1}{D_{ba}} - \frac{1}{2}\ln \frac{1}{D_{bc}}\right)$$

$$= 2 \times 10^{-7} I_b \ln \frac{\sqrt{D_{ba}D_{bc}}}{D_S} \text{ linkages/m} \tag{5-57}$$

and for phase c as

$$\lambda_c = 2 \times 10^{-7} I_c \ln \frac{\sqrt{D_{ca}D_{cb}}}{D_S} \text{ linkages/m} \tag{5-58}$$

It is apparent that these three linkages will differ, due to the various conductor distances involved. But if the linkages differ, the inductances will be unequal, unequal voltage drops will appear in the three lines, and this implies unbalanced operation with probable unbalanced currents, contrary to the assumption above.

While there have been constructed some high-voltage lines in which this condition is permitted because of the cost of eliminating it, it is more customary to correct the unbalance by *transposing*

the line wires, so that over the full length of a line each phase will occupy each conductor position for an equal distance. A complete transposition cycle is illustrated in Fig. 5-14. The three tower or cross arm positions are indicated by 1, 2, 3, and the three conductors by a, b, c. Each wire or phase occupies each position for one-third of the cycle length, and the conductors rotate in position at the one-third and two-third points in the cycle. Such a transposition

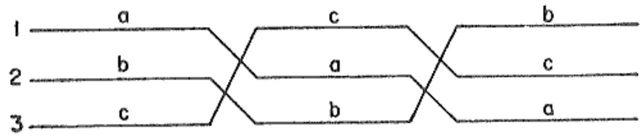

Fig. 5-14. Transposition diagram for a three-phase line.

is dictated not only to achieve balance of the phases, but also to reduce inductive interference with nearby telephone lines. This latter point even dictates that symmetrical three-phase lines should be transposed, as well.

The linkages per meter about conductor a may be computed for each of the three sections or positions, totaled, and divided by 3 to get the average value over the transposition cycle. Then

$$\lambda_a = 2 \times 10^{-7}\frac{I_a}{3}\left(\ln \frac{\sqrt{D_{ab}D_{ac}}}{D_S} + \ln \frac{\sqrt{D_{ba}D_{bc}}}{D_S} + \ln \frac{\sqrt{D_{ca}D_{cb}}}{D_S}\right)$$

$$= 2 \times 10^{-7} I_a \ln \frac{\sqrt[3]{D_{ab}D_{bc}D_{ac}}}{D_S} \text{ linkages/m} \tag{5-59}$$

Computations of average λ_b and λ_c would contain the same logarithm, proving the balance obtained by the method of transposition.

The average inductance per meter of any phase, with transpositions, will then be

$$L = 2 \times 10^{-7} \ln \frac{\sqrt[3]{D_{ab}D_{bc}D_{ac}}}{D_S} \text{ henrys/m} \tag{5-60}$$

The numerator $\sqrt[3]{D_{ab}D_{bc}D_{ac}}$ is the GMD among the three conductors. The factor D_S is, of course, computed for the particular cable used by the methods of Section 5-8. Thus the GMD method of Section 5-7 may be applied and will lead more quickly to the correct result. It is, as was stated in its derivation, a general method of solution, and will be further demonstrated in the next section.

5-11. Multicircuit lines

Occasionally single-phase or polyphase lines are built with multiple wires, connected in parallel. The four-wire line is an example at radio frequencies. The calculation of inductance may be readily carried out by the general methods of Section 5-7.

Consideration of Fig. 5-15, for the four-wire or double-circuit single-phase case, shows it to be similar to the common two-wire line, but each *conductor* of the simple case is now composed of two *wires*, a and a', b and b'. It is assumed that all four conductors are of the same radius r, and material. The symmetry of the arrangement indicates that transpositions are not required for balance, however in many applications they are used in any case, to reduce induced voltages in nearby telephone or other circuits, or to achieve balance of the circuits to ground or other conductors which happen to run nearby.

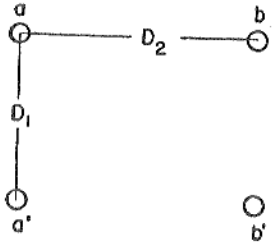

Fig. 5-15. Double-circuit or four-wire single-phase line; a and a' in parallel, b and b' in parallel.

Using the general GMD relation of Eq. 5-49 as

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_S} \text{ henrys/m}$$

will give the inductance of a conductor (the pair of wires a and a', or b and b') which carries the total current of one line $I = I_a + I_{a'}$. Computing D_S for the pair of wires a and a',

$$\begin{aligned} D_S &= \sqrt[4]{r\epsilon^{-1/4} r\epsilon^{-1/4} D_1 D_1} \\ &= \sqrt[2]{0.7788 r D_1} \end{aligned} \tag{5-61}$$

showing the use of the self-GMR of the two round conductors and the distances D_1 from a to a' and from a' to a.

Computing the GMD of the one conductor (a,a') to the other (b,b') by use of all the four distances gives

$$D_M = \sqrt[8]{(D_2 \sqrt{D_2^2 + D_1^2})^4} = \sqrt{D_2 \sqrt{D_2^2 + D_1^2}} \tag{5-62}$$

The complete expression for the inductance of *both conductors*, or

of the line of Fig. 5-15 is then

$$L = 4 \times 10^{-7} \ln \frac{\sqrt{D_2 \sqrt{D_2^2 + D_1^2}}}{\sqrt{0.7788 r D_1}} \tag{5-63}$$

where r is the radius of a wire.

Frequently one polyphase line with a vertical arrangement of

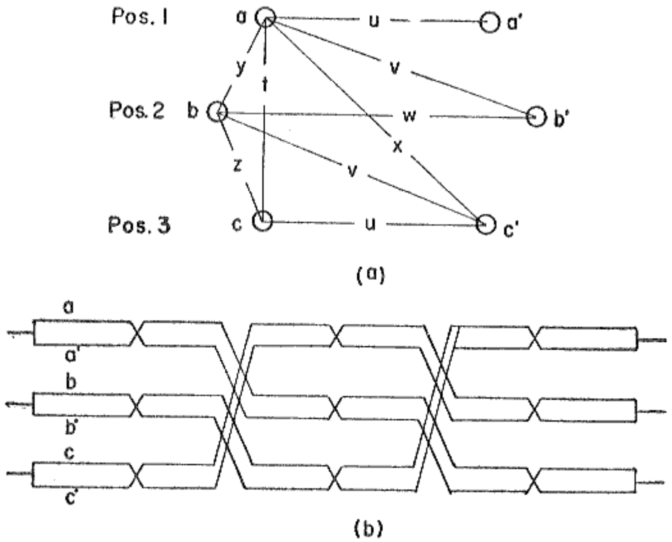

Fig. 5-16. (a) Double-circuit three-phase line; (b) transposition cycle for (a).

conductors will be built on one side of a tower, and at a later date when the electric load has increased, a second three-phase circuit will be added on the other side of the tower, giving a common form of double-circuit three-phase line shown in Fig. 5-16.

The conductors are assumed to be alike, with spacings as given in the figure, and transposed. Then using the GMD relations of Eq. 5-49, the value of D_S for a phase (of two wires) in any position over the transposition cycle, is

$$\begin{aligned} D_S &= \sqrt[12]{(r\epsilon^{-1/4} r\epsilon^{-1/4} u^2)(r\epsilon^{-1/4} r\epsilon^{-1/4} w^2)(r\epsilon^{-1/4} r\epsilon^{-1/4} u^2)} \\ &= \sqrt[6]{(r\epsilon^{-1/4})^3 u^2 w} = \sqrt[6]{(0.7788r)^3 u^2 w} \end{aligned} \tag{5-64}$$

The value of D_M for phase a (wires a,a') when in position 1 involves the distance from wires a and a' to all the other wires, as

$$D_{M1} = \sqrt[8]{(vxyt)^2}$$

When phase a is in position 2:

$$D_{M2} = \sqrt[8]{(v^2yz)^2}$$

and when in position 3:

$$D_{M3} = \sqrt[8]{(vxzt)^2}$$

The value of D_M, taken as a mean over all three portions of the transposition cycle, is then

$$D_M = \sqrt[24]{(vxyt)^2(v^2yz)^2(vxzt)^2} = \sqrt[6]{v^2xyzt} \tag{5-65}$$

The inductance, over a transposition cycle, is

$$L = 2 \times 10^{-7} \ln \frac{D_M}{D_S} = 2 \times 10^{-7} \ln \frac{\sqrt[6]{v^2xyzt}}{\sqrt[6]{(0.7788r)^3u^2w}} \tag{5-66}$$

henrys per meter for a phase of a double-circuit three-phase line of the construction of Fig. 5-16.

It is desirable to minimize the inductance of a given line. To do this would require that D_S be large and D_M small. Thus the dimensions u and w or the spacing between circuits should be large, and y and z or the spacing between phases should be as small as electrical clearances under icing and wind conditions permit.

5-12. Potentials in a system of conductors

The potential between any point x and a further point y, in the electric field of a charged wire a in space, carrying a charge of q_a coulombs per meter, can be written by the method of Eq. 5-28, as

$$V = -\int_x^y \frac{q_a}{2\pi r\epsilon}\,dr = \frac{q_a}{2\pi\epsilon} \ln \frac{y}{x} \text{ volts} \tag{5-67}$$

Now consider an arbitrary grouping of isolated conductors as was shown in Fig. 5-7. If arbitrary charges $q_A, q_B, q_C, \ldots, q_N$ are placed on these isolated conductors, each will assume a potential due to its own charge and to the fields from the other conductors.

The potential difference between two conductors will be due to the charge on the first conductor, plus that due to the charge on the second conductor, plus that due to any other charged conductors in the region. If the charges are known, it is possible to find the potential and the capacitance of each conductor since $C = q/V$.

The potential difference between any two conductors, A and B for example, is then obtained by applying Eq. 5-67, so that

$$E_{AB} = \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AB}}{r_A} + q_B \ln \frac{r_B}{D_{BA}} + q_C \ln \frac{D_{CB}}{D_{CA}} + \cdots + q_N \ln \frac{D_{NB}}{D_{NA}}\right)$$

and similar expressions could be written for the potentials of each of the other conductors, all with A as reference, as

$$E_{AC} = \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AC}}{r_A} + q_B \ln \frac{D_{BC}}{D_{BA}} + q_C \ln \frac{r_C}{D_{CA}} + \cdots + q_N \ln \frac{D_{NC}}{D_{NA}}\right)$$

. . . .

. . . .

$$E_{AN} = \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AN}}{r_A} + q_B \ln \frac{D_{BN}}{D_{BA}} + \cdots + q_N \ln \frac{r_N}{D_{NA}}\right) \tag{5-68}$$

This method assumes the charges will be known, whereas in the operation of electric systems it is the voltages which are known and not the charges. Considering the q as unknowns in the equations above, it can be seen that there are N unknowns, but only $N - 1$ equations are available for solution. To overcome this difficulty it may be assumed that the sum of all the charges is zero. That is

$$q_A + q_B + q_C + \cdots + q_N = 0 \tag{5-69}$$

and this is true for any ordinary line under normal operating conditions. Solution for the charges with known potentials is then possible, from which the capacitance value will follow.

Several special cases will be considered.

5-13. Capacitance of the symmetrical three-phase line

Assuming a balanced set of voltages applied to the equilateral three-phase line of Fig. 5-17, with each wire of radius r, equations of the form of the set of Eq. 5-68 may be written for the charges. If

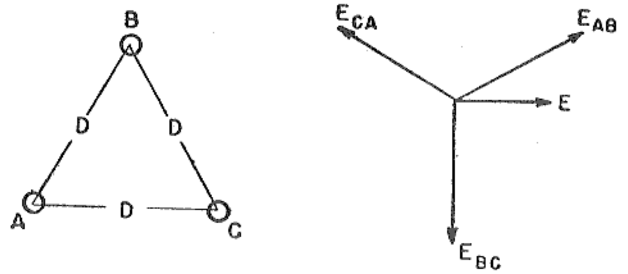

Fig. 5-17. Symmetrical three-phase line.

the line to neutral voltage is $E/\underline{0°}$, then

$$E_{AB} = E\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln\frac{D}{r} + q_B \ln\frac{r}{D} + q_C \ln\frac{D}{D}\right) \qquad (5\text{-}70)$$

$$E_{AC} = -E_{CA} = E\left(\frac{3}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln\frac{D}{r} + q_B \ln\frac{D}{D} + q_C \ln\frac{r}{D}\right) \qquad (5\text{-}71)$$

and also

$$q_A + q_B + q_C + \ . \ . \ . \ + q_N = 0$$

From these equations it is possible to solve for q_A as

$$q_A = \frac{2\pi\epsilon E}{\ln\frac{D}{r}}$$

and since E is a line-to-neutral potential, the line-to-neutral capacitance is given by $C = q_A/E$ or

$$C = \frac{2\pi\epsilon}{\ln\frac{D}{r}} \quad \text{farads/m} \qquad (5\text{-}72)$$

This is equivalent to

$$C = \frac{1}{18 \times 10^9 \ln\frac{D}{r}} \quad \text{farads/m to neutral} \qquad (5\text{-}73)$$

after use of the space value of ϵ. Writing the expression on a per mile basis gives

$$C = \frac{0.0388}{\log_{10}\frac{D}{r}} \quad \mu\text{f/mile to neutral} \qquad (5\text{-}74)$$

5-14. Capacitance of an unsymmetrical three-phase line

The unsymmetrical line requires transposition to achieve balanced charges and capacitances. For one particular portion of the transposition cycle the conductors of such a line have the arrangement of Fig. 5-18. Given a radius r for all the wires, the potentials may be written as before:

$$E_{AB} = E\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln\frac{D_{AB}}{r} + q_B \ln\frac{r}{D_{BA}} + q_C \ln\frac{D_{CB}}{D_{CA}}\right)$$

$$E_{AC} = E\left(\frac{3}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln\frac{D_{AC}}{r} + q_B \ln\frac{D_{BC}}{D_{BA}} + q_C \ln\frac{r}{D_{CA}}\right)$$

It is also required that

$$q_A + q_B + q_C = 0$$

Solution of these equations involves very considerable labor. It is necessary to solve for the charges in the first third of the transposition cycle from the above. Then in the second part of the cycle, A occupies the place of B in the first portion, so that q_{A2} equals q_{B1} rotated through 120°. In the third portion of the cycle, A is in the position of C in the first part, and q_{A3} equals q_{C1} rotated through $-120°$. Having thus obtained q_{A1}, q_{A2}, q_{A3}, an average value may be taken and the average value of capacitance per meter over the transposition cycle obtained. This will be a somewhat complicated expression.

