Chapter 4

FILTERS

Resonant circuits that will select relatively narrow bands of frequencies and reject others have already been discussed. Certain other reactive networks are available that will freely pass desired bands of frequencies while almost totally suppressing other bands of frequencies. Such reactive networks, called *filters*, were first investigated by G. A. Campbell and O. J. Zobel of the Bell Telephone Laboratories.

An ideal filter would pass all frequencies in a given band without reduction in magnitude, and totally suppress all other frequencies. Such ideal performance is not possible but can be approached with complex designs, if the need warrants. Filter circuits are widely used and vary in complexity from the relatively simple power-supply filter of the a-c operated radio receiver to complex filter sets used to separate the various voice channels in carrier-frequency telephone circuits. Whenever alternating currents occupying different frequency bands are to be separated, filter circuits have an application.

Here analysis of filter circuits is carried out on the basis of certain definitions from the general field of electric network theory, *under the assumption of symmetrical network sections.*

4-1. The neper; the decibel

In filter circuits and other electric networks it is frequently convenient to appraise the performance of a circuit in terms of the ratio of input-current to output-current magnitude. If the input and output image impedances, or the ratios of voltage to current at input and output of the network, are equal, then the ratios of the input to output currents, or input to output voltages, may equally well be written

$$\left|\frac{I_1}{I_2}\right| = \left|\frac{V_1}{V_2}\right| \tag{4-1}$$

If several networks are used in succession as in Fig. 4-1, the over-all

performance may be appraised as

$$\left|\frac{V_1}{V_2}\right| \times \left|\frac{V_2}{V_3}\right| \times \left|\frac{V_3}{V_4}\right| \times \cdots \left|\frac{V_{n-1}}{V_n}\right| = \left|\frac{V_1}{V_n}\right| \tag{4-2}$$

which may also be stated as

$$A_1\underline{/\alpha} \times A_2\underline{/\beta} \times A_3\underline{/\gamma} \times A_4\underline{/\delta} = A_1A_2A_3A_4\underline{/\alpha + \beta + \gamma + \delta}$$

both processes employing multiplication of magnitudes. In general, the process of addition or subtraction may be carried out with

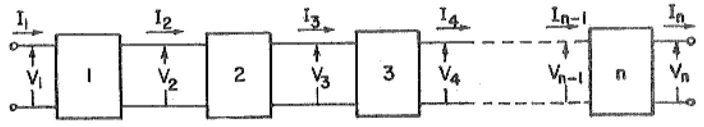


Fig. 4-1. A succession of n networks in cascade.

greater ease than the process of multiplication or division. It is therefore of interest to note that

$$\epsilon^a \times \epsilon^b \times \epsilon^c \times \cdots \epsilon^n = \epsilon^{a+b+c+\cdots n}$$

is an application in which addition is substituted for multiplication. If the voltage ratios of Eq. 4-2 are defined as

$$\left|\frac{V_1}{V_2}\right| = \epsilon^a; \quad \left|\frac{V_2}{V_3}\right| = \epsilon^b; \quad \left|\frac{V_3}{V_4}\right| = \epsilon^c; \quad \text{etc.}$$

then Eq. 4-2 becomes

$$\left|\frac{V_1}{V_n}\right| = \epsilon^{a+b+c+\cdots n}$$

and if the natural logarithm (ln)* of both sides is taken, then

$$\ln\left|\frac{V_1}{V_n}\right| = a + b + c + \cdots n \tag{4-3}$$

Consequently, if the ratio of each individual network is given as ϵ to an exponent, the logarithm of the current or voltage ratio for all the networks in series is very easily obtained as the simple sum of the various exponents. It has become common, for this reason, to

* In this text the abbreviation "ln" will be used to indicate the natural logarithm, whereas "log" will indicate the logarithm to the base 10.

define

$$\left|\frac{V_1}{V_2}\right| = \left|\frac{I_1}{I_2}\right| = \epsilon^{N} \tag{4-4}$$

under conditions of equal impedance associated with input and output circuits. The unit of "N" has been given the name *neper* and defined as

$$\text{N nepers} = \ln\left|\frac{V_1}{V_2}\right| = \ln\left|\frac{I_1}{I_2}\right| \tag{4-5}$$

Two voltages, or currents, differ by one neper when one of them is ϵ times as large as the other.

Obviously, ratios of input to output power may also be expressed in this fashion. That is,

$$\frac{P_1}{P_2} = \epsilon^{2N}$$

The number of nepers represents a convenient measure of the power loss or gain of a given network. Losses or gains of successive networks then may be introduced by addition or subtraction of their appropriate N values.

The telephone industry proposed and has popularized a similar unit based on logarithms to the base 10, naming the unit the *bel* for Alexander Graham Bell. The bel is defined as the logarithm of a power ratio,

$$\text{number of bels} = \log\frac{P_1}{P_2}$$

It has been found that a unit one-tenth as large is more convenient, and the smaller unit is called the *decibel*, abbreviated "db," defined as

$$\text{db} = 10\log\frac{P_1}{P_2} \tag{4-6}$$

For the case of equal impedances in input and output circuits,

$$\text{db} = 20\log\frac{I_1}{I_2} = 20\log\frac{V_1}{V_2} \tag{4-7}$$

Equating the values for the power ratios,

$$\epsilon^{2N} = 10^{\text{db}/10}$$

and taking the logarithm of both sides,

$$8.686\ \text{N} = \text{db}$$

or

$$1\ \text{neper} = 8.686\ \text{db}$$

is obtained as the relation between nepers and decibels.

Most of the energy transmitted through electric networks, lines, and filters is ultimately converted to acoustic energy and heard by the human ear as sound. Further substantiation of the logic of a logarithmic unit for measurement of energy to be ultimately delivered to the ear is furnished by an important property of the human ear. This property may be stated as "the ear hears logarithmically." More elaborately, the ear is observed to obey the Weber-Fechner law, which states, "The change in stimulus necessary to produce a perceptible change in response is proportional to the stimulus already existing." The ear hears sound intensities on a proportional, or logarithmic, scale and not on a linear one. The loudness L may be expressed as

$$L = K\log_b S$$

where K is a constant of proportionality, b is the base of the logarithms, and S is the sound power.

A sound of original power S_1 and loudness L_1 may be caused to increase in power to S_2 units, with loudness L_2. The change in loudness is then

$$\Delta L = L_2 - L_1 = K\log_b S_2 - K\log_b S_1$$

$$\Delta L = K\log_b\frac{S_2}{S_1} \tag{4-8}$$

Upon assigning values to K and b, the change in power is expressed in

nepers, if $K = 0.5$ and the base $b = \epsilon$

bels, if $K = 1.0$ and the base $b = 10$

decibels, if $K = 10$ and the base $b = 10$

The logic of the logarithmic units, the neper and the decibel, is then apparent.

The use of these logarithmic units may be illustrated by a few examples. For instance, under one condition the output of a network is 2 watts. Under a changed condition the output is 3.2 watts. The output is then said to have changed by

$$10 \log \frac{3.2}{2} = 2.04 \text{ db}$$

If this were acoustic power reaching the ear, the change would appear as a noticeable increase, since 1 decibel represents approximately the minimum perceptible audible change.

If instead of increasing the power to 3.2 watts it had been reduced to 0.5 watt, then

$$10 \log \frac{0.5}{2.0} = -10 \log \frac{2.0}{0.5} = -6.02 \text{ db}$$

the minus sign indicating that a reduction in power has taken place.

Although the decibel as discussed is a *power ratio*, it can be used for absolute measurements if a certain reference, or zero level, for P_1 is adopted beforehand and known or stated. Various reference levels which have been used in the telephone and broadcasting industries are 1, 6, 10, and 12.5 milliwatts. Until a uniform agreement on a single standard for *zero level* has been reached, it will always be necessary to state specifically which level is meant as reference.

Using the 6-milliwatt level as reference, the original 2-watt output of the network above is

$$10 \log \frac{2}{0.006} = 10 \times 2.523 = 25.23 \text{ db above reference}$$

After the change to 3.2 watts the output is 27.27 db above zero, the reduction to 0.5 watt output changing the level to 19.21 db above the reference. These figures would again indicate that a loss of −6.02 db had taken place.

When referred to a reference, negative decibel values are powers smaller than the reference level.

When the concept of logarithmic power ratios is carried into other fields, various standard references are adopted. As an example, in acoustic measurements sound levels are measured in decibels with reference or zero power level of 10^{-16} watt per square centimeter.

4-2. Characteristic impedance of symmetrical networks

When $Z_1 = Z_2$ or the two series arms of a T network are equal, or $Z_a = Z_c$ and the shunt arms of a π network are equal, the networks are said to be *symmetrical*.

Filter networks are ordinarily set up as symmetrical sections, basically of the T or π type, such as shown at (b) and (d), Fig. 4-2.

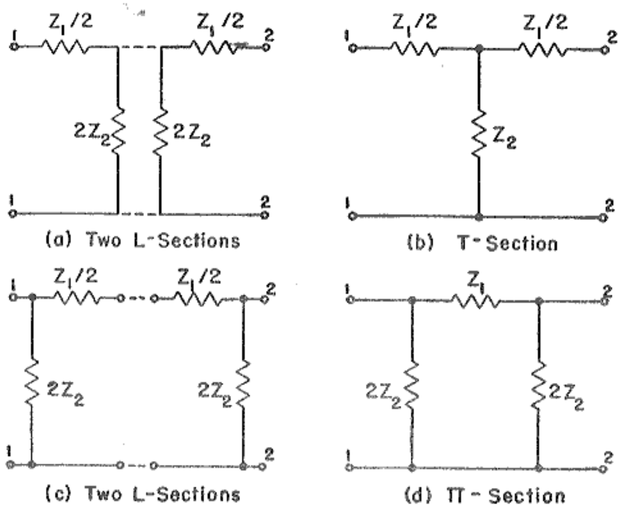


Fig. 4-2. The T and π sections as derived from unsymmetrical L sections, showing notation used in symmetrical network analysis.

Attention is called to the peculiarities of notation employed on the various arms. This peculiarity is largely dictated by custom, arising from the fact that both T and π networks can be considered as built of unsymmetrical L half sections, connected together in one fashion for the T network, and oppositely for the π network as at (a) and (c), Fig. 4-2. A series connection of several T or π networks leads to so-called "ladder networks," which are indistinguishable one from the other except for the end or terminating L half sections, as can be seen in Fig. 4-3.

For a symmetrical network the image impedances Z_{1i} and Z_{2i}, of Eqs. 3-15 and 3-16, are equal to each other, and the image impedance is then called the *characteristic impedance* or the *iterative impedance*, Z_0. That is, if a symmetrical T network is terminated in Z_0, its input impedance will also be Z_0, or its impedance transformation ratio is unity. The term iterative impedance is apparent if the terminating impedance Z_0 is considered as the input impedance

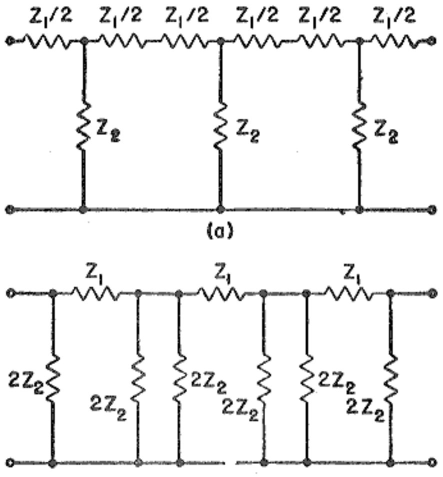


Fig. 4-3. (a) Ladder network made from T sections; (b) ladder network built from π sections. The parallel shunt arms will be combined.

of a chain of similar networks, in which case Z_0 is iterated at the input to each network.

The value of Z_0 for a symmetrical network can be easily determined. For the T network of Fig. 4-4(a), terminated in an impedance Z_0, the input impedance is

$$Z_{1\,\mathrm{in}} = \frac{Z_1}{2} + \frac{Z_2(Z_1/2 + Z_0)}{Z_1/2 + Z_2 + Z_0} \qquad (4\text{-}9)$$

It can be assumed that if Z_0 is properly chosen in terms of the network arms, it should be possible to make $Z_{1\,\mathrm{in}}$ equal to Z_0. Requiring this equality gives

$$Z_0 = \frac{Z_1^2/4 + Z_1Z_2 + Z_2Z_0 + Z_1Z_0/2}{Z_1/2 + Z_2 + Z_0}$$

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2$$

For the symmetrical T section, then,

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2} = \sqrt{Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)} \qquad (4\text{-}10)$$

becomes the characteristic impedance. This result could also have been immediately obtained from Eqs. 3-15 and 3-16 for the image

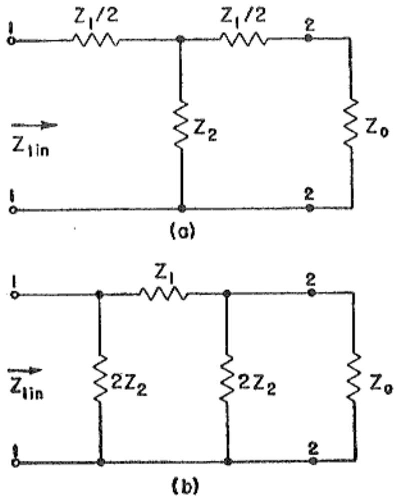


Fig. 4-4. Determination of Z_0: (a) for a T section; (b) for a π section.

impedance of a T section, by using the values of the arms of Fig. 4-4.

Similarly, for the π section of Fig. 4-4 (b) the input impedance is

$$Z_{1\,\mathrm{in}} = \frac{\left[Z_1 + \left(\frac{2Z_2Z_0}{2Z_2 + Z_0}\right)\right]2Z_2}{Z_1 + \frac{2Z_2Z_0}{2Z_2 + Z_0} + 2Z_2}$$

Requiring that $Z_{1\,\mathrm{in}} = Z_0$ leads to

$$Z_{0\pi} = \sqrt{\frac{Z_1Z_2}{1 + Z_1/4Z_2}} \qquad (4\text{-}11)$$

which is the characteristic impedance of the symmetrical π section.

In Chapter 1, certain information concerning networks was developed from measurements of Z_{oc} and Z_{sc}. If these measurements are made on the T section of (a), Fig. 4-4, exclusive of the load Z_0, then

$$Z_{1oc} = Z_{oc} = \frac{Z_1}{2} + Z_2$$

$$Z_{1sc} = Z_{sc} = \frac{Z_1}{2} + \frac{Z_1Z_2/2}{Z_1/2 + Z_2}$$

$$Z_{oc}Z_{sc} = \frac{Z_1^2}{4} + Z_1Z_2 = Z_{0T}^2 \tag{4-12}$$

Similar work for the π section leads to

$$Z_{oc}Z_{sc} = \frac{4Z_2^2Z_1}{Z_1 + 4Z_2} = Z_{0\pi}^2$$

Therefore, for a symmetrical network,

$$Z_0 = \sqrt{Z_{oc}Z_{sc}} \tag{4-13}$$

This result could have been directly obtained from the image impedance relations of Section 3-3. It is a valuable relationship, since it supplies an easy experimental means of determining the Z_0 of any symmetrical network.

4-3. Current and voltage ratios as exponentials; the propagation constant

In Section 4-1, under the assumption of equal input and output impedances, which may now be interpreted as a Z_0 termination on the network, the absolute value of the ratio of input current to output current of a given *symmetrical network* was defined as an exponential function,* for the purpose of simplifying network calculations. Obviously, the magnitude ratio does not express the

* In the general case of unsymmetrical 4-terminal networks, terminated on an image basis, it is customary to define a *transfer constant* θ, by

$$\epsilon^{2\theta} = \frac{E_1I_1}{E_2I_2} = \frac{\text{input volt-amperes}}{\text{output volt-amperes}}$$

or

$$\theta = \frac{1}{2}\ln\frac{E_1I_1}{E_2I_2}$$

where θ is in general a complex number. For symmetrical networks $Z_{1i} = Z_{2i} = Z_0$, and with a termination of Z_0, the above discussion follows, with γ customarily replacing θ and implying symmetry and Z_0 termination.

complete network performance, the phase angle between the currents being needed as well. The use of the exponential can be extended to include the phasor current ratio *if it be defined that, under the condition of Z_0 termination,*

$$\frac{I_1}{I_2} = \epsilon^{\gamma} \tag{4-14}$$

where γ is a complex number defined as

$$\gamma = \alpha + j\beta \tag{4-15}$$

Hence

$$\frac{I_1}{I_2} = \epsilon^{\gamma} = \epsilon^{\alpha+j\beta}$$

To illustrate further, if $I_1/I_2 = A\underline{/\beta}$, then

$$A = |I_1/I_2| = \epsilon^{\alpha}$$

$$\underline{/\beta} = \epsilon^{j\beta}$$

Since the input and output impedances are equal under the Z_0 termination, it is also true that

$$\frac{V_1}{V_2} = \epsilon^{\gamma}$$

The term γ has been given the name *propagation constant.* The exponent α is known as the *attenuation constant,* since it determines the magnitude ratio between input and output quantities, or the attenuation produced in passing through the network. The units of α are nepers. The exponent β is the *phase constant* as it determines the phase angle between input and output quantities, or the shift in phase introduced by the network. The units of β are radians.

If a number of sections all having a common Z_0 value are cascaded, the ratio of currents is

$$\frac{I_1}{I_2} \times \frac{I_2}{I_3} \times \frac{I_3}{I_4} \times \cdots = \frac{I_1}{I_n}$$

from which

$$\epsilon^{\gamma_1} \times \epsilon^{\gamma_2} \times \epsilon^{\gamma_3} \times \cdots = \epsilon^{\gamma_n}$$

and taking the natural logarithm,

$$\gamma_1 + \gamma_2 + \gamma_3 + \cdots = \gamma_n \tag{4-16}$$

Thus the over-all propagation constant is equal to the sum of the individual propagation constants.

4-4. Hyperbolic trigonometry

It is assumed that the student is familiar with some of the properties of hyperbolic functions, at least for real angles. Hyperbolic angles also have geometric meaning, being related to a hyperbola in the same way that trigonometric functions are related to a circle. This is illustrated in Fig. 4-5, wherein the hyperbola is the locus for the radius r, and $\sinh u = a/r$, $\cosh u = b/r$, $\tanh u = a/b$.

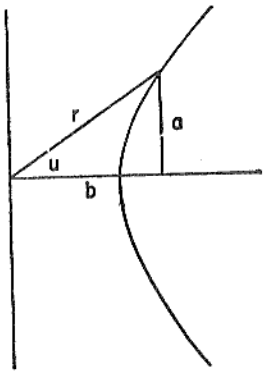


Fig. 4-5. Meaning of a hyperbolic angle.

As they will be used here, hyperbolic functions simplify the writing of certain exponential relations, and knowledge of their limits is particularly useful. A few properties are here summarized and extended to the case of complex angles:

$$\sinh u = \frac{\epsilon^u - \epsilon^{-u}}{2} \tag{4-17}$$

$$\cosh u = \frac{\epsilon^u + \epsilon^{-u}}{2} \tag{4-18}$$

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{1}{\coth u} \tag{4-19}$$

$$\cosh^2 u - \sinh^2 u = 1 \tag{4-20}$$

The values of the functions at the limits $u = 0$, and $u = \infty$ are

	$u = 0$	$u = \infty$
$\sinh u$	0	∞
$\cosh u$	1	∞
$\tanh u$	0	1

For u large, $\sinh u = \cosh u$. If u is imaginary or $u = jw$, then

$$\sinh jw = \frac{\epsilon^{jw} - \epsilon^{-jw}}{2} = j \sin w \tag{4-21}$$

$$\cosh jw = \frac{\epsilon^{jw} + \epsilon^{-jw}}{2} = \cos w \tag{4-22}$$

Expressions for complex angles, where $u = a + jb$, can be ob-

tained by expansion. That is

$$\begin{aligned}\sinh (a + jb) &= \sinh a \cosh jb + \cosh a \sinh jb \\ &= \sinh a \cos b + j \cosh a \sin b \end{aligned} \tag{4-23}$$

$$\begin{aligned}\cosh (a + jb) &= \cosh a \cosh jb + \sinh a \sinh jb \\ &= \cosh a \cos b + j \sinh a \sin b \end{aligned} \tag{4-24}$$

A few useful half-angle identities, which can be proved from the above are:

$$\sinh \frac{u}{2} = \sqrt{\frac{1}{2}(\cosh u - 1)} \tag{4-25}$$

$$\cosh \frac{u}{2} = \sqrt{\frac{1}{2}(\cosh u + 1)} \tag{4-26}$$

$$\sinh u = 2 \sinh \frac{u}{2} \cosh \frac{u}{2} \tag{4-27}$$

A considerable number of hyperbolic functions will prove useful in the sections to follow.

4-5. Properties of symmetrical networks

Use of the definition of γ, and the introduction of ϵ^γ as the current ratio for a Z_0 terminated network, leads to further useful results.

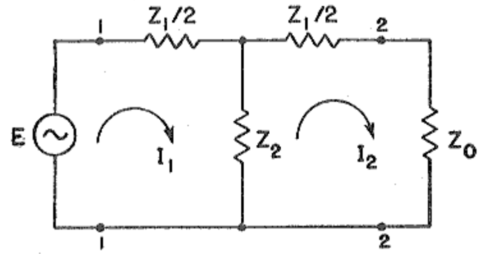


Fig. 4-6. Symmetrical network with generator and load.

In Fig. 4-6, the T network is considered equivalent to any connected symmetrical network, and is terminated in a load Z_0. The mesh equations are

$$E = I_1\left(\frac{Z_1}{2} + Z_2\right) - I_2Z_2 \tag{4-28}$$

$$0 = -I_1Z_2 + I_2\left(\frac{Z_1}{2} + Z_2 + Z_0\right) \tag{4-29}$$

The current ratic for the two meshes, which is equal to ϵ^γ by definition, can be obtained from the second equation as

$$\frac{I_1}{I_2} = \frac{Z_1/2 + Z_2 + Z_0}{Z_2} = \epsilon^\gamma \tag{4-30}$$

After thus introducing ϵ^γ, the above may be written

$$Z_0 = Z_2(\epsilon^\gamma - 1) - \frac{Z_1}{2} \tag{4-31}$$

From Eq. 4-10 for the characteristic impedance,

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2 \tag{4-32}$$

If Z_0 is eliminated by use of Eq. 4-32 in Eq. 4-31, there results

$$Z_2(\epsilon^\gamma - 1)^2 - Z_1\epsilon^\gamma = 0$$

$$\epsilon^{2\gamma} - 2\epsilon^\gamma + 1 = \frac{Z_1}{Z_2}\epsilon^\gamma$$

$$\frac{\epsilon^\gamma + \epsilon^{-\gamma}}{2} = 1 + \frac{Z_1}{2Z_2}$$

$$\cosh\gamma = 1 + \frac{Z_1}{2Z_2} \tag{4-33}$$

Equation 4-33 and its other derived forms will be of considerable value in the study of filters.

By use of the identity, Eq. 4-20, that

$$\cosh^2\gamma - \sinh^2\gamma = 1$$

it is possible to write

$$\sinh\gamma = \frac{Z_0}{Z_2} \tag{4-34}$$

Combining Eqs. 4-33 and 4-34 leads to

$$\tanh\gamma = \frac{Z_0}{Z_1/2 + Z_2} \tag{4-35}$$

By use of Eq. 4-25 it is possible to write

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{1}{2}\left(1 + \frac{Z_1}{2Z_2} - 1\right)}$$

$$= \sqrt{\frac{Z_1}{4Z_2}} \tag{4-36}$$

an expression which will serve to predict filter performance.

The propagation constant γ can be related to the network parameters by use of Eq. 4-10, for Z_{0T}, in Eq. 4-30 as

$$\epsilon^\gamma = 1 + \frac{Z_1}{2Z_2} + \sqrt{\left(\frac{Z_1}{2Z_2}\right)^2 + \frac{Z_1}{Z_2}} \tag{4-37}$$

Taking the natural logarithm

$$\gamma = \ln\left[1 + \frac{Z_1}{2Z_2} + \sqrt{\left(\frac{Z_1}{2Z_2}\right)^2 + \frac{Z_1}{Z_2}}\right] \tag{4-38}$$

For a network of pure reactances this is not difficult to compute. For an impedance it may be noted that the logarithm of a complex quantity $B/\alpha = \ln B + j\alpha$.