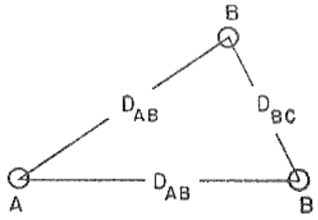

Fig. 5-18. Unsymmetrical three-phase line.

However, it will be found that an expression based on the GMD value for the given conductor arrangement will give very good agreement with the precise relation, while being simple. Such a GMD relation for the capacitance of the line of Fig. 5-18, over the transposition cycle, is

$$C = \frac{2\pi\epsilon}{\ln \dfrac{\sqrt[3]{D_{AB}D_{BC}D_{CA}}}{r}}$$

$$= \frac{2\pi\epsilon}{\ln \dfrac{D_M}{r}} \quad \text{farads/m to neutral} \tag{5-75}$$

It should be noted that while the numerator of the logarithm is the D_M value computed as in the preceding sections, the denominator of the logarithm is the radius of the conductor. The D_S value used for computation of inductance appeared because of the internal flux linkages of the conductor. However, in electrostatics the charge resides on the conductor surface and there are no internal effects; thus the appearance of the radius term is logical.

The expression above for an unsymmetrical conductor arrangement will agree with the precise relation, obtainable by use of the method outlined, within 1 per cent. For cases with symmetry the result will be exact, and this is illustrated by Eq. 5-72, in which the GMD of the three conductors was D, so that use of Eq. 5-75 would have lead directly to Eq. 5-72.

Thus for any arrangement of conductors, over a transposition cycle, Eq. 5-75 may be written with $\epsilon = \epsilon_v$ as

$$C = \frac{1}{18 \times 10^9 \ln \dfrac{D_M}{r}} \quad \text{farads/m to neutral} \tag{5-76}$$

$$= \frac{0.0388}{\log_{10} \dfrac{D_M}{r}} \quad \mu\text{f/mile to neutral} \tag{5-77}$$

5-15. Effect of ground

The presence of the conducting ground near an overhead transmission line causes a slight change in the capacitance value. The effect may be studied by the *method of images*, illustrated in Fig. 5-19.

If two parallel wires have equal and opposite charges, there will be a zero-potential plane halfway between the wires. This field configuration will be identical to that between a wire and an equipotential ground plane, so that the wire over a conducting ground problem may be studied by use of the two-wire configuration.

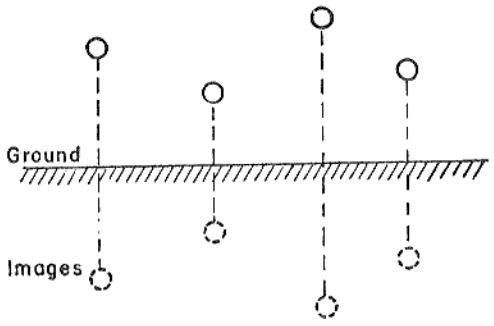

Fig. 5-19. Illustrating the method of images.

The charges may then be computed as due to the potentials between lines and images, but in terms of potentials from lines to ground.

The effect on the capacitance is of the order of 1 or 2 per cent, and is of the same order as other uncertainties in the capacitance value due to the presence of towers and other objects near the line. Because of these uncertainties in actual capacitance value, the effect of ground is usually neglected.

5-16. Relation between L and C values

It is interesting to note that if the capacitance for a two-wire line *in space*, from Eq. 5-30,

$$C = \frac{\pi\epsilon_v}{\ln \dfrac{d}{a}}$$

is multiplied by the *external inductance* of the same two-wire line, from Eq. 5-14, as

$$L_e = \frac{\mu_v}{\pi} \ln \frac{d}{a}$$

there results a constant

$$L_e C = \mu_v \epsilon_v$$

Since μ_v and ϵ_v have space values of $4\pi \times 10^{-7}$ and $10^{-9}/36\pi$ in the rationalized MKS system, this product is

$$L_eC = \mu_v\epsilon_v = \frac{1}{9 \times 10^{16}} = \frac{1}{c^2} \qquad (5\text{-}78)$$

Actually this is universally true, regardless of the arrangement or number of line conductors. Thus if the value of the *external inductance* of any line be computed by the GMD methods given in detail in previous sections, the accompanying value of capacity may be immediately found by Eq. 5-78.

It should be noted that the external inductance is found from any of the GMD relations by use of the D_M term only in the logarithm, since D_S is concerned with the internal inductance. If the line is surrounded with a dielectric, appropriate values and change of constant in Eq. 5-78 will still yield useful results.

PROBLEMS

5-1. A line is made up of two No. 8 (0.1285 in. diameter) copper wires with a center-to-center spacing of 6 in. in air. Find the inductance, capacitance, and d-c resistance per mile of line. Specify separately the inductances due to internal and external linkages.

5-2. A telephone cable pair of No. 16 (0.051 in. diameter) copper wire is insulated with 0.006 in. of paper on each wire and twisted. If the dielectric is entirely of paper, find the capacitance and inductance per mile of line. If the frequency is 796 c, what shunt susceptance and series reactance are represented by the line?

5-3. A coaxial cable is made with a central conductor of No. 8 (0.1285 in. diameter) copper wire with the outer conductor of copper tubing 0.435 in. outside diameter and 0.032 in. wall thickness. The dielectric is polyethylene. Compute the shunt susceptance and series reactance per meter for a frequency of 1000 mc.

5-4. What is the inductance per loop mile of a line spaced 5 ft on centers, and having one conductor with a radius of 0.075 in. and the other of 0.15 in?

5-5. Find the D_S value for a copper cable of 6 strands of radius a, surrounding a central nonconducting core. Give the answer in terms of the radius of the cable.

5-6. A double-circuit single-phase line has the four conductors at the corners of a square 12 in. on a side. The conductors are No. 8, B & S copper wire. Find the voltage drop per thousand feet, when the line is carrying 30 kva at 2300 v. Paralleled conductors are at adjacent corners of the square, and the frequency is 60 cps.

5-7. If the paralleled conductors are on diagonally opposite corners of the square, repeat Prob. 5-6. Which arrangement gives the more desirable form of connection?

5-8. By approximating the rectangular area of Fig. 5-20 with inscribed circles, show that as the number of circles increases, the D_S value of the rectangle approaches the theoretical value of Eq. 5-52. Try at least 2 and 8 circles.

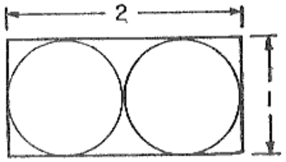

Fig. 5-20.

5-9. Find the D_S value for a stranded cable of four strands in terms of the cable area.

5-10. A rectangular conductor has a cross section of 1.5 million cir mils, and length to width ratio of 3:1. What will be the radius of a circular conductor having the same internal inductance per unit length? What will be the ratio of d-c resistances per unit length?

5-11. A horizontally spaced three-phase line has conductor spacing of 25 ft, each conductor being 37-strand, 500,000 cir mil copper cable. Find the inductance and capacitance per mile, assuming proper transpositions.

5-12. The above line is operated at 220 kv, for a length of 100 miles, handling 150,000 kva. If equal line currents are assumed, find the reactive line drop per mile if no transpositions are used.

5-13. A double-circuit three-phase line is hung in the arrangement

O	O	O	O	O	O
a	a'	b	b'	c	c'

Conductors a and a', b and b', c and c' are paralleled. Spacing between wires of a phase is 15 in., spacing between a' and b, and b and c is 15 ft each. Each wire is a copper 7-strand No. 3/0 cable of 167,800 cir mils. Assuming transposition, compute the inductance and capacitance per phase per mile.

5-14. The Hoover Dam-Los Angeles line has hollow copper conductors made of interlocking segments, giving essentially a tube of 1.40 in. outside diameter and 512,000 cir mil area. The line is spaced horizontally, with 32.5 ft between conductors. Assuming transposition, find the reactance per mile per phase at 60 c.

5-15. The line of Prob. 5-14 operates at 287 kv phase voltage. Compute the capacitive charging current and kva per phase for the 275-mile line.

5-16. A double-circuit three-phase line is arranged on the points of a hexagon. If wires on opposite points are paralleled and the system is transposed, find the inductive reactance per mile per phase at 60 c. Each wire is 37-strand 500,000 cir mil copper cable.

5-17. If either adjacent wires or opposite wires of the hexagonal arrangement of Prob. 5-16 may be paralleled, specify the arrangement giving the greatest capacitance, phase to neutral.

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Chapter 6

TRANSMISSION-LINE THEORY

Because of the distributed nature of the line constants, special methods must be developed for the analysis of long lines or circuits with distributed constants. This chapter will consider the transmission of electric energy over a line, with attention to the physical phenomena and the terminology associated with the distributed constant circuit viewpoint. In Chapters 9 and 10 similar phenomena will be discussed from the electromagnetic field viewpoint, leading to similar results.

6-1. A line of cascaded T sections

It is convenient to approach the study of the line through use of some of the network theory developed in Chapter 4. Therefore consider a number of identical and symmetrical T networks connected in series as in Fig. 6-1. If the final section is terminated in

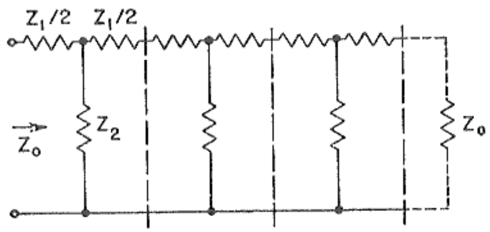

Fig. 6-1. A line of cascaded symmetrical sections of impedance.

its characteristic impedance, the input impedance at the first section is Z_0. Each section is terminated by the input impedance of the following section; and since the last section has a Z_0 termination, all sections are so terminated. The value of the characteristic impedance has been derived for a T section as

$$Z_{0T} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)} \qquad (6\text{-}1)$$

If there are n such terminated sections and if the input and output currents are I_s and I_r, respectively, then

$$\frac{I_s}{I_r} = \epsilon^{n\gamma} \tag{6-2}$$

where γ is the propagation constant for one T section. As discussed in Chapter 4, γ is in general complex and is equal to $\alpha + j\beta$. From Eq. 4-37, ϵ^{γ} can be evaluated as

$$\epsilon^{\gamma} = \epsilon^{\alpha + j\beta} = 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2}\left(1 + \frac{Z_1}{4Z_2}\right)} \tag{6-3}$$

The above reasoning is not invalidated if the number of T sections, n, between source and load, is allowed to increase without limit. A uniform transmission line can be considered as made up of an infinity of T sections, each of infinitesimal size, each element including its proportionate share of the distributed inductance, capacitance, resistance, and leakance per unit of line length. Thus certain methods of network analysis, developed for lumped networks in Chapter 4, are fundamental to distributed networks as well.

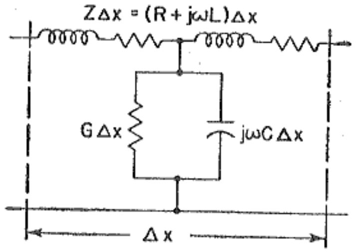

Fig. 6-2. The constants for an incremental length of transmission line.

The constants of an incremental length Δx of a line are indicated in Fig. 6-2. The series constants $Z = R + j\omega L$ are stated in ohms per unit length of line, and $Y = G + j\omega C$ are in mhos per unit length of line. Thus one T section, representing an incremental length Δx of the line, has a series impedance $Z\,\Delta x$ ohms and a shunt admittance $Y\,\Delta x$ mhos. The characteristic impedances of all the incremental sections are alike, since the sections are alike; and thus the characteristic impedance of any small section is that of the line as a whole. The characteristic impedance of a line of distributed constants can then be obtained from Eq. 6-1 for one section as

$$Z_0 = \sqrt{\frac{Z\,\Delta x}{Y\,\Delta x}\left(1 + \frac{Z\,\Delta x\; Y\,\Delta x}{4}\right)}$$

$$= \sqrt{\frac{Z}{Y}\left(1 + \frac{ZY\,\overline{\Delta x^2}}{4}\right)} \tag{6-4}$$

Allowing Δx to approach zero in the limit the value of Z_0 for the line of distributed constants is obtained as

$$Z_0 = \sqrt{\frac{Z}{Y}} \text{ ohms} \tag{6-5}$$

It should be noted that since Z and Y are defined in terms of unit length of line, the ratio Z/Y is independent of the length units chosen.

The radical in Eq. 6-3 may be expanded by the binomial theorem as

$$\sqrt{\frac{Z_1}{Z_2}\left(1 + \frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{Z_1}{Z_2}}\left[1 + \frac{1}{2}\left(\frac{Z_1}{4Z_2}\right) - \frac{1}{8}\left(\frac{Z_1}{4Z_2}\right)^2 + \cdots\right]$$

so that Eq. 6-3 may be written

$$\epsilon^{\gamma} = 1 + \sqrt{\frac{Z_1}{Z_2}} + \frac{1}{2}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^2 + \frac{1}{8}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^3 - \frac{1}{128}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^5 + \cdots$$

When applied to the incremental length of line Δx, then $Z_1 = Z\,\Delta x$, $Z_2 = 1/Y\,\Delta x$, and the propagation constant applying is $\gamma\,\Delta x$. Writing the above equation for $\epsilon^{\gamma\Delta x}$ gives

$$\epsilon^{\gamma\Delta x} = 1 + \sqrt{ZY}\,\Delta x + \frac{1}{2}(\sqrt{ZY})^2\,\overline{\Delta x^2} + \frac{1}{8}(\sqrt{ZY})^3\,\overline{\Delta x^3}$$
$$- \frac{1}{128}(\sqrt{ZY})^5\,\overline{\Delta x^5} + \cdots \tag{6-6}$$

If the series expansion is used for an exponential, $\epsilon^{\gamma\Delta x}$ can also be stated as

$$\epsilon^{\gamma\Delta x} = 1 + \gamma\,\Delta x + \frac{\gamma^2\,\overline{\Delta x^2}}{2!} + \frac{\gamma^3\,\overline{\Delta x^3}}{3!} + \cdots \tag{6-7}$$

Equating the two values for $\epsilon^{\gamma\Delta x}$ and canceling the unity terms,

$$\gamma\,\Delta x + \frac{\gamma^2\,\overline{\Delta x^2}}{2} + \frac{\gamma^3\,\overline{\Delta x^3}}{6} + \cdots$$
$$= \sqrt{ZY}\,\Delta x + \frac{(\sqrt{ZY})^2\,\overline{\Delta x^2}}{2} + \frac{(\sqrt{ZY})^3\,\overline{\Delta x^3}}{8} + \cdots$$

Division by Δx leaves

$$\gamma + \frac{\gamma^2\,\Delta x}{2} + \frac{\gamma^3\,\overline{\Delta x^2}}{6} + \cdots$$
$$= \sqrt{ZY} + \frac{(\sqrt{ZY})^2\,\Delta x}{2} + \frac{(\sqrt{ZY})^3\,\overline{\Delta x^2}}{8} + \cdots$$

Allowing Δx to approach zero in the limit it is seen that all terms but two vanish so that

$$\gamma = \sqrt{ZY} \tag{6-8}$$

This is the value for the line of distributed constants, since all elemental lengths are alike. Since Z and Y are in terms of unit length, γ is a value per unit length of line.

6-2. The transmission line—general solution

A circuit with distributed parameters requires a method of analysis somewhat different from that employed in circuits of lumped constants. Since a voltage drop occurs across each series increment of line, the voltage applied to each increment of shunt admittance is a variable and thus the shunted current is a variable along the line. Hence the line current around the loop is not a constant, as is assumed in lumped constant circuits, but varies from point to point along the line. Differential circuit equations that describe this action will be written for the steady state, from which general circuit equations can be obtained.

The notation used will be defined as follows:

R = series resistance, ohms per unit length of line (includes both wires)
L = series inductance, henrys per unit length of line
C = capacitance between conductors, farads per unit length of line
G = shunt leakage conductance between conductors, mhos per unit length of line
ωL = series reactance, ohms per unit length of line
$Z = R + j\omega L$ = series impedance, ohms per unit length of line
ωC = shunt susceptance, mhos per unit length of line
$Y = G + j\omega C$ = shunt admittance, mhos per unit length of line
s = distance to the point of observation, measured from the receiving end of the line
I = current in the line at any point
E = voltage between conductors at any point
l = length of line

Figure 6-3 illustrates a line that in the limit may be considered as made up of cascaded infinitesimal T sections, one of which is shown.

This elemental section is of length ds and carries a current I. The series line impedance being Z ohms per unit, the series impedance of the element is Zds ohms, and the voltage drop in the length ds is

$$dE = IZds$$

or

$$\frac{dE}{ds} = IZ \tag{6-9}$$

The shunt admittance per unit of length of line is Y mhos, so that

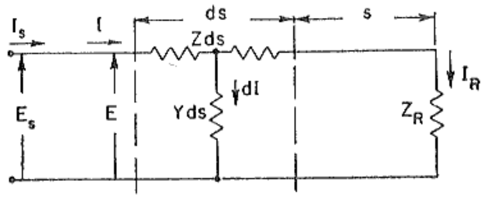

Fig. 6-3. A long line, with the elements of one of the infinitesimal sections shown.

the admittance of the element of line is Yds mhos. The current dI that flows across the line or from one conductor to the other is

$$dI = EYds$$

or

$$\frac{dI}{ds} = EY \tag{6-10}$$

Equations 6-9 and 6-10 may be differentiated with respect to s:

$$\frac{d^2E}{ds^2} = Z\frac{dI}{ds}, \qquad \frac{d^2I}{ds^2} = Y\frac{dE}{ds}$$

Then

$$\frac{d^2E}{ds^2} = ZYE \tag{6-11}$$

$$\frac{d^2I}{ds^2} = ZYI \tag{6-12}$$

These are the differential equations of the transmission line, fundamental to circuits of distributed constants. If $E = E_0\epsilon^{j\omega t}$ and $I = I_0\epsilon^{j\omega t}$ they may also be shown to be forms of the wave equation which will be discussed in Chapter 9.

Solution by conventional methods follows directly. In terms of the operator m, Eq. 6-11 becomes

$$(m^2 - ZY)E = 0$$

$$m = \pm\sqrt{ZY} \tag{6-13}$$

This result indicates two solutions, one for the plus sign and the other for the minus sign before the radical. The solutions of the differential equations then are

$$E = A\epsilon^{\sqrt{ZY}s} + B\epsilon^{-\sqrt{ZY}s} \tag{6-14}$$

$$I = C\epsilon^{\sqrt{ZY}s} + D\epsilon^{-\sqrt{ZY}s} \tag{6-15}$$

where A, B, C, and D are arbitrary constants of integration.

Since distance is measured from the receiving end of the line, it is possible to assign conditions such that at

$$s = 0, \qquad I = I_R, \qquad E = E_R$$

Then Eqs. 6-14 and 6-15 become

$$\left.\begin{aligned} E_R &= A + B \\ I_R &= C + D \end{aligned}\right\} \tag{6-16}$$

A second set of boundary conditions is not available, but the same set may be used over again if a new set of equations are formed by differentiation of Eqs. 6-14 and 6-15. Thus

$$\frac{dE}{ds} = A\sqrt{ZY}\,\epsilon^{\sqrt{ZY}s} - B\sqrt{ZY}\,\epsilon^{-\sqrt{ZY}s}$$

From Eq. 6-9, this becomes

$$IZ = A\sqrt{ZY}\,\epsilon^{\sqrt{ZY}s} - B\sqrt{ZY}\,\epsilon^{-\sqrt{ZY}s}$$

$$I = A\sqrt{\frac{Y}{Z}}\,\epsilon^{\sqrt{ZY}s} - B\sqrt{\frac{Y}{Z}}\,\epsilon^{-\sqrt{ZY}s} \tag{6-17}$$

In a similar manner,

$$\frac{dI}{ds} = C\sqrt{ZY}\,\epsilon^{\sqrt{ZY}s} - D\sqrt{ZY}\,\epsilon^{-\sqrt{ZY}s}$$

$$E = C\sqrt{\frac{Z}{Y}}\,\epsilon^{\sqrt{ZY}s} - D\sqrt{\frac{Z}{Y}}\,\epsilon^{-\sqrt{ZY}s} \tag{6-18}$$

At $s = 0$, Eqs. 6-17 and 6-18 become

$$I_R = A\sqrt{\frac{Y}{Z}} - B\sqrt{\frac{Y}{Z}} \tag{6-19}$$

$$E_R = C\sqrt{\frac{Z}{Y}} - D\sqrt{\frac{Z}{Y}} \tag{6-20}$$

Simultaneous solution of Eqs. 6-16 with Eqs. 6-19 and 6-20, along with the fact that $E_R = I_R Z_R$ and that $\sqrt{Z/Y}$ has been identified as the Z_0 of the line, leads to solutions for the constants of the above equations as

$$A = \frac{E_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2}\left(1 + \frac{Z_0}{Z_R}\right)$$

$$B = \frac{E_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2}\left(1 - \frac{Z_0}{Z_R}\right)$$

$$C = \frac{I_R}{2} + \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2}\left(1 + \frac{Z_R}{Z_0}\right)$$

$$D = \frac{I_R}{2} - \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2}\left(1 - \frac{Z_R}{Z_0}\right)$$

The solution of the differential equations of the transmission line may then be written

$$E = \frac{E_R}{2}\left[\left(1 + \frac{Z_0}{Z_R}\right)\epsilon^{\sqrt{ZY}s} + \left(1 - \frac{Z_0}{Z_R}\right)\epsilon^{-\sqrt{ZY}s}\right] \tag{6-21}$$

$$I = \frac{I_R}{2}\left[\left(1 + \frac{Z_R}{Z_0}\right)\epsilon^{\sqrt{ZY}s} + \left(1 - \frac{Z_R}{Z_0}\right)\epsilon^{-\sqrt{ZY}s}\right] \tag{6-22}$$

Then

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\left[\epsilon^{\sqrt{ZY}s} + \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\sqrt{ZY}s}\right] \tag{6-23}$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left[\epsilon^{\sqrt{ZY}s} - \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\sqrt{ZY}s}\right] \tag{6-24}$$

These are a final and very useful form of the equations for voltage and current at any point on a transmission line, and are solutions to the wave equation of Chapter 9.

Equations 6-21 and 6-22 may also be arranged as

$$E = E_R\left(\frac{\epsilon^{\sqrt{ZY}s} + \epsilon^{-\sqrt{ZY}s}}{2}\right) + I_R Z_0\left(\frac{\epsilon^{\sqrt{ZY}s} - \epsilon^{-\sqrt{ZY}s}}{2}\right)$$

$$I = I_R\left(\frac{\epsilon^{\sqrt{ZY}s} + \epsilon^{-\sqrt{ZY}s}}{2}\right) + \frac{E_R}{Z_0}\left(\frac{\epsilon^{\sqrt{ZY}s} - \epsilon^{-\sqrt{ZY}s}}{2}\right)$$

and these equations can be recognized as

$$E = E_R \cosh \sqrt{ZY}\, s + I_R Z_0 \sinh \sqrt{ZY}\, s \tag{6-25}$$

$$I = I_R \cosh \sqrt{ZY}\, s + \frac{E_R}{Z_0} \sinh \sqrt{ZY}\, s \tag{6-26}$$

Equations 6-25 and 6-26 constitute a second very useful form for the voltage and current values at any point on a transmission line. Although these equations are extensively used, it is believed that the exponential forms of Eqs. 6-23 and 6-24 lead to a clearer physical picture of the phenomena occuring on a line. They also lead to easier calculation with complex values of $\sqrt{ZY}$.

6-3. Physical significance of the equations; the infinite line

Equation 6-26 may be written for the sending-end current I_S of a line of length l as

$$I_S = I_R \left(\cosh \sqrt{ZY}\, l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY}\, l\right)$$

If the line is terminated in $Z_R = Z_0$, then

$$I_S = I_R(\cosh \sqrt{ZY}\, l + \sinh \sqrt{ZY}\, l)$$

from which

$$\frac{I_S}{I_R} = \epsilon^{\sqrt{ZY}\, l} = \epsilon^{\gamma l} \tag{6-27}$$

by reason of the determination that $\sqrt{ZY} = \gamma$. The result of Eq. 6-27 is simply a restatement, for the line, of the basic relation assumed between input and output currents in Chapter 4. The *propagation constant* γ is defined *per unit length* of line. As before, it is complex from the nature of its equivalence to $\sqrt{ZY}$, or $\gamma = \alpha + j\beta$.

Division of Eq. 6-25 by 6-26 leads to an expression for the input impedance of the line of length l as

$$Z_S = \frac{E_S}{I_S} = Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l}\right) \tag{6-28}$$

This equation may be recognized as identical with Eq. 4-39 derived from network theory for a lumped constant circuit. In different form, this result may be obtained by division of Eq. 6-23 by 6-24

and by use of the fact that $E_R/I_R = Z_R$. Then

$$Z_S = \frac{E_S}{I_S} = Z_0 \left[\frac{\epsilon^{\gamma l} + \left(\dfrac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \left(\dfrac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma l}}\right] \tag{6-29}$$

As a particular case, it is of interest to find the value of Z_S, the sending-end input impedance, when the line is terminated in its characteristic impedance; that is, when $Z_R = Z_0$. Equation 6-29 then reduces to

$$Z_S = Z_0 \tag{6-30}$$

which conforms to the definition of characteristic impedance for lumped networks and establishes the validity of the operations in Section 6-1, in which Z_0 was identified as $\sqrt{Z/Y}$ for a T section of elemental magnitude in a distributed constant circuit.

Unity has been established between the lumped-constant and distributed-constant circuits in that the description of circuit performance through Z_0 and γ has been found to hold for both cases and thus is a truly general method of description of circuit performance.