The input impedance of any T network, terminated in any impedance Z_R, may also be written in terms of hyperbolic functions of γ. Writing

$$Z_{in} = Z_{11} - \frac{Z_{12}^2}{Z_{22}}$$

and substituting the required mesh relations from Fig. 4-6, with Z_0 replaced by Z_R, then

$$Z_{in} = \frac{Z_1}{2} + Z_2 - \frac{Z_2^2}{Z_1/2 + Z_2 + Z_R}$$

$$= \frac{Z_1^2/4 + Z_1Z_2 + (Z_1/2 + Z_2)Z_R}{Z_1/2 + Z_2 + Z_R}$$

Use of Eqs. 4-32 and 4-35 leads to

$$Z_{in} = \frac{Z_0^2 + Z_RZ_0/\tanh\gamma}{Z_0/\tanh\gamma + Z_R}$$

$$= Z_0\left(\frac{Z_R\cosh\gamma + Z_0\sinh\gamma}{Z_0\cosh\gamma + Z_R\sinh\gamma}\right) \tag{4-39}$$

This is the input impedance of a symmetrical T network terminated in a load Z_R, in terms of the propagation constant and Z_0 of the network.

For a short-circuited network $Z_R = 0$. The input impedance is then Z_{sc} where, from the above equation,

$$Z_{sc} = Z_0 \tanh \gamma \qquad (4\text{-}40)$$

For an open circuit $Z_R = \infty$ in the limit, and Z_{oc} is then

$$\underset{\lim Z_R \to \infty}{Z_{oc}} = \frac{Z_0}{\tanh \gamma} \qquad (4\text{-}41)$$

From these two equations it can be seen that

$$\tanh \gamma = \sqrt{\frac{Z_{sc}}{Z_{oc}}} \qquad (4\text{-}42)$$

and

$$Z_0 = \sqrt{Z_{oc} Z_{sc}}$$

which has already been proved from the properties of the characteristic impedance.

In Chapter 1, open-circuit and short-circuit measurements were used to describe the performance of a network. In this chapter, two new parameters, the characteristic impedance Z_0, and the propagation constant γ, have been introduced, and the properties of the network have been developed in terms of these new parameters. The last few equations are relations between the two sets of parameters.

4-6. Filter fundamentals; pass and stop bands

Ideally it is desired that a filter network transmit or *pass* a desired frequency band without loss, whereas it should *stop* or completely *attenuate* all undesired frequencies. The propagation constant $\gamma = \alpha + j\beta$, being a function of frequency by Eq. 4-38, can supply information on the ability of the filter to perform as desired. If $\alpha = 0$ or $I_1 = I_2$, then there is no attenuation, only a phase shift, in transmitting a signal through the filter, and operation is in a *pass band* of frequencies. When α has a positive value, then I_2 is smaller in magnitude than I_1, attenuation has occurred and operation is in an attenuation or *stop band* of frequencies.

The propagation constant γ may be conveniently studied by use

of Eq. 4-36:

$$\sinh \frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} \qquad (4\text{-}43)$$

It will first be assumed that the network contains only pure reactances, and thus $Z_1/4Z_2$ will be real, and either positive or negative, depending on the type of reactance used for Z_1 and Z_2. Expanding gives

$$\sinh \frac{\gamma}{2} = \sinh \left(\frac{\alpha}{2} + \frac{j\beta}{2}\right)$$

$$= \sinh \frac{\alpha}{2} \cos \frac{\beta}{2} + j \cosh \frac{\alpha}{2} \sin \frac{\beta}{2} \qquad (4\text{-}44)$$

as an equation containing much information.

If Z_1 and Z_2 *are the same type of reactance* then $|Z_1/4Z_2| > 0$, or the ratio is positive and real. This requires that $\sinh \gamma/2$ be real, which means that the imaginary term in Eq. 4-44 must equal zero and that

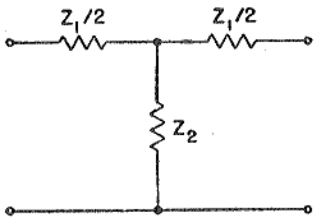


Fig. 4-7. Symmetrical T network.

(a) $\cosh \frac{\alpha}{2} \sin \frac{\beta}{2} = 0$

(b) $\sinh \frac{\alpha}{2} \cos \frac{\beta}{2} = \sqrt{\frac{Z_1}{4Z_2}}$

are simultaneously satisfied.

From (a),

$$\sin \frac{\beta}{2} = 0; \qquad \beta = n\pi \qquad \text{where } n = 0, 2, 4, \ldots$$

From (b), since $\cos \beta/2 = 1$, then

$$\sinh \frac{\alpha}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

and the attenuation will be given by

$$\alpha = 2 \sinh^{-1} \sqrt{\frac{Z_1}{4Z_2}} \qquad (4\text{-}45)$$

Thus the *condition that* $|Z_1/4Z_2| > 0$ *implies a stop or attenuation band of frequencies.*

If Z_1 *and* Z_2 *are opposite types of reactance* then $Z_1/4Z_2$ is negative, $|Z_1/4Z_2| < 0$, and the radical of Eq. 4-43 is imaginary. The real term in Eq. 4-44 must then be zero, so that

(c) $$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = 0$$

(d) $$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

must be satisfied.

Two conditions are possible from the above:

I. $\sinh\frac{\alpha}{2} = 0$; therefore

$$\alpha = 0; \qquad \beta \neq 0; \qquad \sin\frac{\beta}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

II. $\cos\frac{\beta}{2} = 0$; therefore $\sin\frac{\beta}{2} = \pm 1$ and

$$\alpha \neq 0; \qquad \beta = (2n-1)\pi; \qquad \cosh\frac{\alpha}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

Condition I leads to a pass band, or region of zero attenuation, which is limited by the upper limit on the sine, or by $\sin\frac{\beta}{2} = 1$, or it is required that

$$-1 < \frac{Z_1}{4Z_2} < 0$$

The phase angle in this pass band will be given by

$$\beta = 2\sin^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{4-46}$$

Condition II leads to a stop or attenuation band since $\alpha \neq 0$. The phase angle is π, and the attenuation is given by

$$\alpha = 2\cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{4-47}$$

Because the hyperbolic cosine has no value below unity, it appears that the region in which condition II applies is a stop band where

$$\frac{Z_1}{4Z_2} < -1$$

Values of $Z_1/4Z_2$ can then be classified into three regions, with corresponding values of α and β, these regions being bounded by $Z_1/4Z_2$ values of $+\infty$, 0, -1 and $-\infty$, as given below:

$Z_1/4Z_2 =$	$+\infty$ to 0	0 to -1	-1 to $-\infty$
Reactance type:	Same	Opposite	Opposite
Band:	Stop	Pass	Stop
α:	$2\sinh^{-1}\sqrt{\frac{Z_1}{4Z_2}}$	0	$2\cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}}$
β:	π	$2\sin^{-1}\sqrt{\frac{Z_1}{4Z_2}}$	π

The frequencies at which the network changes from a pass network to a stop network, or vice versa, are called *cutoff frequencies.* These frequencies occur when

$$\left.\begin{aligned} \frac{Z_1}{4Z_2} &= 0 \quad \text{or} \quad Z_1 = 0 \\ \frac{Z_1}{4Z_2} &= -1 \quad \text{or} \quad Z_1 = -4Z_2 \end{aligned}\right\} \tag{4-48}$$

where Z_1 and Z_2 are opposite types of reactance.

Since Z_1 and Z_2 may have a number of configurations, as L and C elements, or as parallel and series combinations, a variety of types of performance are possible.

The elements considered above were assumed pure reactances, and design is ordinarily carried out on this basis. Measurements of actual performance are then made and adjustments introduced into the design to compensate for deviation of the results from the ideal. In addition to minimizing the losses of physical elements it is also necessary to reduce stray electric and magnetic couplings between elements to obtain more nearly the predicted performance.

4-7. Behavior of the characteristic impedance

It has been shown that for a symmetrical T network

$$Z_{0T} = \sqrt{Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)}$$

In a network made up entirely of pure reactances this expression for the characteristic impedance becomes

$$Z_{0T} = \sqrt{-X_1X_2\left(1 + \frac{X_1}{4X_2}\right)} \qquad (4\text{-}49)$$

where the X terms will carry their own signs, the minus sign under the radical being due to j^2.

In Section 4-6 it was shown that a stop band exists where X_1 and X_2 are the same type of reactance. The ratio $X_1/4X_2$ will be real and positive, and the characteristic impedance will be a pure reactance in this attenuation region.

A pass band was shown to exist where X_1 and X_2 were of opposite reactance types and $-1 < X_1/4X_2 < 0$. Placing these conditions in Eq. 4-49 results in the product X_1X_2 being negative, with the bracketed term positive. The over-all sign under the radical will be positive and Z_0 will be real, and thus able to absorb power from a source.

A stop band exists with X_1 and X_2 of opposite types, but with $X_1/4X_2 < -1$. This implies that the product X_1X_2 is a negative term, and that the bracketed term is negative. When combined with the negative sign present in Eq. 4-49, the over-all sign under the radical will be negative and Z_0 will be a pure reactance in this stop region.

It has been shown that in a pass band Z_0 is real and positive. If the reactive network is terminated with a resistive $Z_0 = R_0$, then the input impedance is R_0, and the network can accept power and will transmit it to the resistive load without loss or attenuation. If the network is supplied by a source having R_0 as its internal impedance, the system will be matched at each set of terminals, and maximum power will be delivered from generator to load.

In a stop band Z_0 has been shown to be reactive. If the network is terminated in its reactive Z_0, it will appear as a totally reactive circuit and as such cannot accept or transmit power, since there is no resistive element in which the power may be dissipated. The network may transmit voltage or current, but with a 90° phase angle between the two and with considerable attenuation.

Similar reasoning may be applied to the Z_0 for a π network if

it is noted that

$$Z_{0\pi} = \frac{Z_1Z_2}{Z_{0T}} \qquad (4\text{-}50)$$

and Z_1Z_2 is always real for Z_1 and Z_2 as pure reactances. Thus it is seen that the conditions developed for pass and stop bands for T sections likewise apply for π sections.

4-8. The constant-*k* low-pass filter

If Z_1 and Z_2 of a reactance network are unlike reactance arms, then

$$Z_1Z_2 = k^2$$

where k is a constant independent of frequency. Networks or filter sections for which this relation holds are called *constant-k* filters.

As a special case, let $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$, then the product

$$Z_1Z_2 = \frac{L}{C} = R_k^2 \qquad (4\text{-}51)$$

The term R_k is used since k must be real if Z_1 and Z_2 are of opposite type. A T section so designed would appear as at (a), Fig. 4-8.

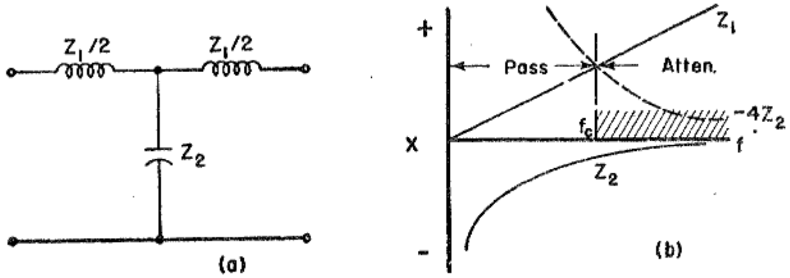


Fig. 4-8. (a) Low-pass filter section; (b) reactance curves demonstrating that (a) is a low-pass section or has a pass band between $Z_1 = 0$ and $Z_1 = -4Z_2$.

The reactances of Z_1 and $4Z_2$ will vary with frequency as sketched at (b), Fig. 4-7. The curve representing $-4Z_2$ may be drawn and compared with the curve for Z_1. It has been shown by Eq. 4-48 that a pass band starts at the frequency at which $Z_1 = 0$ and runs to the frequency at which $Z_1 = -4Z_2$. Thus the reactance curves show that a pass band starts at $f = 0$ and continues to some higher frequency f_c. All frequencies above f_c lie in a stop, or attenuation, band. Thus the network is called a *low-pass* filter.