Even though a line of infinite length seems hypothetical, actually much may be learned from a study of such a line. The input impedance of an infinite line may be found by letting l approach infinity in Eq. 6-28 or 6-29. The result is

$$Z_S = Z_0 \tag{6-31}$$

Thus, by comparison of the results shown in Eqs. 6-30 and 6-31, *a line of finite length, terminated in a load equivalent to its characteristic impedance, appears to the sending-end generator as an infinite line.* A finite line terminated in Z_0 and an infinite line are indistinguishable by measurements at the source. Since the use of

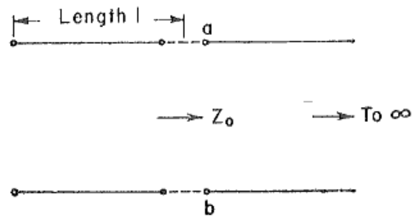

Fig. 6-4. A length l taken from an infinite line.

terminations equal to Z_0 is quite common, the Z_0-terminated finite line may be studied by analysis of the infinite line.

This result may be reasoned qualitatively by consideration of the line of infinite length of Fig. 6-4. This line may have terminals a,b placed at a finite distance l from the sending end. The remainder of the line is still infinite in length, so that the input impedance at terminals a,b is Z_0. Thus the length l of the line represents a finite length of line terminated in its characteristic impedance.

Restating Eqs. 6-23 and 6-24,

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\left[\epsilon^{\gamma s} + \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right]$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left[\epsilon^{\gamma s} - \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right]$$

For a portion of an infinite line, or a *finite line terminated in its characteristic impedance*,

$$Z_R = Z_0$$

so that these equations reduce to

$$E = E_R\epsilon^{\gamma s} \quad (6\text{-}32)$$

$$I = I_R\epsilon^{\gamma s} \quad (6\text{-}33)$$

It is apparent that the voltage and current values change with distance s from the receiving end because of the factor $\epsilon^{\gamma s}$.

These expressions may be readily written for distance s measured from the sending end, with E_s and I_s the voltage and current values at that point, giving

$$E_R = E_s\epsilon^{-\gamma s}, \qquad I_R = I_s\epsilon^{-\gamma s}$$

If the terminals a,b be considered as any point on the infinite line, then E and I at any point are expressed in terms of E_s and I_s as

$$E = E_s\epsilon^{-\gamma s} \quad (6\text{-}34)$$

$$I = I_s\epsilon^{-\gamma s} \quad (6\text{-}35)$$

Since $\gamma = \alpha + j\beta$, then

$$E = E_s\epsilon^{-\alpha s}\epsilon^{-j\beta s} \quad (6\text{-}36)$$

$$I = I_s\epsilon^{-\alpha s}\epsilon^{-j\beta s} \quad (6\text{-}37)$$

Since the sending end values will be functions of time as $E_s = E_{so}\epsilon^{j\omega t}$ and $I_s = I_{so}\epsilon^{j\omega t}$, then it is seen that Eqs. 6-36 and 6-37 are functions of both distance and time. This is a property of any solution to the wave equation.

It is to be seen that as one measures along this line from the

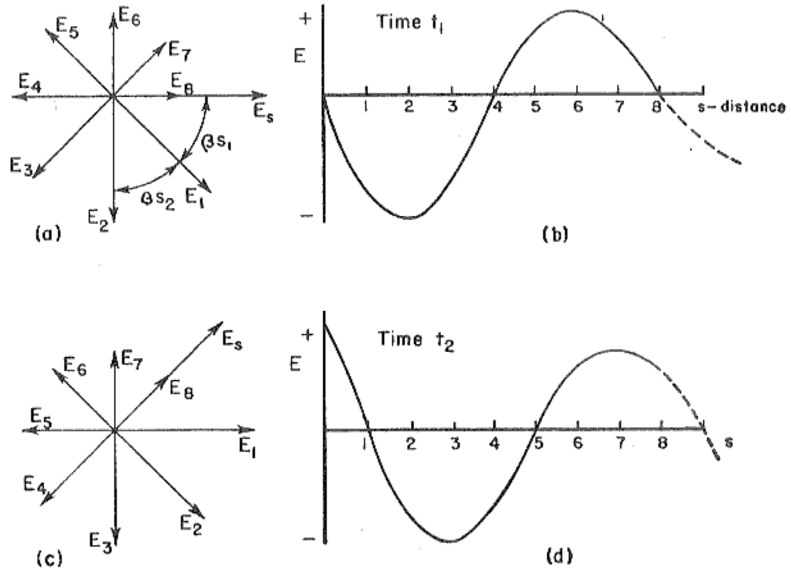

Fig. 6-5. (a) Voltage phasors at $\frac{1}{8}$-wavelength points along a line at a particular time instant; (b) a plot of the phasors of (a), showing voltage on the line as a function of distance; (c) voltage phasors one-eighth of a cycle later than (a); (d) plot of voltage along the line for the time instant of (c) showing the movement of the wave along the line.

sending end, the voltage and current become progressively smaller in magnitude because of the factor $\epsilon^{-\alpha s}$. The logic of calling α the *attenuation constant* is apparent. Also, the current and voltage phases lag progressively more and more as s increases, because of the increasing angle inherent in $\epsilon^{-j\beta s} = \angle -\beta s$.

It is essential that the student understand the use of $-s$ or s in measuring in the direction of, or counter to the direction of, the incident energy. Free use will be made of both forms as needed.

If uniformly spaced points are selected on the line or if uniformly increasing s values, such as $s_1, s_2, s_3, \ldots$, are chosen, then voltage phasors may be drawn for each point as in (a), Fig. 6-5. These phasors may be considered as representing maximum instantaneous voltage values, and are seen to vary in magnitude as $\epsilon^{-\alpha s}$ and in phase by uniform angle increments. It may then be assumed that all the phasors are rotating in the counterclockwise direction with an angular velocity ω equal to that of the generating source at the sending end. At a given time t_1, the rotation may be stopped and the instantaneous values of voltage plotted as a function of distance, as in (b), Fig. 6-5. This plot shows a portion of an infinite line, with the instantaneous voltage conditions existing at each point along the line. Such an oscillating and attenuating voltage condition exists over the whole length of the line.

At (c), Fig. 6-5, the rotating phasors have been stopped at a time t_2, which is one-eighth of a cycle later than time t_1. The instantaneous values of voltage are again plotted as a function of distance in (d). Comparison of sketches (d) and (b) shows that the wave of (b) has moved to the right and become that of (d). This movement discloses the *existence of a voltage wave traveling down the line from the generator.* The student may verify this action by plotting the waves on the line for intermediate or later instants of time.

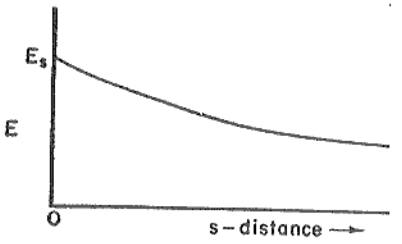

Fig. 6-6. Voltage along an infinite line as measured by an effective-reading meter.

Thus propagation takes place along the line in a wave motion, the amplitude of voltage or current being constantly reduced due to attenuation $\epsilon^{-\alpha s}$, the phase of the voltage or current constantly changing because of the phase factor $\epsilon^{-j\beta s}$. The phasors may be thought of as rotating in space, in which case the locus of the ends of the phasors appears as a tapering corkscrew. The projection of this corkscrew on a plane then constitutes the wave pattern at any instant.

This discussion has been in terms of instantaneous values. If instruments reading effective values are connected along the line, they will not show the phase angles, and thus their readings along

the line would correspond to the curves for

$$E = E_s\epsilon^{-\alpha s}, \qquad I = I_s\epsilon^{-\alpha s}$$

as shown in Fig. 6-6.

Although the analysis has been given here in terms of current and voltage, it should be remembered that actually energy is being propagated along the line, its transfer taking place in the electric and magnetic fields in the region surrounding the line. The waves of current and voltage are merely convenient means of observation of the fields present. A discussion of the line from the field viewpoint is given in Chapter 12.

6-4. Wavelength; velocity of propagation

The distance the wave travels along the line while the phase angle is changing through 2π radians is called a *wavelength.* In Fig. 6-5(b), the distance from the sending end to point 8 is thus one wavelength. From the definition above, if the wavelength is represented by the symbol λ, then at a distance such that $s = \lambda$,

$$\beta\lambda = 2\pi$$

and

$$\lambda = \frac{2\pi}{\beta} \tag{6-38}$$

Since the change of 2π in phase angle represents one cycle in time and occurs in a distance of one wavelength, then

$$x = vt$$

and

$$\lambda = \frac{v}{f} \tag{6-39}$$

From this and Eq. 6-38, the velocity can be expressed in terms of the line constants as

$$v = \lambda f = \frac{2\pi f}{\beta}$$

$$v = \frac{\omega}{\beta} \tag{6-40}$$

This is the *velocity of propagation* along the line, based on observations of the change in phase along the line. It is measured in miles per second if β is in radians per mile, or in meters per second if β is in radians per meter.

If it be remembered that

$$Z = R + j\omega L$$
$$Y = G + j\omega C$$

then

$$\gamma = \alpha + j\beta = \sqrt{ZY}$$
$$= \sqrt{RG - \omega^2 LC + j\omega(LG + CR)} \quad (6\text{-}41)$$

Squaring both sides,

$$\alpha^2 + j2\alpha\beta - \beta^2 = RG - \omega^2 LC + j\omega(LG + CR)$$

Equating the reals and solving for α^2 gives

$$\alpha^2 = \beta^2 + RG - \omega^2 LC \quad (6\text{-}42)$$

Equating the imaginaries and squaring yields

$$4\alpha^2\beta^2 = \omega^2(LG + CR)^2$$

after which substitution of Eq. 6-42 gives

$$\beta^4 + \beta^2(RG - \omega^2 LC) - \frac{\omega^2}{4}(LG + CR)^2 = 0$$

A solution for β, neglecting the negative values, follows as

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}} \quad (6\text{-}43)$$

and use of Eq. 6-42 leads to a value for

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}} \quad (6\text{-}44)$$

In a perfect line $R = 0$ and $G = 0$. Equation 6-43 then would be

$$\beta = \omega\sqrt{LC} \quad (6\text{-}45)$$

and the velocity of propagation for such an ideal line is given by

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \text{ m/sec} \quad (6\text{-}46)$$

thus showing that the line parameter values fix the velocity of

propagation. Using Eqs. 5-14 and 5-30 for L and C values for an open-wire line, the LC product becomes

$$LC = \frac{\pi\epsilon}{\ln(d/a)}\left(\frac{\mu}{4\pi} + \frac{\mu_v}{\pi}\ln\frac{d}{a}\right)$$

For a line of nonmagnetic material with air spacing

$$LC = \frac{\mu_v\epsilon_v}{4\ln(d/a)} + \mu_v\epsilon_v \quad (6\text{-}47)$$

Then

$$v = \frac{1}{\sqrt{\mu_v\epsilon_v\left[\dfrac{1}{4\ln(d/a)} + 1\right]}} \text{ m/sec} \quad (6\text{-}48)$$

It should be noted that the first term on the right of Eq. 6-47 and the first term in the bracket under the radical of Eq. 6-48 are present because of the internal inductance of the conductors. If this internal inductance is reduced, as by skin effect, the velocity increases and reaches the limiting condition of

$$v = \frac{1}{\sqrt{\mu_v\epsilon_v}} \text{ m/sec}$$

which in space becomes

$$3 \times 10^8 = c \text{ m/sec} \quad (6\text{-}49)$$

which is identified as the velocity of light in space. The velocity of propagation in an actual line is slowed below this value by the effect of internal inductance, resistance, and leakage of the line.

The above substantiates the method of calculation of C presented in Section 5-16, and shows the constant in Eq. 5-78 to be equal to c^2.