The cutoff frequency f_c may be readily determined, since at that point

$$Z_1 = -4Z_2, \qquad j\omega_c L = \frac{4j}{\omega_c C}$$

$$f_c = \frac{1}{\pi\sqrt{LC}} \tag{4-52}$$

This expression may be used to develop certain relations applicable to the low-pass network. Then $\sinh \gamma/2$ may be evaluated as

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{-\omega^2 LC}{4}} = \frac{j\omega\sqrt{LC}}{2}$$

and in view of Eq. 4-52 this is

$$\sinh\frac{\gamma}{2} = j\frac{f}{f_c} \tag{4-53}$$

Then if the frequency f is in the pass band or $f/f_c < 1$, so that $-1 < Z_1/4Z_2 < 0$, then

$$\frac{f}{f_c} < 1, \qquad \alpha = 0, \qquad \beta = 2\sin^{-1}\left(\frac{f}{f_c}\right)$$

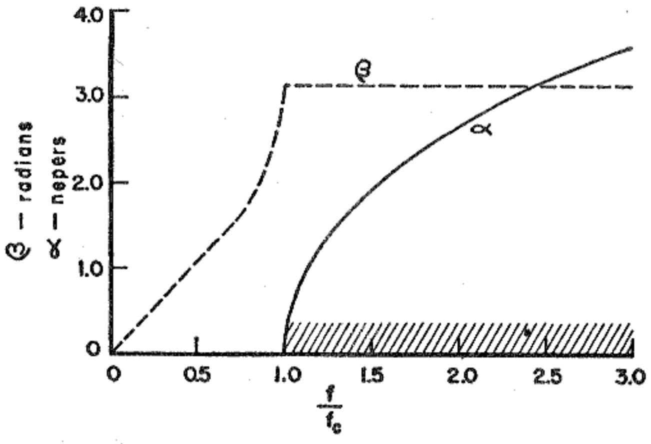


Fig. 4-9. Variation of α and β with frequency for the low-pass section.

whereas if frequency f is in the attenuation band or $f/f_c > 1$, so that $Z_1/4Z_2 < -1$, then

$$\frac{f}{f_c} > 1, \qquad \alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right), \qquad \beta = \pi$$

thereby allowing determination of α and β. The variation of α and β is plotted in Fig. 4-9 as a function of f/f_c. This method shows that the attenuation α is zero throughout the pass band but rises gradually from the cutoff frequency at $f/f_c = 1.0$ to a value of ∞ at infinite frequency. The phase shift β is zero at zero frequency and increases gradually through the pass band, reaching π at f_c and remaining at π for all higher frequencies.

The characteristic impedance of a T section was obtained as

$$Z_{0T} = \sqrt{Z_1 Z_2\left(1 + \frac{Z_1}{4Z_2}\right)}$$

which becomes
$$Z_{0T} = \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2 LC}{4}\right)} \tag{4-54}$$

for the low-pass constant-k section under discussion. By use of Eq. 4-52 the characteristic impedance of a low-pass filter may be stated as

$$Z_{0T} = \sqrt{\frac{L}{C}\left[1 - \left(\frac{f}{f_c}\right)^2\right]} \tag{4-55}$$

$$= R_k\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{4-56}$$

in accordance with the definition of R_k in Eq. 4-51. Values of Z_{0T}/R_k are plotted against f/f_c in Fig. 4-10. It may be seen that Z_{0T} varies throughout the pass band, reaching a value of zero at cutoff, then becomes imaginary in the attenuation band, rising to infinite reactance at infinite frequency.

A low-pass filter may be designed from a knowledge of the cutoff frequency desired and the load resistance to be supplied. It is desirable that the Z_0 in the pass band match the load; but because of the nature of the Z_0 curve in Fig. 4-10, this result can occur at only one frequency. This match may be arranged to occur at any frequency which it is desired to favor by an impedance match.

For reasons which will appear in Section 4-13, the load is chosen as $R = R_k = \sqrt{L/C}$, which will favor zero frequency for a low-pass filter.

The design of a low-pass filter may be readily carried out. From

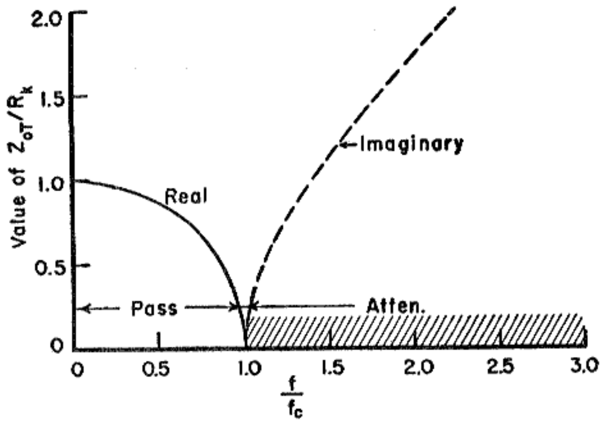


Fig. 4-10. Variation of Z_{0T}/R_k with frequency for the low-pass section.

the relation that at cutoff

$$Z_1 = -4Z_2$$

it is seen that

$$\omega_c L = \frac{4}{\omega_c C}$$

Using the cutoff frequency equation changes this to

$$\pi^2 f_c^2 LC = 1$$

and use of the relation $R = \sqrt{L/C}$ gives for the value of the shunt capacitance arm

$$C = \frac{1}{\pi f_c R} \tag{4-57}$$

By similar methods the inductance for Z_1 is obtained as

$$L = \frac{R}{\pi f_c} \tag{4-58}$$

Since the design is based on an impedance match at zero frequency only, power transfer to a matched load will drop at higher

pass-band frequencies. This condition may be undesirable in certain applications, and a remedy will be discussed in Section 4-13.

A network such as is described here is called a *prototype section*. It may be employed when a sharp cutoff is not required, although cutoff may be sharpened by using a number of such networks in cascade. This is not usually an economic use of circuit elements, and introduces excessive losses over other available methods of raising the attenuation near the cutoff frequency.

4-9. The constant-*k* high-pass filter

If the positions of inductance and capacitance are interchanged to make $Z_1 = -j/\omega C$ and $Z_2 = j\omega L$, then Z_1Z_2 will still be given by

$$Z_1Z_2 = k^2$$

and the filter design obtained will be of the constant-k type. The T section will then appear as in (a), Fig. 4-11. The reactances of Z_1 and Z_2 are sketched as functions of frequency in (b), and Z_1

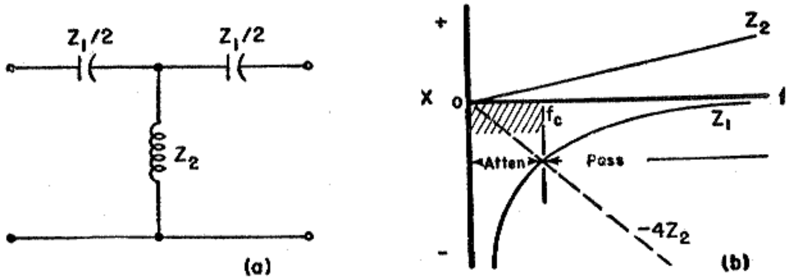


Fig. 4-11. (a) High-pass filter section; (b) reactance curves demonstrating that (a) is a high-pass section or has a pass band between $Z_1 = 0$ and $Z_1 = -4Z_2$.

is compared with $-4Z_2$, showing a cutoff frequency at the point at which Z_1 equals $-4Z_2$, with a pass band from that frequency to infinity where $Z_1 = 0$. The network is thus a *high-pass* filter. All frequencies below f_c lie in an attenuation, or stop, band.

The cutoff frequency is determined as the frequency at which $Z_1 = -4Z_2$, or

$$\frac{-j}{\omega_c C} = -j4\omega_c L, \qquad 4\omega_c^2 LC = 1$$

$$f_c = \frac{1}{4\pi\sqrt{LC}} \tag{4-59}$$

Using the above expression

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{-\frac{1}{4\omega^2LC}} = \frac{-j}{2\omega\sqrt{LC}} = -j\frac{f_c}{f} \qquad (4\text{-}60)$$

The region in which $f_c/f < 1$ is a pass band, so that the variation of γ inside and outside the pass band will be identical with the values for the low-pass filter, and the curves of Fig. 4-9 will apply if the abscissa be considered as calibrated in terms of f_c/f, except that the phase angle β will be negative, changing from 0 at infinite frequency or $f_c/f = 0$, to $-\pi$ at cutoff or $f_c/f = 1$.

The high-pass filter may be designed by again choosing a resistive load R equal to R_k such that

$$R = R_k = \sqrt{\frac{L}{C}} \qquad (4\text{-}61)$$

From the relation that at cutoff $Z_1 = -4Z_2$ it was shown that

$$4\omega_c^2LC = 1$$

and again $L/C = R^2$, so that the value of the capacitance for Z_1, the series element, is

$$C = \frac{1}{4\pi f_c R} \qquad (4\text{-}62)$$

It should be noted that since $Z_1/2$ is the value of each series arm, the capacity used in each series $Z_1/2$ element should be $2C$. By similar methods the value for the inductance for Z_2, the shunt arm, is

$$L = \frac{R}{4\pi f_c} \qquad (4\text{-}63)$$

The characteristic impedance for the *high-pass filter* may be transformed to

$$Z_{0T} = R_k\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad (4\text{-}64)$$

4-10. The *m*-derived T section

The constant-*k* prototype filter section, though simple, has two major disadvantages. The attenuation does not rise very rapidly at cutoff, so that frequencies just outside the pass band are not

appreciably attenuated with respect to frequencies just inside the pass band. Also, the characteristic impedance varies widely over the pass band, so that a satisfactory impedance match is not possible. In cases where an impedance match is not important, the attenuation may be built up near cutoff by cascading or connecting a number of constant-*k* sections in series.

It is more economical to attempt to raise the attenuation near cutoff by other means. Consider first the circuit of (a), Fig. 4-12. The reactance curves sketched at (b) show that this circuit is a low-pass filter. However, it can be seen that the shunt arm is a series

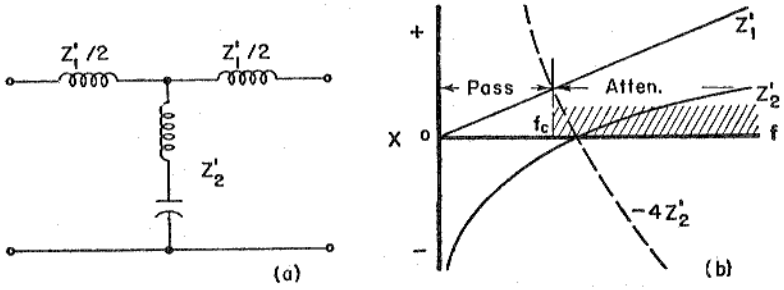


Fig. 4-12. (a) Derivation of a low-pass section having a sharp cutoff action; (b) reactance curves for (a).

circuit resonant at a frequency above f_c. At this resonant frequency the shunt arm appears as a short circuit on the network, or the attenuation becomes infinite. This frequency of infinite or high attenuation is called f_∞; and by reason of the requirement that below f_c the shunt circuit appear as a capacitance, the frequency of resonance, f_∞, will always be higher in value than f_c. If, then, f_∞ can be chosen arbitrarily close to f_c, the attenuation near cutoff may be made high.

The attenuation above f_∞ will fall to low values, so that if high attenuation is desired over the whole attenuation band, it is necessary to use a section such as in Fig. 4-12 for high attenuation near cutoff, in series with a prototype section to provide high attenuation at frequencies well removed from cutoff. For satisfactory matching of several such types of filters in series, it is necessary that the Z_0 of all be identical at all points in the pass band. They will consequently also all have the same pass band.

The network of Fig. 4-12 may be derived by assuming that

$$Z_1' = mZ_1 \tag{4-65}$$

the primes indicating the *derived section.* It is then necessary to find the value for Z_2' such that $Z_0' = Z_0$. Setting the characteristic impedances equal,

$$Z_0' = Z_0$$

$$\frac{(mZ_1)^2}{4} + mZ_1Z_2' = \frac{Z_1^2}{4} + Z_1Z_2$$

$$Z_2' = \frac{Z_2}{m} + \frac{1-m^2}{4m} Z_1 \tag{4-66}$$

It then appears that the shunt arm Z_2' consists of two impedances in series, as shown in Fig. 4-13. As required, the characteristic impedance and f_c remain equal to those of the T section prototype containing Z_1 and Z_2 values.

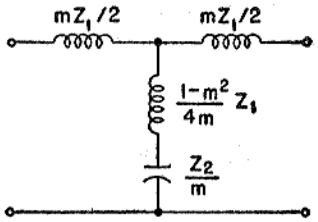


Fig. 4-13. The m-derived low-pass filter.

Since m is arbitrary, it is possible to design an infinite variety of filter networks meeting the required conditions on Z_0 and f_c. However, Z_2 will be opposite in sign to Z_1, and it is desired that this relation continue in the two series impedances given by Eq. 4-66 for the Z_2' arm. Equation 4-66 then indicates that $(1 - m^2)/4m$ must be positive, forcing the terms $1 - m^2$ and m always to be positive. Thus m must always be chosen so that

$$0 < m < 1$$

Filter sections obtained in this manner are called *m-derived sections.*

The shunt arm is to be chosen so that it is resonant at some frequency f_∞ above f_c. This means that at the resonant frequency

$$\left|\frac{Z_2}{m}\right| = \left|\frac{1-m^2}{4m} Z_1\right| \tag{4-67}$$

and for the *low-pass filter*

$$\frac{1}{2\pi f_\infty mC} = \frac{1-m^2}{4m} 2\pi f_\infty L$$

$$f_\infty = \frac{1}{\pi\sqrt{(1-m^2)LC}}$$

Since the cutoff frequency for the low-pass filter is

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

the frequency of infinite attenuation will be

$$f_\infty = \frac{f_c}{\sqrt{1-m^2}} \tag{4-68}$$

from which

$$m = \sqrt{1-(f_c/f_\infty)^2} \tag{4-69}$$

This equation determines the m to be used for a particular f_∞.

Similar relations for the *high-pass filter* can be derived as

$$f_\infty = f_c\sqrt{1-m^2} \tag{4-70}$$

and

$$m = \sqrt{1-(f_\infty/f_c)^2} \tag{4-71}$$

The m-derived section is designed following the design of the prototype T section. The use of a prototype and one or more m-derived sections in series results in a *composite filter.* If a sharp cutoff is desired, an m-derived section may be used with f_∞ near f_c, followed by as many m-derived sections as desired to place frequencies of high attenuation where needed to suppress various signal components or to produce a high attenuation over the entire attenuation band.

The variation of attenuation over the attenuation band for a *low-pass m-derived section* in the stop band is dependent on the sign of the reactances or

$$\alpha = 2\cosh^{-1}\sqrt{\left|\frac{Z_1}{4Z_2}\right|} \quad \text{or} \quad \alpha = 2\sinh^{-1}\sqrt{\left|\frac{Z_1}{4Z_2}\right|}$$

$$f_c < f < f_\infty \qquad\qquad f_\infty < f$$

For $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$ for the prototype, then

$$\left|\frac{Z_1}{4Z_2}\right| = \frac{m\omega L}{4[1/m\omega C - \omega L(1-m^2)/4m]}$$

so that for $f_c < f < f_\infty$

$$\alpha = 2\cosh^{-1}\frac{mf/f_c}{\sqrt{1-f^2/f_\infty^2}} \quad (4\text{-}72)$$

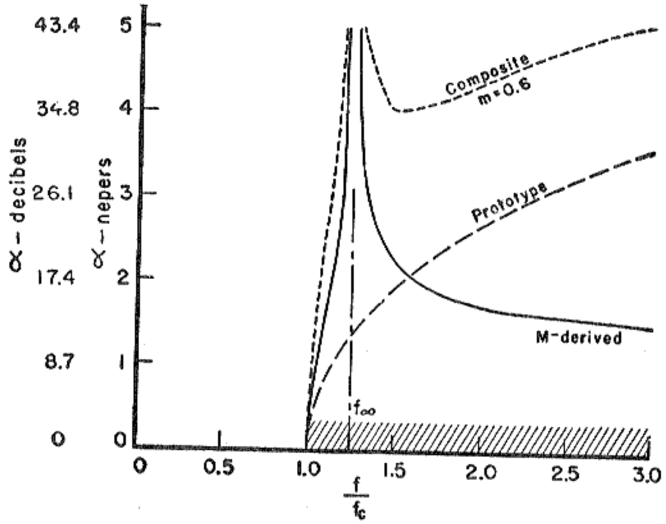


Fig. 4-14. Variation of attenuation for the prototype and m-derived sections, and the composite result of the two in series.

and for $f_\infty < f$

$$\alpha = 2\sinh^{-1}\frac{mf/f_c}{\sqrt{f^2/f_\infty^2 - 1}} \quad (4\text{-}73)$$

The value of α may be determined from this expression. Figure 4–14 is a plot of α against f/f_c for $m = 0.6$, which gives a value of f_∞ equal to 1.25 times the cutoff frequency f_c. The great increase in sharpness of cutoff for the m-derived section over the prototype is apparent. The higher attenuation over the whole attenuation band obtained by use of a prototype section and an m-derived section in series as a composite filter is also readily seen.

Again following the procedure of Section 4–8, the phase shift

constant β may be determined; in the pass band, from

$$\beta = 2\sin^{-1}\sqrt{\left|\frac{Z_1}{4Z_2}\right|}$$

$$= 2\sin^{-1}\frac{mf/f_c}{\sqrt{1-(f^2/f_c^2)(1-m^2)}} \quad (4\text{-}74)$$

In the attenuation band, up to f_∞, β has the value π. Above f_∞ the value of β drops to zero, because the shunt arm becomes inductive above resonance. The phase shift of the m-derived section is plotted as a function of f/f_c in Fig. 4-15.

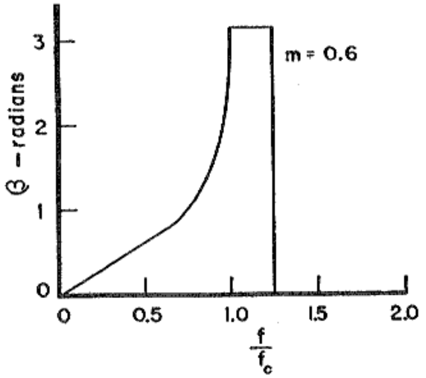


Fig. 4-15. Variation of phase shift β, for the m-derived filter.

This material demonstrates the ability of the m-derived section to overcome the lack of a sharp cutoff in the simple prototype filter. Although it may be noted that the sharpness of cutoff increases for small values of m, the attenuation beyond the point of peak attenuation becomes smaller for small m. This emphasizes the necessity of supplementing the m-derived section with a prototype section in series to raise the attenuation for frequencies well removed from cutoff.

4-11. The m-derived π section

An m-derived π section may also be obtained. The characteristic impedance of the π section is

$$Z_{0\pi} = \frac{Z_1 Z_2}{\sqrt{Z_1 Z_2(1 + Z_1/4Z_2)}}$$

The characteristic impedances of the prototype and m-derived sections are to be equal so that they may be joined without mismatch. By use of the transformation for the shunt arm,

$$Z_2' = \frac{Z_2}{m} \tag{4-75}$$

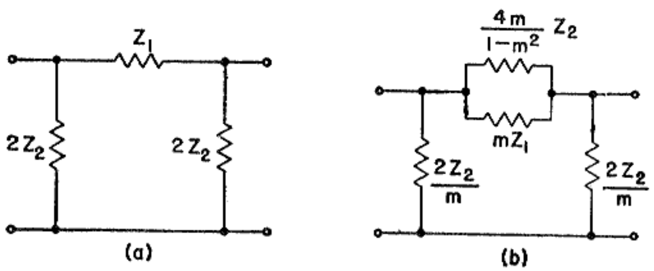


Fig. 4-16. (a) Usual symmetrical π section; (b) the m-derived π filter.

it is possible to equate the characteristic impedances as

$$\frac{Z_1'Z_2/m}{\sqrt{Z_1(Z_2/m)(1 + Z_1'm/4Z_2)}} = \frac{Z_1Z_2}{\sqrt{Z_1Z_2(1 + Z_1/4Z_2)}}$$

from which

$$Z_1' = \frac{1}{\dfrac{1}{mZ_1} + \dfrac{1}{\dfrac{4m}{1-m^2}Z_2}} \tag{4-76}$$

It is apparent that the series arm Z_1' is represented by two impedances in parallel, one being mZ_1, the other being $4m/(1 - m^2)Z_2$ in value.

Equations 4-75 and 4-76 thus give the values to be used in designing the m-derived π section. The circuit is drawn in Fig. 4-16.

4-12. Variation of characteristic impedance over the pass band

It has been shown in Section 4-8 that for a low-pass T section

$$Z_{0T} = R_k\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{4-77}$$

The characteristic impedance for a π section is

$$Z_{0\pi} = \frac{Z_1Z_2}{\sqrt{Z_1Z_2(1 + Z_1/4Z_2)}}$$

Since $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$ for the *low-pass filter,* use of the cutoff frequency expression permits the characteristic impedance of

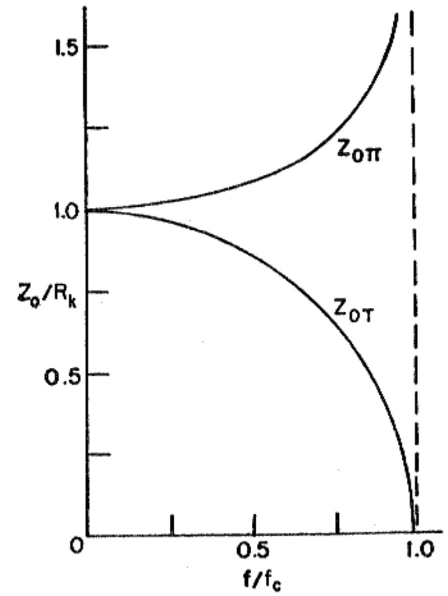


Fig. 4-17. Manner of variation of Z_0 over the pass band for the T and π networks.

the low-pass π section to be expressed as

$$Z_{0\pi} = \frac{L/C}{\sqrt{L/C[1 - (f/f_c)^2]}}$$

$$= \frac{R_k}{\sqrt{1 - (f/f_c)^2}} \tag{4-78}$$

The π-section characteristic impedance is plotted over the pass band in Fig. 4-17 as a function of f/f_c and is compared with the curve for the T section, reproduced from Fig. 4-10.

The curves show that the characteristic impedance of neither section is sufficiently constant over the pass band that a load equal to R_k will give a satisfactory impedance match.

4-13. Termination with m-derived half sections

In Chapter 3, reactance L sections were designed that would transform a given resistance to a more desired value. The problem

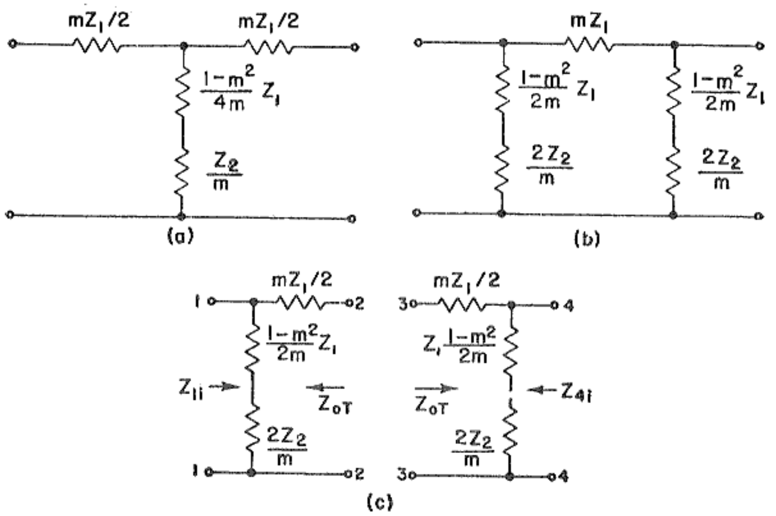


Fig. 4-18. (a) m-derived T section; (b) π section formed by rearranging constants of (a); (c) circuit of (b) split into two L sections.

of satisfactorily terminating or matching a T or π filter to a given load could be performed by an L-section network at one frequency. The real problem, however, is that of causing an L section to change its characteristics with frequency in such a way that the filter is approximately matched to its load at all frequencies over the pass band. Zobel discovered that an m-derived half section or L section could be made to have the desired properties over most of the pass band.