6-5. An example

The relative magnitudes of quantities to be encountered in line calculations are indicated in the following example:

A generator of 1.0 volt, 1000 cycles, supplies power to a 100-mile open-wire line terminated in Z_0 and having the following parameters:

$$R = 10.4 \text{ ohms per mile}$$
$$L = 0.00367 \text{ henry per mile}$$
$$G = 0.8 \times 10^{-6} \text{ mho per mile}$$
$$C = 0.00835\ \mu\text{f per mile}$$

The line constants then are

$$Z = R + j\omega L = 10.4 + j23.0 = 25.2\underline{/66^\circ}$$

$$Y = G + j\omega C = (0.8 + j52.5)10^{-6} = 52.6 \times 10^{-6}\underline{/90^\circ}$$

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{25.2\underline{/66^\circ}}{52.6 \times 10^{-6}\underline{/90^\circ}}} = 692\underline{/-12^\circ} \text{ ohms}$$

$$\gamma = \sqrt{ZY} = \sqrt{25.2\underline{/66^\circ} \times 52.6 \times 10^{-6}\underline{/90^\circ}} = 0.0363\underline{/78^\circ}$$

$$\alpha = 0.0363 \cos 78^\circ = 0.00755 \text{ neper per mile}$$

$$\beta = 0.0363 \sin 78^\circ = 0.0355 \text{ radian per mile}$$

$$v = \frac{\omega}{\beta} = \frac{6280}{0.0355} = 177{,}000 \text{ miles per second}$$

$$\lambda = \frac{2\pi}{\beta} = 177 \text{ miles}$$

Since the line is terminated in Z_0, then $Z_s = Z_0$, so that

$$I_s = \frac{E_s}{Z_0} = \frac{1.0}{692\underline{/-12^\circ}} = 0.00145\underline{/12^\circ} \text{ amp}$$

$$\frac{I_R}{I_s} = \epsilon^{-\gamma l} = \epsilon^{-\alpha l}\epsilon^{-j\beta l} = \epsilon^{-0.755}\epsilon^{-j3.55}$$

But $\epsilon^{-j3.55}$ is equivalent to an angle of -3.55 radians, or -203.8 deg from the tables in the rear of the book. Then

$$I_R = I_s\,\epsilon^{-0.755}\underline{/-203.8^\circ}$$
$$= 0.00145\underline{/12^\circ} \times 0.472\underline{/-203.8^\circ}$$
$$= 0.000685\underline{/-191.8^\circ} \text{ amp } (E_s \text{ reference})$$

The received voltage is

$$E_R = I_R Z_0$$
$$= 0.000685\underline{/-191.8} \times 692\underline{/-12^\circ}$$
$$= 0.474\underline{/-203.8^\circ} \text{ volts } (E_s \text{ reference})$$

The received power is

$$P_R = E_R I_R \cos\theta = 318 \times 10^{-6} \text{ watts}$$

6-6. Wave-form distortion

The value of the attenuation constant α has been determined in Section 6-4 as

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}}$$

In general, α is a function of frequency. All frequencies transmitted on a line will then not be attenuated equally. A complex applied voltage, such as a voice voltage containing many frequencies, will not have all frequencies transmitted with equal attenuation, and the received wave form will not be identical with the input wave form at the sending end. This variation is known as *frequency distortion.*

The phase constant β was shown in Section 6-4 to be

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}}$$

and can be seen to be a complicated function of frequency, in general. Since the velocity of propagation has been stated as

$$v = \frac{\omega}{\beta}$$

it is apparent that ω and β do not both involve frequency in the same manner and that the velocity of propagation will in general be some function of frequency. All frequencies applied to a transmission line will not have the same time of transmission, some frequencies being delayed more than others. For an applied voice-voltage wave the received wave form will not be identical with the input wave form at the sending end, since some components will be delayed more than those of other frequencies. This phenomenon is known as *delay* or *phase distortion.*

Frequency distortion is reduced in the transmission of high-quality radio broadcast programs over wire lines by use of *equalizers* at the line terminals. These circuits are networks whose frequency and phase characteristics are adjusted to be inverse to those of the lines, resulting in an over-all uniform frequency response over the desired frequency band.

Delay distortion is of relatively minor importance to voice and music transmission because of the characteristics of the ear. It can be very serious in circuits intended for picture transmission, and applications of the coaxial cable have been made to overcome the difficulty. In such cables the internal inductance is low at high frequencies because of skin effect, the resistance is small because of the large conductors, and capacitance and leakance are small because of the use of air dielectric with a minimum of spacers. The velocity of propagation is raised and made more nearly equal for all frequencies.

6-7. The distortionless line

If a line is to have neither frequency nor delay distortion, then α and the velocity of propagation cannot be functions of frequency. In view of the fact that

$$v = \frac{\omega}{\beta}$$

then β must be a direct function of frequency.

Consideration of Eq. 6-43,

$$\beta = \sqrt{\frac{\omega^2LC - RG + \sqrt{(RG - \omega^2LC)^2 + \omega^2(LG + CR)^2}}{2}}$$

shows that if the term under the second radical be reduced to equal

$$(RG + \omega^2LC)^2$$

then the required condition on β is obtained. Expanding the terms under the internal radical and forcing the equality gives

$$R^2G^2 - 2\omega^2LCRG + \omega^4L^2C^2 + \omega^2L^2G^2 + 2\omega^2LCRG + \omega^2C^2R^2 = (RG + \omega^2LC)^2$$

This reduces to

$$\omega^2L^2G^2 - 2\omega^2LCRG + \omega^2C^2R^2 = 0$$

$$(LG - CR)^2 = 0$$

Therefore the condition that will make β a direct function of frequency is

$$LG = CR \tag{6-50}$$

A hypothetical line might be built to fulfill this condition. The line would then have a value of β obtained by use of Eq. 6-50 in

6-43 as

$$\beta = \omega\sqrt{LC}$$

The velocity of propagation is then

$$v = \frac{1}{\sqrt{LC}}$$

which is the same for all frequencies, thus eliminating delay distortion.

Equation 6-44 for

$$\alpha = \sqrt{\frac{RG - \omega^2LC + \sqrt{(RG - \omega^2LC)^2 + \omega^2(LG + CR)^2}}{2}}$$

may be made independent of frequency if the term under the internal radical is forced to reduce to

$$(RG + \omega^2LC)^2$$

Analysis shows that the condition of Eq. 6-50, $LG = CR$, will produce the desired result, so that it is possible to make α and the velocity independent of frequency simultaneously. Applying the condition of Eq. 6-50 to the expression for α gives

$$\alpha = \sqrt{RG}$$

which is independent of frequency, thus eliminating frequency distortion on the line.

Unfortunately, such a hypothetical line is not practical with distributed parameters, but the analysis points the way to the solution of Section 6-9. To achieve the condition

$$LG = CR$$

or

$$\frac{L}{C} = \frac{R}{G} \tag{6-51}$$

requires a very large value of L, since G is small. If G is intentionally increased, α and the attenuation are increased, resulting in poor line efficiency. To reduce R raises the size and cost of the conductors above economic limits, so that the hypothetical results cannot be achieved.

6-8. The telephone cable

In the ordinary telephone cable the wires are insulated with paper and twisted in pairs. This construction results in negligible values of inductance and conductance so that reasonable assumptions in the audio range of frequencies are that

$$Z = R \tag{6-52}$$

$$Y = j\omega C \tag{6-53}$$

Equation 6-41 stated that

$$\gamma = \sqrt{RG - \omega^2 LC + j\omega(LG + CR)}$$

With $L = G = 0$, this equation becomes

$$\gamma = \sqrt{j\omega CR} = \sqrt{\frac{j2\omega CR}{2}}$$

$$\gamma = \alpha + j\beta = (1 + j1)\sqrt{\frac{\omega CR}{2}}$$

Therefore

$$\alpha = \sqrt{\frac{\omega CR}{2}} \tag{6-54}$$

$$\beta = \sqrt{\frac{\omega CR}{2}} \tag{6-55}$$

Hence the velocity of propagation is

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{CR}} \tag{6-56}$$

It should be observed that both α and the velocity are functions of frequency, such that the higher frequencies are attenuated more and travel faster than the lower frequencies. Very considerable frequency and delay distortion is the result on telephone cable.

6-9. Inductance loading of telephone cables

The analysis of Section 6-7 concerning a distortionless line with distributed parameters suggests a remedy for the severe frequency and delay distortion experienced on long cables. It was indicated that it was necessary to increase the L/C ratio to achieve distortionless conditions. Heaviside suggested that the inductance be

increased, and Pupin developed the theory that made possible this increase in the inductance by *lumped* inductors spaced at intervals along the line. This use of inductance is called *loading* the line. In some submarine cables, distributed or uniform loading is obtained by winding the cable with a high-permeability steel tape such as permalloy. This method is employed because of the practical difficulties of designing lumped loading coils for such underwater circuits.

For simplicity, consider first a uniformly loaded cable circuit for which it may be assumed that $G = 0$ and for which L has been increased so that ωL is large with respect to R. Then

$$Z = R + j\omega L$$

$$Y = j\omega C$$

Since,

$$Z = \sqrt{R^2 + \omega^2 L^2}\,\Big/\frac{\pi}{2} - \tan^{-1}\frac{R}{\omega L} \tag{6-57}$$

then

$$\gamma = \sqrt{ZY}$$

$$= \sqrt{\sqrt{R^2 + \omega^2 L^2}\,\Big/\frac{\pi}{2} - \tan^{-1}\frac{R}{\omega L} \times \omega C\,\Big/\frac{\pi}{2}}$$

$$= \omega\sqrt{LC}\,\sqrt[4]{1 + \frac{R^2}{\omega^2 L^2}}\,\Big/\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L} \tag{6-58}$$

In view of the fact that R is small with respect to ωL, the term $R^2/\omega^2 L^2$ may be dropped, and γ becomes

$$\gamma = \omega\sqrt{LC}\,\Big/\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L} \tag{6-59}$$

If $\theta = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}$,

$$\cos\theta = \cos\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right) = \sin\left(\frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right)$$

For a small angle,

$$\sin\theta = \tan\theta = \theta, \qquad \text{so that} \qquad \cos\theta = \frac{R}{2\omega L}$$

Likewise, $\sin \theta = \sin \left(\frac{\pi}{2} - \frac{1}{2} \tan^{-1} \frac{R}{\omega L} \right) = 1$

Equation 6-59 may then be written

$$\gamma = \omega \sqrt{LC} (\cos \theta + j \sin \theta) = \omega \sqrt{LC} \left(\frac{R}{2\omega L} + j \right)$$

Therefore, for the uniformly loaded cable,

$$\alpha = \frac{R}{2} \sqrt{\frac{C}{L}} \tag{6-60}$$

$$\beta = \omega \sqrt{LC} \tag{6-61}$$

and
$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

It is readily observed that, under the assumptions of $G = 0$ and ωL large with respect to R, the attenuation and velocity are both independent of frequency and the loaded cable will be distortionless. The expression for α shows that the attenuation may be reduced by increasing L, provided that R is not also increased too greatly.

Continuous or uniform loading is expensive and achieves only a small increase in L per unit length. Lumped loading is ordinarily preferred as a means of transmission improvement for cables. The improvement obtainable on open-wire lines is usually not sufficient to justify the extra cost of the loading inductors.

An analysis for the performance of a line loaded at uniform intervals can be obtained by considering a symmetrical section of line from the center of one loading coil to the center of the next, where the loading coil impedance is Z_c. The section of line may be replaced with an equivalent T section (see Section 6-17) having symmetrical series arms. Adopting the notation of filter circuits, one of these series arms is called $Z_1/2$ and is

$$\frac{Z_1}{2} = Z_0 \tanh \frac{N\gamma}{2}$$

where N is the number of miles between loading coils and γ is the propagation constant per mile. Upon including half a loading coil, the equivalent series arm of the loaded section becomes

$$\frac{Z_1'}{2} = \frac{Z_c}{2} + Z_0 \tanh \frac{N\gamma}{2}$$

The shunt Z_2 arm of the equivalent T section is

$$Z_2 = \frac{Z_0}{\sinh N\gamma}$$

An equation relating γ and the circuit elements of a T section was derived as Eq. 4-33, which may be applied to the loaded T section as

$$\cosh N\gamma' = 1 + \frac{Z_1'}{2Z_2}$$

$$= 1 + \frac{Z_c/2 + Z_0 \tanh (N\gamma/2)}{Z_0/\sinh N\gamma} \tag{6-62}$$

By use of exponentials it can be shown that

$$\tanh \frac{N\gamma}{2} = \frac{\cosh N\gamma - 1}{\sinh N\gamma}$$

so that Eq. 6-62 reduces to

$$\cosh N\gamma' = \frac{Z_c}{2Z_0} \sinh N\gamma + \cosh N\gamma \tag{6-63}$$

This expression is known as *Campbell's equation* and permits the determination of a value for γ' of a line section consisting partially of lumped and partially of distributed elements. Campbell's equation makes possible the calculation of the effects of loading coils in reducing attenuation and distortion on lines.

For a cable, Z_2 of Fig. 6-7 is essentially capacitive and the cable capacitance plus lumped inductances appear similar to the circuit of the low-pass filter. It is found that for frequencies below cutoff,

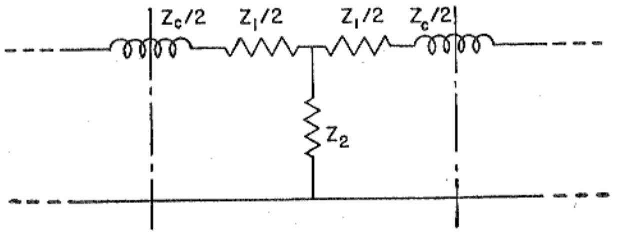

Fig. 6-7. Equivalent T section for part of a line between two lumped loading coils of impedance, Z_c.

given by

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

the attenuation is reduced as expected, but above cutoff the attenuation rises as a result of filter action. This cutoff frequency forms a definite upper limit to successful transmission over cables. It can be raised by reducing L but this expedient allows the attenuation to rise. The cutoff frequency can also be raised by spacing the coils closer together, thus reducing C and more closely approximating the distributed-constant line, but the cost increases rapidly.

In practice, a truly distortionless line is not obtained by loading, because R and L are to some extent functions of frequency. Eddy-current losses in the loading inductors aggravate this condition. However, a major improvement is obtained in the loaded cable over the unloaded cable for a reasonable frequency range.