Consider the π section in (b), Fig. 4-18, formed by use of the elements of the m-derived T section of (a). This π section can be split into two half sections, with values as at (c). The image impedance of the left half section at the 1,1 terminals is given by Eq. 3-17 as

$$Z_{1i} = \sqrt{Z_{1oc}Z_{1sc}}$$

so that

$$Z_{1i} = \sqrt{\frac{[(1-m^2)Z_1/2m + 2Z_2/m]^2(mZ_1/2)}{(1-m^2)Z_1/2m + 2Z_2/m + mZ_1/2}}$$

$$= \left(\frac{1-m^2}{2m}Z_1 + \frac{2Z_2}{m}\right)\sqrt{\frac{(mZ_1/2)^2}{(1-m^2)Z_1^2/4 + Z_1Z_2 + m^2Z_1^2/4}}$$

$$= \left[1 + (1-m^2)\frac{Z_1}{4Z_2}\right]\sqrt{\frac{Z_1Z_2}{1 + Z_1/4Z_2}} \qquad (4\text{-}80)$$

By Eq. 4-11, the above is recognizable as

$$Z_{1i} = \left[1 + (1-m^2)\frac{Z_1}{4Z_2}\right]Z_{0\pi} \qquad (4\text{-}81)$$

Equation 4-81 shows the 1,1 image impedance to be a function of $Z_{0\pi}$ modified by the factor $1 + (1 - m^2)\, Z_1/4Z_2$ and thus to have possibilities of variation of image impedance with values of m.

For the *low-pass* filter, Eq. 4-81 may be written as

$$Z_{1i} = \frac{R_k[1 - (1-m^2)f^2/f_c^2]}{\sqrt{1 - f^2/f_c^2}} \qquad (4\text{-}82)$$

The variation of Z_{1i}, or the image impedance at the 1,1 terminals of Fig. 4-18, is plotted over the pass band for several values of m in Fig. 4-19. It can be seen that, by use of the value $m = 0.6$ for the half section, a nearly constant value of image impedance equal to R_k is obtained at the 1,1 terminals over 85 per cent of the pass band. A source impedance equal to R_k then could be matched satisfactorily on an image basis at the 1,1 terminals over most of the pass band.

A similar variation with m can be developed for the *high-pass* filter.

The image impedance of the left half section of Fig. 4-18(c) at the 2,2 terminals may be written

$$Z_{2i} = \sqrt{Z_{2oc}Z_{2sc}}$$

$$= \sqrt{\left(\frac{mZ_1}{2} + \frac{1-m^2}{2m}Z_1 + \frac{2Z_2}{m}\right)\frac{mZ_1}{2}}$$

$$= \sqrt{Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)} \qquad (4\text{-}83)$$

$$Z_{2i} = Z_{0T} \qquad (4\text{-}84)$$

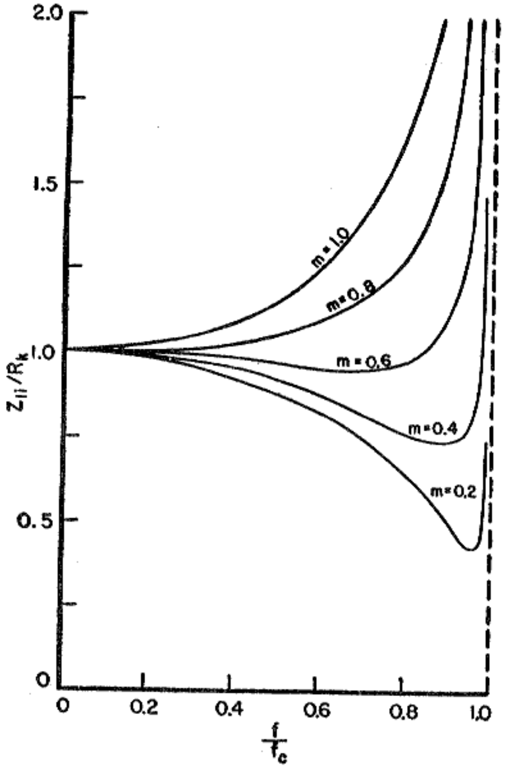


Fig. 4-19. Variation of Z_{1i} of the L section of Fig. 4-18 (c) over the pass band, plotted for various m values.

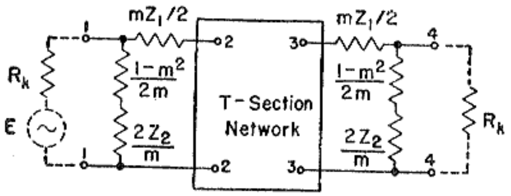


Fig. 4-20. Manner of use of m-derived L sections to terminate a T-section filter. The value of m should be 0.6.

Thus the image impedance looking to the left at the 2,2 terminals, Fig. 4-18(c), is that of a T section. By the same reasoning, the image impedance looking to the right at the 3,3 terminals is also Z_{0T}; and that looking to the left at the 4,4 terminals is the modified value of $Z_{0\pi}$ given by Eq. 4-81.

Thus a generator of internal impedance R_k may be connected to the 1,1 terminals of such a half section and a satisfactory image impedance match obtained over the pass band except close to cutoff. Likewise, a load of value R_k may be connected to the 4,4 terminals with a satisfactory match. Between the 2,2 and 3,3 terminals may be inserted prototype and m-derived T sections designed for a value

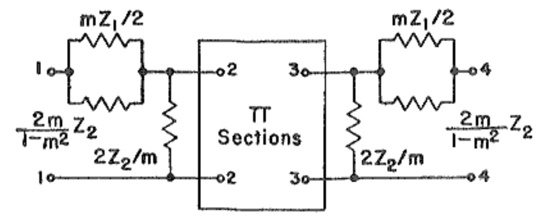


Fig. 4-21. Use of L sections to terminate a π-section filter, with $m = 0.6$.

of $R = R_k$. These sections will be working into terminals of the half sections that have image impedances equal to the Z_0 of the T sections, and thus will be matched. The over-all characteristic impedance will be very nearly constant except near cutoff, at a value equal to R_k, and all sections will be closely matched for maximum power transfer over at least 85 per cent of the pass band. The general arrangement will be as in Fig. 4-20.

The two half m-derived sections, known as *terminating half-sections*, are normally added to the design of any filter to provide uniform termination and matching characteristics. Moreover, they provide a point of high attenuation at a frequency 1.25 that of f_c, thus improving the attenuation characteristics of the filter. If additional attenuation is needed in the stop band or if the cutoff must be made sharper, then additional full m-derived sections with different m values may be added in series with the prototype T sections. Derived sections may be used alone, with nothing between, if attenuation at $f = \infty$ is not important.

The mismatch at frequencies near cutoff tends to slightly decrease the sharpness of the change in attenuation at the cutoff point, producing a small amount of attenuation inside the pass band near cutoff.

Prototype or m-derived sections of the π type may be analyzed in a similar manner. It is then found that if an m-derived π section is rearranged as a T section and split into half sections, these half sections, with $m = 0.6$, will give similarly satisfactory matching of impedances. These half sections are shown in Fig. 4-21.

4-14. Band-pass filters

Occasionally it is desirable to pass a band of frequencies and to attenuate frequencies on both sides of the pass band. The action

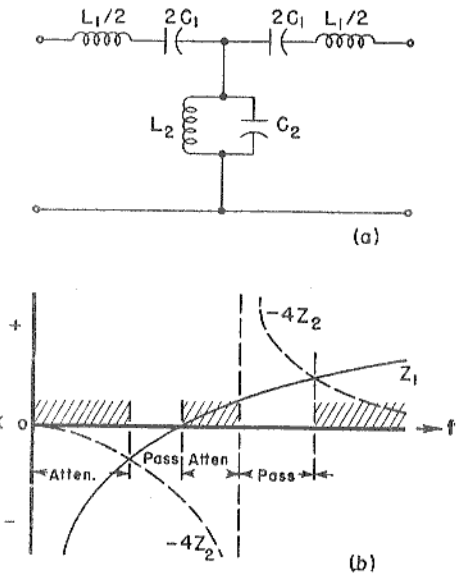


Fig. 4-22. (a) Band-pass filter network; (b) reactance curves showing possibility of two bands.

might be thought of as that of low-pass and high-pass filters in series, in which the cutoff frequency of the low-pass filter is above the cutoff frequency of the high-pass filter, the overlap thus allowing only a band of frequencies to pass. Although such a design would

function, it is more economical to combine the low- and high-pass functions into a single filter section.

Consider the circuit of (a), Fig. 4-22, with a series-resonant series arm and an antiresonant shunt arm. In general, the reactance curves show that two pass bands might exist. If, however, the antiresonant frequency of the shunt arm is made to correspond to the resonant frequency of the series arm, the reactance curves become as shown in Fig. 4-23 and only one pass band appears. For

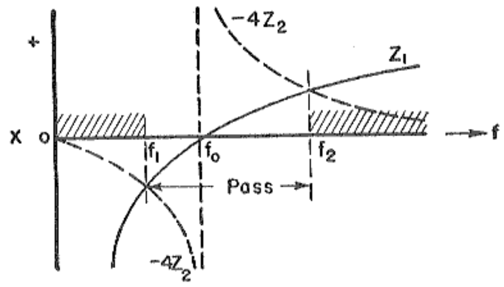


Fig. 4-23. Reactance curves for the band-pass network when resonant and antiresonant frequencies are properly adjusted.

this condition of equal resonant frequencies,

$$\omega_0^2 L_1 C_1 = 1 = \omega_0^2 L_2 C_2 \tag{4-85}$$

or

$$L_1 C_1 = L_2 C_2$$

The impedances of the arms are

$$Z_1 = j\left(\omega L_1 - \frac{1}{\omega C_1}\right) = j\,\frac{(\omega^2 L_1 C_1 - 1)}{\omega C_1} \tag{4-86}$$

$$Z_2 = \frac{j\omega L_2(-j/\omega C_2)}{j(\omega L_2 - 1/\omega C_2)} = \frac{j\omega L_2}{1 - \omega^2 L_2 C_2} \tag{4-87}$$

That a network such as (a), Fig. 4-23 is still a constant-k filter is easily shown as

$$Z_1 Z_2 = -\frac{L_2(\omega^2 L_2 C_1 - 1)}{C_1(1 - \omega^2 L_2 C_2)}$$

and if $L_1 C_1 = L_2 C_2$, then

$$Z_1 Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = R_k^2 \tag{4-88}$$

Thus the previously developed theory still applies.

At the cutoff frequencies,

$$Z_1 = -4Z_2$$

Multiplying by Z_1 gives

$$Z_1^2 = -4Z_1Z_2 = -4R_k^2$$

from which the value of Z_1 at the cutoff frequencies is obtained as

$$Z_1 = \pm j2R_k \tag{4-89}$$

so that

$$Z_1 \text{ at lower cutoff } f_1 = -Z_1 \text{ at upper cutoff } f_2$$

The reactance of the series arm at the cutoff frequencies then can be written by use of the above as

$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = \omega_2 L_1 - \frac{1}{\omega_2 C_1}$$

$$1 - \omega_1^2 L_1 C_1 = \frac{\omega_1}{\omega_2}(\omega_2^2 L_1 C_1 - 1) \tag{4-90}$$

Now from Eq. 4-85,

$$L_1 C_1 = \frac{1}{\omega_0^2} \tag{4-91}$$

so that Eq. 4-90 may be written as

$$1 - \frac{f_1^2}{f_0^2} = \frac{f_1}{f_2}\left(\frac{f_2^2}{f_0^2} - 1\right)$$

$$f_0^2(f_1 + f_2) = f_1 f_2 (f_1 + f_2)$$

$$f_0 = \sqrt{f_1 f_2} \tag{4-92}$$

or the frequency of resonance of the individual arms should be the geometric mean of the two frequencies of cutoff.

If the filter is terminated in a load $R = R_k$, as is customary, then the values of the circuit components can be determined in terms of R and the cutoff frequencies f_1 and f_2. At the lower cutoff frequency,

$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = 2R$$

$$1 - \frac{f_1^2}{f_0^2} = 4\pi R f_1 C_1$$

In view of Eq. 4-92, the expression for C_1 becomes

$$C_1 = \frac{f_2 - f_1}{4\pi R f_1 f_2} \tag{4-93}$$

It follows, then, from Eqs. 4-91 and 4-92, that

$$L_1 = \frac{R}{\pi(f_2 - f_1)} \tag{4-94}$$

From Eq. 4-88, it is possible to obtain the values for the shunt arm as

$$L_2 = C_1 R^2 = \frac{R(f_2 - f_1)}{4\pi f_1 f_2} \tag{4-95}$$

$$C_2 = \frac{L_1}{R^2} = \frac{1}{\pi R(f_2 - f_1)} \tag{4-96}$$

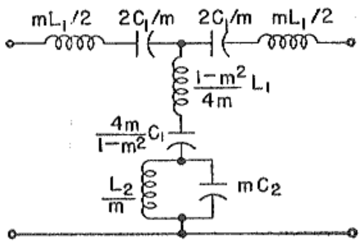


Fig. 4-24. m-derived band-pass section.