6-10. Reflection on a line not terminated in Z_0

Returning to Eqs. 6-23 and 6-24 for the voltages and currents on the line, with *s measured as positive from the receiving end,*

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\left[\epsilon^{\gamma s} + \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right] \qquad (6\text{-}23)$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left[\epsilon^{\gamma s} - \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right] \qquad (6\text{-}24)$$

it may be observed that for the most general case in which Z_R is not equal to Z_0, each equation consists of two terms, one of which varies exponentially with $+s$, the other with $-s$. It has been shown in Section 6-3 that a wave travels from the sending end to the receiving end of the line, decreasing in amplitude as it approaches the receiving end. In a direction back along this wave from the receiving end, the amplitude of the wave would increase as s increases, so that the wave that travels from the sending end to the receiving end can be identified as the component varying with $\epsilon^{\gamma s}$. This wave of voltage or current is known as the *incident wave.*

Hence the second term, varying with $\epsilon^{-\gamma s}$, must represent a wave of voltage or current progressing from the receiving end toward the sending end, and decreasing in amplitude with increased distance

from the load. Such a wave of voltage or current is called the *reflected wave.*

The situation may be more clearly seen in Fig. 6-8. The incident voltage values, with Z_R chosen as an open circuit for convenience,

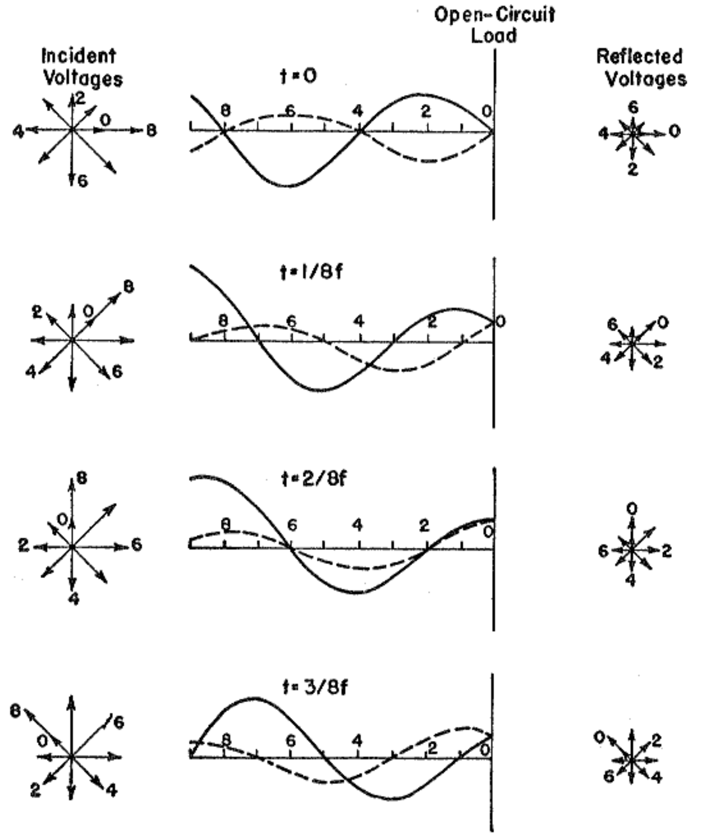

Fig. 6-8. Rotating voltage-phasor systems for incident and reflected waves for an open-circuit termination. The incident wave (solid curve) and reflected wave (dashed curve) are shown for four successive time instants.

are plotted as solid curves, as derived from the rotating maximum value phasors at the left. The waves are plotted over the last wavelength of line adjacent to the receiving end and are for four instants of time, each separated by one-eighth of a cycle. It can be seen that the voltage component varying as $\epsilon^{\gamma s}$ (s measured from the load or receiving end) is really a wave progressing from sending end to receiving end of the line. The dashed wave is derived from the rotating vectors at the right, varying in amplitude as $\epsilon^{-\gamma s}$, and it can be seen that this component is truly represented by a wave progressing from the receiving end toward the source, with initial value equal to the incident voltage at the load (for open circuit). This is the reflected wave.

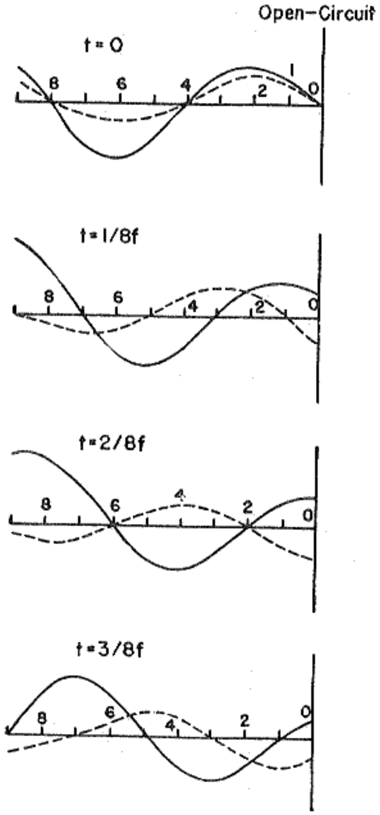

Fig. 6-9. Curves of incident (solid curve) and reflected (dashed curve) current waves for an open-circuited line, at successive time instants.

Therefore the total instantaneous voltage at any point on the line is the vector sum of the voltage of the incident and reflected waves.

From Eq. 6-23 it may be seen that the only difference between the curves for voltage and current (with Z_R infinite or as an open circuit) is in the reversed phase of the reflected current wave. The waves of instantaneous current are plotted in Fig. 6-9. It may be seen that the two current waves are equal and of opposite phase at the open-circuited receiving end, the total instantaneous current at that point always being zero as required by the open circuit.

The relative phase angles of the incident and reflected waves at the load are determined by the term $(Z_R - Z_0)/(Z_R + Z_0)$. Therefore both angle and magnitude of Z_R and Z_0 enter into the determination of the phase angle between the two waves.

In the case of an infinite line ($s = \infty$), or for $Z_R = Z_0$, the second term in the brackets of Eqs. 6-23 and 6-24 becomes zero and the

reflected wave is absent. This effect seems reasonable, since in the case of the infinite line the traveling waves of energy continue in one direction indefinitely. Along such a uniform line of infinite length there is no source of energy or discontinuity to send back a reflected wave along the line. The line terminated in Z_0 behaves in exactly similar fashion, the waves traveling smoothly down the line and the energy being absorbed in the Z_0 load without setting up a reflected wave. In fact, the characteristic impedance may be looked upon as a measure of the natural rate of absorption of energy. Such a line is frequently called a *smooth* line.

It has been mentioned that the quantity actually being transmitted is energy. This energy is conveyed by the electric and magnetic fields traveling or guided along the line. A line terminated in Z_0 shows this value of impedance at all points along the line. For an ideal line with $R = G = 0$ and so terminated, the ratio of voltage to current is a constant given by

$$Z_0 = \frac{E}{I}$$

and the distribution of energy between the electric and magnetic fields is fixed. The energy conveyed in the electric field is

$$W_e = \frac{CE^2}{2} \text{ joules/m}^3$$

and that conveyed in the magnetic field is

$$W_m = \frac{LI^2}{2} \text{ joules/m}^3$$

It will be shown later that for such an ideal line $Z_0 = L/C$. Using this condition for the Z_0-terminated line where $E/I = Z_0$ everywhere, it appears that

$$W_e = W_m$$

(or the electric field energy equals the magnetic field energy) is the physical relation existing everywhere along the ideal line terminated in Z_0.

At a load where $Z_R \neq Z_0$, a different ratio of E_R to I_R

$$Z_R = \frac{E_R}{I_R}$$

is required and a redistribution of energy between the two fields is called for. This redistribution of energy acts as a source to send a reflected wave back along the line.

When the field waves strike an open circuit, for example, the magnetic field must become zero because the current is zero. The energy that was conveyed by the magnetic field cannot be dissipated, so it will appear as additional energy in the electric field, causing an increased voltage to appear. This increased voltage then sets up a returning current wave down the line. At a short circuit it is the electric field that is forced to zero by the condition of zero voltage. The transferred energy causes an increase in the magnetic field, which in turn induces a returning voltage wave down the line.

Any discontinuity in line parameters, such as the junction of an open-wire line to a cable of different Z_0, requires a redistribution of energy between the fields and thus sets up a reflected wave. A long line has an impedance of Z_0 at all points, which then determines the energy distribution in the fields. If the load is also Z_0, the voltage to current ratio is the same in the line and in the load, no energy interchange is required between the fields, and there is no opportunity for reflection to occur.

Reflection is ordinarily considered as undesirable on a transmission line. If the attenuation is not large, the returning wave appears as an echo at the sending end. Also, if reflection is present there is a reduction in efficiency and output because a portion of the received energy is rejected by the load. In passing back down the line as a reflected wave, additional energy is lost because of the R and G of the line. If the impedance of the generator is not Z_0, the reflected wave is reflected again at the sending end, becoming a new incident wave. Energy is thus transmitted back and forth on the line until dissipated in the line losses. Hence a termination in Z_0 with no reflection is desirable.

6-11. Reflection coefficient

The ratio of amplitudes of the reflected and incident *voltage waves at the receiving end of the line* is frequently called the *reflection coefficient.* From Eq. 6-23 with $s = 0$, this ratio is

$$K = \frac{\text{reflected voltage at load}}{\text{incident voltage at load}} = \frac{Z_R - Z_0}{Z_R + Z_0} \tag{6-64}$$

Equations 6-23 and 6-24 then may be written

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}(\epsilon^{\gamma s} + K\epsilon^{-\gamma s}) \tag{6-65}$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}(\epsilon^{\gamma s} - K\epsilon^{-\gamma s}) \tag{6-66}$$

The sign of K, and hence the polarity of the reflected wave, is dependent on the angles and magnitudes of Z_R and Z_0. For a termination of Z_0 the reflection coefficient is zero.

The reflection coefficient will be found to be extremely useful.

6-12. Line calculation

The example of Section 6-5 may be chosen as an illustration of calculation using the transmission line equations with a load other than Z_0. The problem may be stated as follows:

A generator of 1.0 volt, 1000 cycles, supplies power to a 100-mile open-wire line terminated in 200 ohms resistance. The line parameters are

$$R = 10.4 \text{ ohms per mile}$$
$$L = 0.00367 \text{ henry per mile}$$
$$G = 0.8 \times 10^{-6} \text{ mho per mile}$$
$$C = 0.00835\ \mu\text{f per mile}$$

The line constants as computed in Section 6-5 are

$$Z = 25.2\underline{/66^\circ} \text{ ohms per mile}$$
$$Y = 52.6 \times 10^{-6}\underline{/90^\circ} \text{ mho per mile}$$
$$Z_0 = 692\underline{/-12^\circ} \text{ ohms}$$
$$\gamma = 0.0363\underline{/78^\circ}$$
$$\alpha = 0.00755 \text{ neper per mile}$$
$$\beta = 0.0355 \text{ radian per mile}$$
$$\alpha l = 0.755 \text{ neper}$$
$$\beta l = 3.55 \text{ radians} = 203.8^\circ$$

The reflection coefficient K is

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{200 - 692\underline{/-12^\circ}}{200 + 692\underline{/-12^\circ}} = 0.558\underline{/172.8^\circ}$$

From Eqs. 6-29 and 6-64,

$$Z_s = Z_0\left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}}\right)$$

so that values of $\epsilon^{\gamma l}$ and $\epsilon^{-\gamma l}$ are needed.

These values may be readily obtained as

$$\epsilon^{\gamma l} = \epsilon^{\alpha l}\epsilon^{j\beta l} = \epsilon^{0.755}\underline{/203.8^\circ}$$

$$= 2.12\underline{/203.8^\circ}$$

$$\epsilon^{-\gamma l} = \epsilon^{-\alpha l}\epsilon^{-j\beta l} = \epsilon^{-0.755}\underline{/-203.8^\circ}$$

$$= 0.472\underline{/-203.8^\circ}$$

Then

$$Z_s = 692\underline{/-12^\circ}\left(\frac{2.12\underline{/203.8^\circ} + 0.558\underline{/172.8^\circ} \times 0.472\underline{/-203.8^\circ}}{2.12\underline{/203.8^\circ} - 0.558\underline{/172.8^\circ} \times 0.472\underline{/-203.8^\circ}}\right)$$

$$= 692\underline{/-12^\circ}\left(\frac{1.975\underline{/210^\circ}}{2.285\underline{/198.5^\circ}}\right)$$

$$= 692\underline{/-12^\circ} \times 0.864\underline{/11.5^\circ} = 597\underline{/-0.5^\circ}$$

The input current then is

$$I_s = \frac{E_s}{Z_s} = \frac{1.0}{597\underline{/-0.5^\circ}} = 0.00167\underline{/0.5^\circ} \text{ amp, } E_s \text{ reference}$$

The received current I_R then is obtainable from

$$I_s = \frac{I_R(Z_R + Z_0)}{2Z_0}(\epsilon^{\gamma l} - K\epsilon^{-\gamma l})$$

Most of the terms in the above equation have already been computed, so that

$$0.00167\underline{/0.5^\circ} = \frac{I_R(888\underline{/-9.5^\circ})}{1384\underline{/-12^\circ}}(2.285\underline{/198.5^\circ})$$

$$I_R = \frac{2.31\underline{/-11.5^\circ}}{2030\underline{/189^\circ}}$$

$$= 0.00113\underline{/-200.5^\circ} \text{ amp, } E_s \text{ reference}$$

The load voltage E_R then is

$$E_R = I_R Z_R = 0.00113\underline{/-200.5^\circ} \times 200$$

$$= 0.226\underline{/-200.5^\circ} \text{ v, } E_s \text{ reference}$$

The power delivered to the load is

$$P_R = I_R{}^2 R = 0.00113^2 \times 200 = 0.000255 \text{ watt}$$

The power input to the line is

$$P_s = E_s I_s \cos\theta = 1.0 \times 0.00167 \cos 0.5^\circ = 0.00167 \text{ watt}$$

If the efficiency of transmission is of interest; it is

$$\eta = \frac{0.000255}{0.00167} \times 100\% = 15.2\%$$

6-13. Input and transfer impedance

The input impedance of a transmission line has already been obtained as

$$Z_s = Z_0\left(\frac{Z_R \cosh\gamma l + Z_0 \sinh\gamma l}{Z_0 \cosh\gamma l + Z_R \sinh\gamma l}\right) \tag{6-67}$$

In terms of exponentials, this is

$$Z_s = Z_0\left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}}\right) \tag{6-68}$$

If the voltage at the sending-end terminals is known, it is convenient to have the transfer impedance so that the received current can be computed directly. The sending-end voltage E_s is

$$E_s = \frac{E_R(Z_R + Z_0)}{2Z_R}(\epsilon^{\gamma l} + K\epsilon^{-\gamma l})$$

$$= \frac{I_R(Z_R + Z_0)}{2}(\epsilon^{\gamma l} + K\epsilon^{-\gamma l}) \tag{6-69}$$

for which the transfer impedance is

$$Z_T = \frac{E_s}{I_R} = \frac{(Z_R + Z_0)}{2}(\epsilon^{\gamma l} + K\epsilon^{-\gamma l}) \tag{6-70}$$

By substituting for K, Eq. 6-70 becomes

$$Z_T = Z_R\left(\frac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{2}\right) + Z_0\left(\frac{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}{2}\right)$$

which is recognizable as

$$Z_r = Z_R \cosh \gamma l + Z_0 \sinh \gamma l \qquad (6\text{-}71)$$

if the expression is desired in terms of the hyperbolic functions.

6-14. Open- and short-circuited lines

As limiting cases it is convenient to consider lines terminated in open circuit or short circuit, that is, with $Z_R = \infty$ or $Z_R = 0$. The input impedance of a line of length l is

$$Z_s = Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right)$$

and for the short-circuit case $Z_R = 0$, so that

$$Z_{sc} = Z_0 \tanh \gamma l \qquad (6\text{-}72)$$

Before the open-circuit case is considered, the input impedance should be written

$$Z_s = Z_0 \left[\frac{\cosh \gamma l + (Z_0/Z_R) \sinh \gamma l}{(Z_0/Z_R) \cosh \gamma l + \sinh \gamma l} \right]$$

The input impedance of the open-circuited line of length l, with $Z_R = \infty$, is

$$Z_{oc} = Z_o \coth \gamma l \qquad (6\text{-}73)$$

By multiplying Eqs. 6-72 and 6-73 it can be seen that

$$Z_o = \sqrt{Z_{oc} Z_{sc}} \qquad (6\text{-}74)$$

This is the same result as was obtained for a lumped network. Equation 6-74 supplies a very valuable means of experimentally determining the value of Z_0 of a line.

Also, from the same two equations,

$$\tanh \gamma l = \sqrt{\frac{Z_{sc}}{Z_{oc}}} \qquad (6\text{-}75)$$

or

$$\gamma l = \tanh^{-1} \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

Use of this equation in experimental work requires the determination of the hyperbolic tangent of a complex angle. If

$$\tanh \gamma l = \tanh (\alpha + j\beta) l = U + jV$$

then it can be shown that

$$\tanh 2\alpha l = \frac{2U}{1 + U^2 + V^2} \qquad (6\text{-}76)$$

and

$$\tan 2\beta l = \frac{2V}{1 - U^2 - V^2} \qquad (6\text{-}77)$$

The value of β is uncertain as to quadrant. Its proper value may be selected if the approximate velocity of propagation is known.

6-15. Reflection factor and reflection loss

If Z_2 is not equal to Z_1, Fig. 6-10, the impedance mismatch causes a change in the ratio of voltage to current, or of energy transmitted

Fig. 6-10. Generator of impedance Z_1 connected to load Z_2.

by the electric field to that transmitted by the magnetic field, and thus a portion of the energy is reflected by the load. The energy delivered to the load may be less than would be delivered if impedances were matched; it is said that a *reflection loss* has occurred.

The magnitude of the reflection loss may be determined by computing the ratio of the current actually flowing in the load to that which would flow if the impedances were matched at the terminals in question. Image matching between a given generator and load might be obtained by insertion of an ideal transformer and a lossless phase shifter between source and load. According to the theory of the ideal transformer from Section 3-9,

$$\frac{I_1}{I_2} = \sqrt{\frac{Z_2}{Z_1}} \qquad (6\text{-}78)$$

For image matching the magnitude of Z_2 may be adjusted to that of Z_1 by choosing the proper transformer ratio, and the phase angle of Z_2 may be adjusted to that of Z_1 by operation of the phase shifter. Under these theoretical conditions Z_2 is image matched to Z_1, and the current which would then flow through the generator would be

$$I_1 = \frac{E}{2Z_1} \tag{6-79}$$

Then the current I_2' that would flow in the load, or secondary of the transformer, *under image matching*, would be given by use of Eqs. 6-78 and 6-79 as

$$I_2' = \frac{E}{2Z_1}\sqrt{\frac{Z_1}{Z_2}} = \frac{E}{2\sqrt{Z_1Z_2}} \tag{6-80}$$

whereas *without image matching* this current would have been

$$|I_2| = \frac{|E|}{|Z_1 + Z_2|} \tag{6-81}$$

Hence the ratio of the current actually flowing in the load to that which might flow under image matched conditions is

$$\left|\frac{I_2}{I_2'}\right| = \frac{\dfrac{|E|}{|Z_1 + Z_2|}}{\dfrac{|E|}{|2\sqrt{Z_1Z_2}|}} = \frac{|2\sqrt{Z_1Z_2}|}{|Z_1 + Z_2|} \tag{6-82}$$

This ratio indicates the change in current in the load due to reflection at the mismatched junction and is called the *reflection factor*, given the symbol k, where

$$k = \left|\frac{2\sqrt{Z_1Z_2}}{Z_1 + Z_2}\right| \tag{6-83}$$

It should be noted that Z_1 and Z_2 are the impedances seen looking both ways at any junction.

The reflection loss is defined as the number of nepers or decibels by which the current in the load under image matched conditions would exceed the current actually flowing in the load. Thus the reflection loss involves the reciprocal of k, or

$$\text{reflection loss, nepers} = \ln\left|\frac{Z_1 + Z_2}{2\sqrt{Z_1Z_2}}\right| \tag{6-84}$$

$$\text{reflection loss, db} = 20 \log\left|\frac{Z_1 + Z_2}{2\sqrt{Z_1Z_2}}\right| \tag{6-85}$$

The use of the word *reflection* here is unfortunate, and *mismatching loss* would be a better choice, since the factor k measures the relative gain or loss of actual terminations with respect to image matching.

The reflection loss may be plotted in terms of the impedance ratio

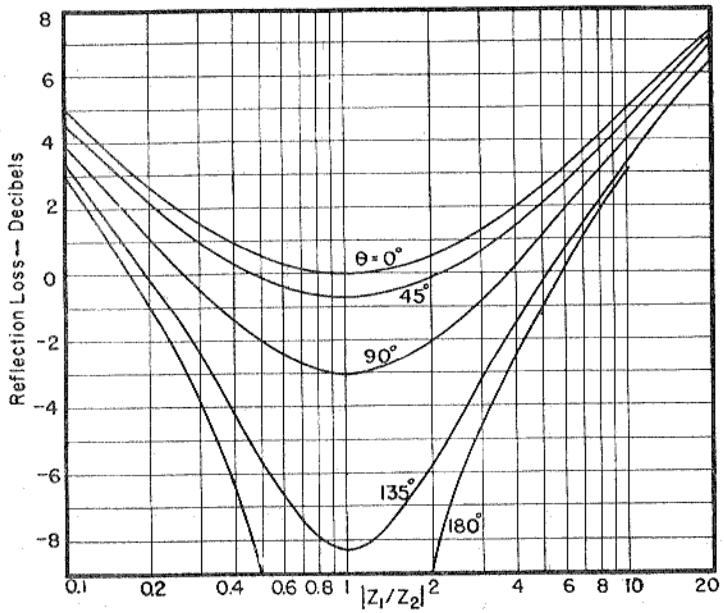

Fig. 6-11. Reflection loss due to mismatch between source and load impedances. The angle θ is the angle of Z_1/Z_2. Note that curves are symmetrical about unity on the abscissa.

$|Z_1/Z_2|$ or $|Z_2/Z_1|$ as in Fig. 6-11. It should be noted that on the logarithmic scale used, the curves are symmetric about the axis $|Z_1/Z_2| = 1$. The angle θ represents the angle of Z_1/Z_2. It can be seen that for certain conditions a negative reflection loss or a reflection gain is obtained.

6-16. Insertion loss

The insertion of a four-terminal network or a line between a generator and a load may improve or diminish the impedance match between source and load, and may also introduce dissipative elements. The net effect may be to improve or reduce the impedance match and thus to increase or decrease the power delivered to the load, resulting in a positive or negative *insertion loss*, due to insertion of the network. *The insertion loss of a line or network is defined as the number of nepers or decibels by which the current in the load is changed by the insertion.* Occasionally the insertion of a network causes an increase in load current, and this would be expected from the matching networks of Chapter 3, if made of low dissipation elements. Such an increase in load current represents a negative loss, or an *insertion gain*.

Therefore the insertion loss is the resultant of several individual losses. In Fig. 6-12, if Z_s is not equal to Z_g at the 1,1 terminals, then

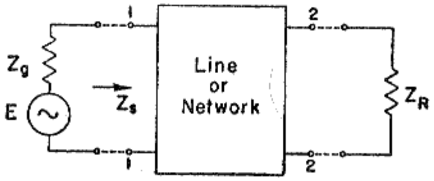

Fig. 6-12. Illustrating insertion loss.

a reflection loss occurs at that point. The line or network also may introduce an attenuation loss. If the impedances are not matched at the 2,2 terminals, then a second reflection loss occurs there. The over-all insertion loss due to insertion of a line or network between a generator and a load can be calculated from the ratio of the load current that would flow if the load were directly connected to the generator to the current actually flowing in the load.

The insertion loss due to a *line* introduced between source and load in Fig. 6-12 can be calculated from the line equations as an example. The sending-end current can be written

$$I_s = \frac{I_R(Z_R + Z_0)}{2Z_0}(\epsilon^{\gamma l} - K\epsilon^{-\gamma l}) \qquad (6\text{-}86)$$

or

$$I_s = \frac{E}{Z_g + Z_s}$$

The input impedance Z_s has been obtained as

$$Z_s = Z_0\left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}}\right)$$

so that

$$I_s = \frac{E(\epsilon^{\gamma l} - K\epsilon^{-\gamma l})}{Z_g(\epsilon^{\gamma l} - K\epsilon^{-\gamma l}) + Z_0(\epsilon^{\gamma l} + K\epsilon^{-\gamma l})} \qquad (6\text{-}87)$$

Substitution of Eq. 6-87 in 6-86 and solution for I_R gives

$$I_R = \frac{2Z_0E}{(Z_R + Z_0)[Z_g(\epsilon^{\gamma l} - K\epsilon^{-\gamma l}) + Z_0(\epsilon^{\gamma l} + K\epsilon^{-\gamma l})]}$$

Introducing the value for K, the reflection coefficient, permits this equation to be written as

$$I_R = \frac{2Z_0E}{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\gamma l} + (Z_0 - Z_g)(Z_R - Z_0)\epsilon^{-\gamma l}} \qquad (6\text{-}88)$$

This is the value of current actually flowing in the load Z_R.

Without the line present, the current I_R' flowing in the load would be

$$I_R' = \frac{E}{Z_g + Z_R} \qquad (6\text{-}89)$$

The ratio of the current that would flow in the load, if generator and load were directly connected, to that flowing with the line inserted is

$$\frac{I_R'}{I_R} = \frac{\dfrac{E}{Z_g + Z_R}}{\dfrac{2Z_0E}{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\gamma l} + (Z_0 - Z_g)(Z_R - Z_0)\epsilon^{-\gamma l}}}$$

$$= \frac{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\alpha l}\epsilon^{j\beta l} + (Z_0 - Z_g)(Z_R - Z_0)\epsilon^{-\alpha l}\epsilon^{-j\beta l}}{2Z_0(Z_g + Z_R)} \qquad (6\text{-}90)$$

If α is large or the line is sufficiently long, the second term in the numerator may be neglected with respect to the first, leaving

$$\frac{I_R'}{I_R} = \frac{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\alpha l}\epsilon^{j\beta l}}{2Z_0(Z_g + Z_R)}$$

Greater physical meaning may be given to the expression if both numerator and denominator are multiplied by $2\sqrt{Z_gZ_R}$, giving

$$\frac{I_R'}{I_R} = \frac{2\sqrt{Z_gZ_R}\,(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\alpha l}\epsilon^{j\beta l}}{4\sqrt{Z_gZ_RZ_0^2}\,(Z_g + Z_R)} \qquad (6\text{-}91)$$

The insertion loss is to be calculated as a function of current magnitudes only, so that after taking absolute values and rearranging, the above expression becomes

$$\left|\frac{I_R'}{I_R}\right| = \frac{|Z_g + Z_0|}{2\sqrt{Z_gZ_0}} \cdot \frac{|Z_R + Z_0|}{2\sqrt{Z_RZ_0}} \cdot \frac{2\sqrt{Z_gZ_R}}{|Z_g + Z_R|} \cdot \epsilon^{\alpha l} \qquad (6\text{-}92)$$

The coefficients on the right side may be recognized as reflection factors, where

$$\frac{2\sqrt{Z_gZ_0}}{|Z_g + Z_0|} = k_s$$

and may be considered as the reflection factor at the 1,1 terminals where the generator may be mismatched at its junction with the line. The line has been assumed long, or α large; thus its input impedance appears to be Z_0. The second term is another reflection factor

$$\frac{2\sqrt{Z_RZ_0}}{|Z_R + Z_0|} = k_R$$

and can be seen as the reflection factor at the 2,2 terminals, or the junction between line and load. The third term is

$$\frac{2\sqrt{Z_gZ_R}}{|Z_g + Z_R|} = k_{sR}$$

and may be considered as a reflection factor occurring if the generator were directly connected to the load. The fourth term is $\epsilon^{\alpha l}$ the loss in the line.