This completes the design of the prototype band-pass filter.

An m-derived band-pass section is also possible. Use of the transformation relations developed in Section 4-10 leads to a network of the form of Fig. 4-24. The shunt arm then consists of series-resonant and antiresonant circuits in series. Plotting reactance curves for these two circuits and adding to obtain the reactance variation of the shunt arm, Z_2 of the filter, gives the dashed curve of Fig. 4-25. The antiresonant frequency of the arm as a whole must, by previous reasoning, be f_0 of the filter. The reactance curve for Z_2 then shows that the shunt arm becomes resonant at a frequency below f_0 and again at a frequency above f_0. At these

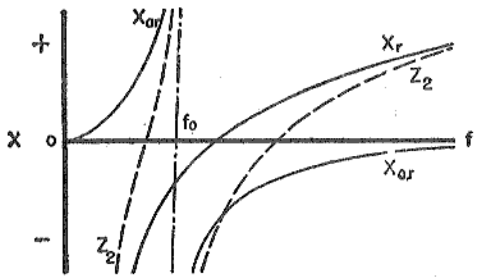


Fig. 4-25. Reactance curves for the shunt arm of the m-derived band-pass section.

frequencies the network is short-circuited, and thus they are frequencies of high attenuation, f_∞. These frequencies of high attenuation are placed on each side of the pass band, and the m-derived section may be used to increase the attenuation near cutoff, as for the high- or low-pass cases.

At one f_∞, the reactances X_r and X_{ar} are equal and opposite, so that

$$j\omega_\infty\left(\frac{1-m^2}{4m}\right)L_1 - \frac{j}{\omega_\infty[4m/(1-m^2)]C_1} = \frac{(j\omega_\infty L_2/m)(-j/\omega_\infty mC_2)}{j(1/\omega_\infty mC_2 - \omega_\infty L_2/m)}$$

$$\frac{1-m^2}{4}(\omega_\infty^2 L_1C_1 - 1) = \frac{\omega_\infty^2 L_2C_1}{\omega_\infty^2 L_2C_2 - 1} \quad (4\text{-}97)$$

In view of the fact that

$$L_1C_1 = L_2C_2 = \frac{1}{\omega_0^2}$$

Eq. 4-97 becomes

$$\frac{1-m^2}{4}\left(\frac{f_\infty^2}{f_0^2} - 1\right)^2 = 4\pi^2 f_\infty^2 L_2C_1 \quad (4\text{-}98)$$

The term L_2C_1 can be evaluated as a function of f_1 and f_2 from Eqs. 4-93 and 4-95:

$$L_2C_1 = \frac{f_2 - f_1}{4\pi R f_1 f_2}\left[\frac{(f_2 - f_1)R}{4\pi f_1 f_2}\right] = \frac{(f_2 - f_1)^2}{16\pi^2 f_0^4}$$

Equation 4-98 then reduces to

$$(1 - m^2)(f_\infty^2 - f_1f_2)^2 = f_\infty^2(f_2 - f_1)^2 \quad (4\text{-}99)$$

$$f_\infty^2 - \frac{(f_2 - f_1)f_\infty}{\sqrt{1 - m^2}} - f_1f_2 = 0$$

Solving for the values of the frequencies of peak attenuation,

$$f_\infty = \frac{f_2 - f_1}{2\sqrt{1 - m^2}} \pm \sqrt{\frac{(f_2 - f_1)^2}{4(1 - m^2)} + f_1f_2} \quad (4\text{-}100)$$

It is apparent that the radical is larger than $(f_2 - f_1)/2\sqrt{1 - m^2}$, and thus one root would appear as a negative frequency that has no physical significance here. Thus the expression for f_∞ should be

reversed so that the two frequencies of peak attenuation are

$$f_{\infty 1} = \sqrt{\frac{(f_2 - f_1)^2}{4(1 - m^2)} + f_1f_2} - \frac{f_2 - f_1}{2\sqrt{1 - m^2}} \quad (4\text{-}101)$$

$$f_{\infty 2} = \sqrt{\frac{(f_2 - f_1)^2}{4(1 - m^2)} + f_1f_2} + \frac{f_2 - f_1}{2\sqrt{1 - m^2}} \quad (4\text{-}102)$$

Equation 4-99 may be solved to determine the value of m, giving

$$m = \sqrt{1 - \left[\frac{f_\infty(f_2 - f_1)}{f_\infty^2 - f_1f_2}\right]^2}$$

$$= \frac{\sqrt{(f_\infty^2 - f_1^2)(f_\infty^2 - f_2^2)}}{f_\infty^2 - f_1f_2} \quad (4\text{-}103)$$

The value of m may be chosen to place either one of the two frequencies of peak attenuation at a desired point, the other frequency of peak attenuation then being definitely fixed. That this is true may be chosen by forming the product for $f_{\infty 1}f_{\infty 2}$ from Eqs. 4-101 and 4-102:

$$f_{\infty 1}f_{\infty 2} = \frac{f_2^2 + 2f_1f_2 + f_1^2 - 4m^2f_1f_2 - f_2^2 + 2f_1f_2 - f_1^2}{4(1 - m^2)} = f_1f_2 \quad (4\text{-}104)$$

$$\sqrt{f_{\infty 1}f_{\infty 2}} = f_0 \quad (4\text{-}105)$$

Thus f_0 is the geometric mean of the frequencies of peak attenuation and, by Eq. 4-92, of the cutoff frequencies as well. If m is selected to place $f_{\infty 1}$ at a desired point, then by Eq. 4-105, $f_{\infty 2}$ is automatically fixed, and vice versa. It is possible to fix both $f_{\infty 1}$ and $f_{\infty 2}$ by the use of two m's or an mm'-derived filter, as shown by Zobel (Reference 2).

An m-derived T section, rearranged as a π, may be split into two half sections and used as terminating half sections. If m is given the value 0.6, then satisfactory impedance matching conditions are maintained over the pass band. This usage follows the previously developed theory for low- or high-pass sections.

4-15. Band-elimination filters

If the series- and parallel-tuned arms of the band-pass filter are interchanged, the result is the band-elimination filter of (a), Fig.

4-26. That this circuit does eliminate or attenuate a given frequency band is shown by the reactance curves for Z_1 and $-4Z_2$ at (b). The action may be thought of as that of a low-pass filter in parallel with a high-pass section, in which the cut-off frequency of the low-pass filter is below that of the high-pass filter.

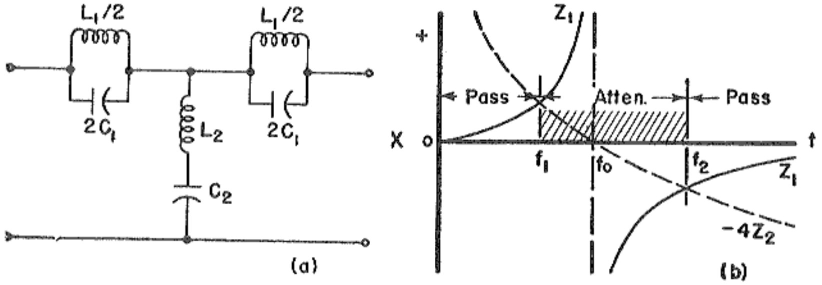


Fig. 4-26. (a) Band-elimination filter; (b) reactance curves showing action of band-elimination section.

As for the band-pass filter, the series and shunt arms are made antiresonant and resonant at the same frequency f_0. Again, it is possible to show that

$$R_k^2 = \frac{L_1}{C_2} = \frac{L_2}{C_1} \tag{4-106}$$

and that

$$f_0 = \sqrt{f_1 f_2} \tag{4-107}$$

At the cutoff frequencies,

$$Z_1 = -4Z_2, \qquad Z_1 Z_2 = -4Z_2^2 = R_k^2$$

$$Z_2 = \pm \frac{jR_k}{2} \tag{4-108}$$

If the filter is terminated in a load $R = R_k$, then at the lower cutoff frequency,

$$Z_2 = j\left(\frac{1}{\omega_1 C_2} - \omega_1 L_2\right) = \frac{jR}{2}$$

Since $L_2 C_2 = 1/\omega_0^2$,

$$1 - \frac{f_1^2}{f_0^2} = \pi f_1 C_2 R$$

$$C_2 = \frac{1}{\pi R}\left(\frac{f_2 - f_1}{f_1 f_2}\right) \tag{4-109}$$

In view of the fact that

$$f_0 = \sqrt{f_1 f_2} = \frac{1}{2\pi\sqrt{L_2 C_2}}$$

then

$$L_2 = \frac{R}{4\pi(f_2 - f_1)} \tag{4-110}$$

By use of Eq. 4-106, the values for the series arm are obtained as

$$L_1 = \frac{R(f_2 - f_1)}{\pi f_1 f_2} \tag{4-111}$$

$$C_1 = \frac{1}{4\pi R(f_2 - f_1)} \tag{4-112}$$

Sections of the m-derived form may also be obtained.

4-16. Filter-circuit design

As an example of the manner in which a complete filter may be designed, it will be convenient to carry out the calculations on a typical filter to meet the following specifications:

A composite low-pass filter is to be terminated in 500 ohms resistance. It must have a cutoff frequency of 1000 cycles, with very high attenuation at 1065, 1250, and ∞ cycles. The prototype is designed first as

$$L = \frac{R}{\pi f_c} = \frac{500}{\pi \times 1000} = 0.159 \text{ henry}$$

$$C = \frac{1}{\pi f_c R} = \frac{1}{\pi \times 1000 \times 500} = 0.636 \text{ microfarad}$$

This prototype section meets the specification for high attenuation at infinity. The assembly of the section is illustrated in the circuit of Fig. 4-27, with inductance of $L/2$ in each series arm.

The m-derived section to provide high attenuation at $f_\infty = 1065$ cycles may then be designed:

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1 - \left(\frac{1000}{1065}\right)^2}$$

$$= \sqrt{1 - 0.882} = 0.343$$

$$\frac{mL}{2} = \frac{0.343 \times 0.159}{2} = 0.0273 \text{ h}$$

$$\frac{1 - m^2}{4m} L = \frac{1 - 0.118}{1.372} \times 0.159 = 0.102 \text{ h}$$

$$mC = 0.343 \times 0.636 = 0.218 \text{ microfarad}$$

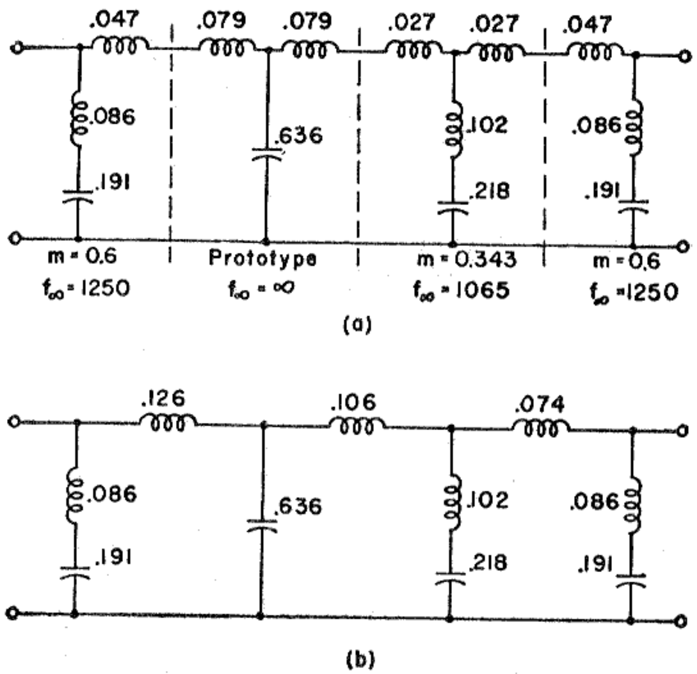


Fig. 4-27. (a) Filter as designed; (b) as it would be combined.

These values are then assembled as in Fig. 4-27.

The section providing high attenuation at 1250 cycles will have a value of m given by

$$m = \sqrt{1 - \left(\frac{1000}{1250}\right)^2} = \sqrt{1 - 0.640} = 0.6$$

and this will be appropriate for use in the terminal half sections.