The current ratio then is

$$\left|\frac{I_R'}{I_R}\right| = \frac{k_{sR}}{k_sk_R}\,\epsilon^{\alpha l} \qquad (6\text{-}93)$$

The insertion loss may then be obtained by taking the logarithm of the current ratio:

$$\text{insertion loss, nepers} = \ln\frac{1}{k_s} + \ln\frac{1}{k_R} - \ln\frac{1}{k_{sR}} + \alpha l \qquad (6\text{-}94)$$

$$\text{insertion loss, db} = 20\left(\log\frac{1}{k_s} + \log\frac{1}{k_R} - \log\frac{1}{k_{sR}} + 0.4343\alpha l\right) \qquad (6\text{-}95)$$

The presence of the term $\ln 1/k_{sR}$ may be perplexing. This loss has been identified as the reflection loss that would occur if the generator were connected directly to the load. As such, it was eliminated by definition, since it is not due to *insertion* of the line, and accordingly the equation shows this loss as subtracted.

If the line conditions do not justify the assumptions concerning α or l, the insertion loss may be determined by direct computation of the current in the load, with and without the line or network present.

The expression above may also apply directly to a single section network, by use of $l = 1$, and the proper value of α applying to the network.

6-17. T and π sections equivalent to lines

In Chapter 1, relations were developed permitting the design of an equivalent T section from measurements on a network. These relations were

$$Z_1 = Z_{1oc} - \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}$$
$$Z_2 = Z_{2oc} - \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}$$
$$Z_3 = \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}$$

The input impedances of open- and short-circuited lines were developed in Section 6-14 as

$$Z_{1oc} = \frac{Z_0}{\tanh \gamma l} = Z_0\left(\frac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}\right)$$

$$Z_{1sc} = Z_0 \tanh \gamma l = Z_0\left(\frac{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}\right)$$

Since a line is a symmetrical network,

$$Z_{1oc} = Z_{2oc}$$

The Z_3 or shunt element of a T section that will be equivalent, in so far as external voltages and currents are concerned, to the long line can then be readily obtained as

$$Z_3 = \sqrt{\frac{Z_0}{\tanh \gamma l}\left(\frac{Z_0}{\tanh \gamma l} - Z_0 \tanh \gamma l\right)}$$
$$= \frac{Z_0}{\sinh \gamma l} \qquad (6\text{-}96)$$

The series elements for the equivalent section then are

$$Z_1 = Z_2 = Z_{1oc} - Z_3 = Z_0\left(\frac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}} - \frac{2}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}\right)$$

$$= Z_0\left[\frac{(\epsilon^{\gamma l/2} - \epsilon^{\gamma l/2})^2}{(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})(\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2})}\right] = Z_0\left(\frac{\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2}}{\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2}}\right)$$

$$Z_1 = Z_2 = Z_0 \tanh\frac{\gamma l}{2} \qquad (6\text{-}97)$$

The T-section equivalent for the long line, made up of these elements, is shown in Fig. 6-13(a). It is useful in certain types of line calculations.

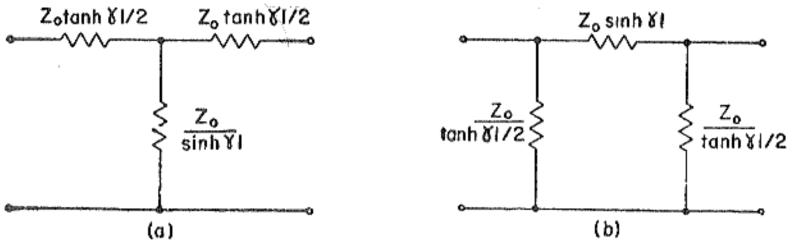

Fig. 6-13. (a) T-section equivalent circuit for a transmission line; (b) same as a π section.

A π-section equivalent for the line may likewise be determined from the terminal measurements as in Chapter 1. Because of symmetry,

$$Z_A = Z_C = \frac{Z_{2oc}Z_{1sc}}{Z_{2oc} - \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}}$$

$$= \frac{Z_0^2}{Z_0\left(\dfrac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}\right) - \dfrac{2Z_0}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}}$$

$$= \frac{Z_0(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})(\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2})}{(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})^2}$$

$$Z_A = Z_C = \frac{Z_0}{\tanh(\gamma l/2)} \qquad (6\text{-}98)$$

The Z_B arm can be easily obtained as

$$Z_B = \frac{Z_{2oc}Z_{1sc}}{\sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}}$$

$$= \frac{Z_0^2}{Z_0/\sinh\gamma l} = Z_0 \sinh\gamma l \qquad (6\text{-}99)$$

The equivalent π section for a line is shown at (b), Fig. 6-13.

6-18. Distance to a line fault

The input impedance of a line is

$$Z_s = Z_0\left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}}\right)$$

$$= Z_0\left(\frac{\epsilon^{\alpha l}\epsilon^{j\beta l} + K\epsilon^{-\alpha l}\epsilon^{-j\beta l}}{\epsilon^{\alpha l}\epsilon^{j\beta l} - K\epsilon^{-\alpha l}\epsilon^{-j\beta l}}\right) \qquad (6\text{-}100)$$

Upon substituting for $\epsilon^{j\beta l}$ and $\epsilon^{-j\beta l}$,

$$Z_s = Z_0\left[\frac{(\epsilon^{\alpha l} + K\epsilon^{-\alpha l})\cos\beta l + j(\epsilon^{\alpha l} - K\epsilon^{-\alpha l})\sin\beta l}{(\epsilon^{\alpha l} - K\epsilon^{-\alpha l})\cos\beta l + j(\epsilon^{\alpha l} + K\epsilon^{-\alpha l})\sin\beta l}\right]$$

$$= Z_0\left[\frac{(\epsilon^{\alpha l} + K\epsilon^{-\alpha l}) + j(\epsilon^{\alpha l} - K\epsilon^{-\alpha l})\tan\beta l}{(\epsilon^{\alpha l} - K\epsilon^{-\alpha l}) + j(\epsilon^{\alpha l} + K\epsilon^{-\alpha l})\tan\beta l}\right] \qquad (6\text{-}101)$$

This expression shows that the input impedance magnitude of the line is periodic and of period $\beta l = \pi$. Thus the input impedance oscillates between maximum and minimum values as the length of line is increased. Measurements of input impedance magnitude against length of line appear as in Fig. 6-14.

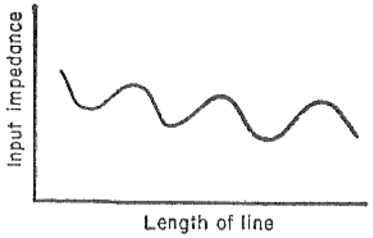

Fig. 6-14. Variation of input impedance of a line not terminated in Z_0, as the electrical length of line is increased.

Since it is βl or the electrical length that is important, the value of βl may also be changed by increasing the frequency applied to the line. The variation of impedance will appear as in Fig. 6-14, with length replaced by frequency. By the above reasoning, the difference between two maximum impedance points represents a change in electrical length of one-half wavelength. For the lower of the two frequencies of maximum impedance, the distance s to the reflecting point in terms of wavelengths is

$$s = p\lambda_1 \tag{6-102}$$

For the next higher frequency at which maximum input impedance occurs, the distance s is the same, but electrically the line is one-half wavelength longer, so that

$$s = \left(p + \frac{1}{2}\right)\lambda_2 \tag{6-103}$$

Using Eq. 6-102,

$$s = \left(\frac{s}{\lambda_1} + \frac{1}{2}\right)\lambda_2 = \frac{\lambda_1\lambda_2}{2(\lambda_1 - \lambda_2)}$$

Since $v = \lambda f$,

$$s = \frac{v_1 v_2}{2(v_1 f_2 - v_2 f_1)} \tag{6-104}$$

The velocities may be determined from $v = \omega/\beta$ by use of the known line parameters. Equation 6-104 then allows determination of the distance to a line fault or point of reflection by measurements made from one end of the line.

In many cases the velocity does not change appreciably for the frequency range between f_1 and f_2, so that Eq. 6-104 may be simplified by assuming $v_1 = v_2$, giving

$$s = \frac{v}{2(f_2 - f_1)} \tag{6-105}$$

as the distance to the reflecting point. This reflecting point may be a line fault whose location is desired.

PROBLEMS

6-1. A simulated line is composed of T sections of pure resistance, $Z_1 = Z_2 = 50$ ohms, $Z_3 = 4000$ ohms.

(a) Find α and Z_0.

(b) A line composed of 50 such sections in series is terminated in its characteristic impedance. A generator of 1 v, 400 ohms internal resistance is at the sending end. Find I_s and I_R.

(c) What is the decibel loss in the line?

6-2. A 60-mile length of 0.104 in. diameter open-wire line (see Table 4) is terminated in Z_0. A generator of 600 ohms internal resistance and 1 v, 800 c, is connected to the sending end. Find I_s, I_R and power output of generator, and power delivered to the load. Also determine wavelength and velocity of propagation.

TABLE 4

CHARACTERISTICS OF CERTAIN TELEPHONE LINES AND CABLES
(per loop mile)

Type	R ohms	L henrys	C μf	G μmhos	Wire spacing, in.
0.165-in. diameter open wire	4.11	0.00311	0.00996	0.14	8
0.128-in. diameter open wire	6.74	0.00353	0.00871	0.29	12
0.104-in. diameter open wire	10.15	0.00393	0.00797	0.29	18
19-gauge cable	85.8	0.001	0.062	1.5	..
16-gauge cable	42.1	0.001	0.062	1.5	..
19-gauge cable, loaded*	92.2	0.078	0.062	1.5	..

* Coil spacing, 6000 feet; inductance 88 mh; resistance 7.3 ohms.

6-3. Over a range of 100 to 10,000 c, plot the values of α, β, and velocity for a mile of 19-gauge cable.

6-4. A line of 0.165-in. open wire is 100 miles long and terminated in Z_0. Find Z_0, α, β, γ, v, and λ for a 500-c signal.

6-5. How much voltage must be applied to the sending end of a 30-mile line having $R = 21.4$ ohms per mile, $L = 0.001$ h per mile, $C = 0.062$ μf per mile, $G = 0.868$ μmho per mile, and terminated in Z_0, if the received power is to be at -20-db level (reference $= 0.006$ watt, $\omega = 5000$)?

6-6. A line is one wavelength long and is short-circuited. If it is of 16-gauge cable and 1 v, 800 c, is applied to it, compute values of incident, reflected, and total voltage at $\lambda/8$ intervals along the line.

6-7. A line of 16-gauge cable is 60 miles long and is terminated in a load of $400 + j300$ ohms. The received power is to be at a level of -10 db (0.001 watt reference). If the frequency is 796 cycles, find

(a) Sending end voltage, current, and power.

(b) Power loss in the line.

(c) Wavelength and velocity of propagation.

6-8. How much inductive loading (per mile) is required to make a 16-gauge cable (Table 4) distortionless? Assume no increase in R.

6-9. To what value must the shunt conductance per mile have to be changed to make the 0.128-in. open-wire line of Table 4 distortionless? How much attenuation, in decibels per mile, is then added?

6-10. If the line of Prob. 6-7 is short-circuited at the receiving end, find the sending-end current for a generator of 1 v, 500 ohms internal resistance, with $\omega = 4000$. Find the received current also.

6-11. A 16-gauge cable, 50 miles long, is inserted between a generator of 2 v, 600 ohms, $\omega = 5000$, and a load of $400 + j400$ ohms. Find the decibel loss in power in the load due to the presence of the cable.

6-12. A 19-gauge cable 32 miles long is supplied by a generator of 2 v, 400 ohms, 1200 c, and terminated in Z_0. Find the insertion loss of the line, in decibels.

6-13. Thirty miles of 0.165-in. open wire is supplied by a generator of 2 v, 600 ohms resistance, and loaded with $300 + j400$ ohms. The frequency is 1000 c. Find

(a) Values of I_R, I_s, E_R, and received power.

(b) Value of the incident and reflected voltages at $s = 0$, $s = 10$, $s = 20$, $s = 30$ miles.

6-14. A 50-mile line has the following measurements made at 1200 c:

$$Z_{1oc} = 200/\underline{-42^\circ} \text{ ohms}; \quad Z_{1sc} = 1890/\underline{22^\circ} \text{ ohms}$$

Find the value of Z_0, α, β, and v for this line. The approximate velocity is 20,000 miles per second.

6-15. Plot a curve of input impedance magnitude vs length of line by 20-mile steps up to 200 miles for the 0.104-in. open-wire line at a frequency of 1000 c. The line is open-circuited.

6-16. A line of 16-gauge cable is to be loaded to improve performance with $\omega = 5000$.

(a) Find Z_0, α, β, v, λ unloaded.

(b) The line is loaded with $L = 0.246$ h and $R = 7.3$ ohms at intervals of 7.88 miles. Assume this loading to be distributed. Recalculate Z_0, α, β, v, λ.

(c) Discuss effects of differences noted in line values calculated in (b).

6-17. Calculate the decibel attenuation in 100 miles of the cable of Prob. 6-16, with and without loading.

6-18. Determine the power delivered to a load of 250 ohms resistance by 30 miles of the line of Prob. 6-16 for both the loaded and unloaded case. The line is supplied by a generator of 1 v, $\omega = 5000$, zero internal resistance.

6-19. Design an equivalent T section for 40 miles of 0.104-in. open-wire line at 1000 c.

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