Then the circuit elements should be

$$\frac{mL}{2} = \frac{0.6 \times 0.159}{2} = 0.0474 \text{ henry}$$

$$\frac{1 - m^2}{2m} L = \frac{1 - 0.36}{1.2} \times 0.159 = 0.086 \text{ henry}$$

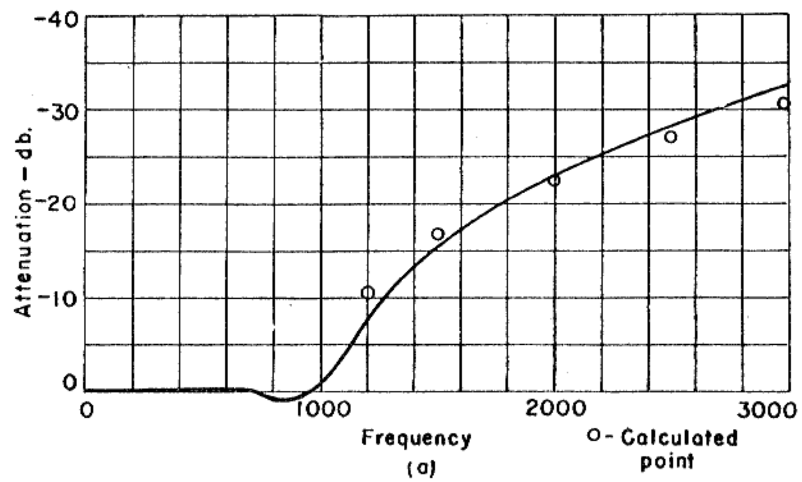
$$\frac{mC}{2} = \frac{0.6 \times 0.636}{2} = 0.191 \text{ microfarad}$$

These sections are all shown individually in Fig. 4-27(a). For economy, it is customary to combine elements wherever possible. The series inductors may then be added, resulting in the final design of (b). In making a mechanical assembly it is necessary to avoid electric and magnetic couplings. To aid in this and to improve the magnetic circuit, the inductors are ordinarily wound as toroids on ring cores. Magnetic materials of very high permeability are used, these usually being high-nickel alloys such as permalloy, or powdered iron or permalloy materials in compressed form. The values of Q obtained should be as high as possible so that the filter performance will closely approximate the performance calculated for pure reactances.

4-17. Filter performance

To illustrate the sort of approach to the theoretical ideal which is possible in filter design, laboratory filters were assembled in accordance with the designs of Fig. 4-27. The inductors used were toroids on compressed molybdenum-permalloy dust cores, and had Q values of approximately 40. These inductors would not be considered as having very high Q by commercial standards, as values of 100 to 300 are available. The filters were designed for 500 ohms resistance termination, and were so used for each measurement.

Attenuation measurements were made over the pass and stop bands on each of the filter sections. The attenuation of the prototype section, terminated in 500 ohms, is shown in (a), Fig. 4-28. The presence of resistance and the insertion loss of the section causes a rounding of the attenuation curve near cutoff, but otherwise the shape of the curve reasonably fits the theoretical curve of Fig. 4-9. Calculation of attenuation, based on pure reactances and the



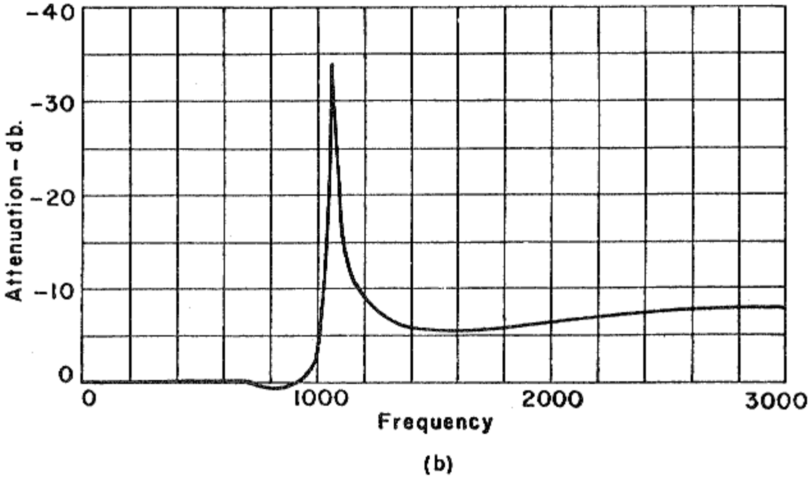


Fig. 4-28. (a) Attenuation of the prototype: $f_c = 1000$ cycles, $R_k = 500$ ohms. (b) Attenuation of the m-derived section: $m = 0.346$, $f_c = 1000$, $f_\infty = 1065$, $R_k = 500$ ohms. (c) Action of composite filter of Fig. 4-27(b).

theoretical equations of Section 4-8, gives 16.7 db attenuation at 1500 cycles or $f/f_c = 1.5$. The measured curve shows 15.5 db, which is a reasonable check. Calculated values of α for a pure

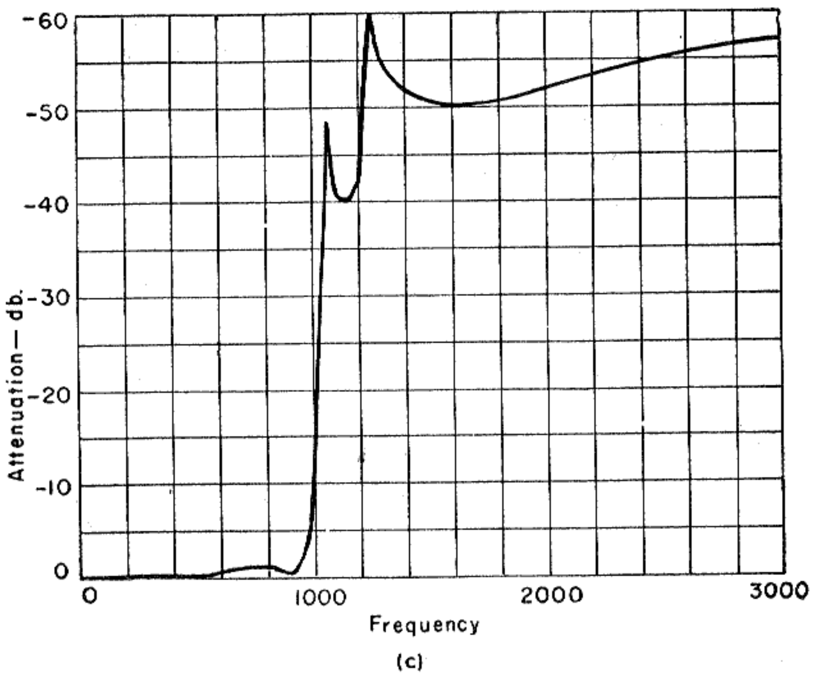


Fig. 4-28. (*cont.*).

reactance network are shown as marked points for a few frequencies in Fig. 4-28(a), for comparison with the measurements.

Figure 4-28(b) illustrates the measured attenuation for the m-derived section alone, with $m = 0.343$. This section was designed to have $f_\infty = 1065$ cycles and a cutoff of 1000 cycles, and a peak of attenuation of -34 db is obtained at the frequency specified. The usually undesirable low α values above f_∞ are also found, the slight rise at 3000 cycles probably being due to increased losses in the coils used.

Figure 4-28(c) shows the over-all attenuation of the complete filter of Fig. 4-27, when the prototype and m-derived sectons of (a) and (b) are combined with terminal half sections having $m = 0.6$ and terminated in 500 ohms. The terminal half sections give an additional value of high attenuation at $f_\infty = 1250$ cycles.

Due to probable slight mismatches and losses, there is a small irregularity in the pass band, but reasonably sharp cutoff characteristics are obtained.

4-18. Crystal filters

The lattice structure can also be shown to have filter properties. Considering the network of Fig. 4-29,

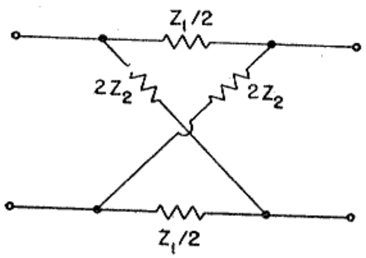


Fig. 4-29. Lattice filter section.

$$Z_{oc} = \frac{(Z_1/2 + 2Z_2)^2}{2(Z_1/2 + 2Z_2)} = \frac{Z_1}{4} + Z_2 \qquad (4\text{-}113)$$

$$Z_{sc} = \frac{Z_1Z_2}{Z_1/2 + 2Z_2} + \frac{Z_1Z_2}{Z_1/2 + 2Z_2} = \frac{Z_1Z_2}{Z_1/4 + Z_2} \qquad (4\text{-}114)$$

The characteristic impedance of the lattice section then is

$$Z_{0L} = \sqrt{Z_{oc}Z_{sc}} = \sqrt{Z_1Z_2} \qquad (4\text{-}115)$$

Thus if the section elements are reactive, Z_{0L} is real, or a pass band exists for frequencies for which Z_1 and Z_2 are of opposite sign. Over ranges where Z_1 and Z_2 have the same sign, an attenuation band exists.

Propagation can be investigated further by noting that

$$\tanh \gamma = \sqrt{\frac{Z_{sc}}{Z_{oc}}} = \sqrt{\frac{Z_1}{Z_2}}\left[\frac{1}{1 + Z_1/4Z_2}\right] \qquad (4\text{-}116)$$

It may be noted that Z_{0L} depends on the product of Z_1 and Z_2, whereas γ depends on the ratio of Z_1 to Z_2. This feature permits somewhat greater versatility in design of the lattice section over the T or π section, especially for filters in which certain of the elements are constructed of piezoelectric crystals. These crystals have a resonant frequency of mechanical vibration dependent on certain of their dimensions; and because of the very high equivalent Q of the crystals, it is possible to make very narrow band filters and filters in which the attenuation rises very rapidly at cutoff.

The equivalent electric circuit of a quartz mechanical-filter

crystal is shown in Fig. 4-30(a), which shows a possibility of both resonance and antiresonance occurring. The inductance L_x is very large, being in henrys for crystals resonating near 500 kc, so that while R_x may approximate a few hundred or few thousand ohms, the effective Q may be in the range of 10,000 to 30,000. Considering the properties of resonant circuits, such as Q would provide a band width of 20 to 50 cycles at 500 kc.

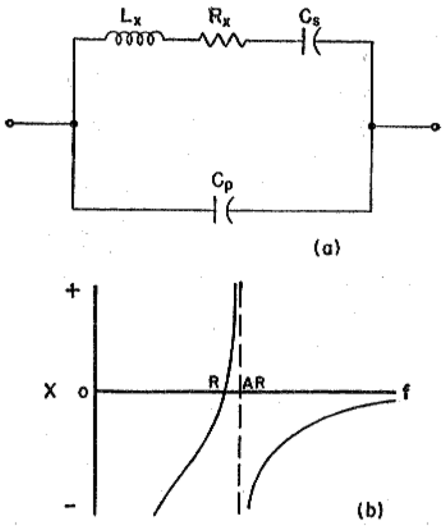


Fig. 4-30. (a) Equivalent electrical circuit for a piezoelectric crystal; (b) reactance curves for the circuit of (a).

The resistance of the crystal is due largely to mechanical damping introduced by the electrodes and by the surrounding atmosphere. By placing a crystal in an evacuated container, the value of Q can be notably increased. The electrodes are normally electroplated onto the crystal faces and need not introduce much damping.

Capacitance C_s is the equivalent series capacitance of the crystal forming a resonant circuit with L_x. Capacitance C_p is the parallel capacitance introduced by the crystal electrodes. The values of C_s and C_p are such that $C_p \gg C_s$, so that the resonant and antiresonant frequencies of the circuit lie very close together, differing by a fraction of 1 per cent of the resonant frequency. The reactance-curve sketch of Fig. 4-30(b) shows the resonant frequency below the antiresonant one. By placing adjustable capacitors in parallel with the crystal, C_p can be increased, resulting in the antiresonant frequency being moved closer to the resonant point.

Since the crystal represents either a resonant or antiresonant circuit, it may be used to replace the normal elements of the band-pass or band-elimination filter. As previously shown for band-pass action, the resonant frequency of one arm must equal the antiresonant frequency of the other arm. The pass band with crystal

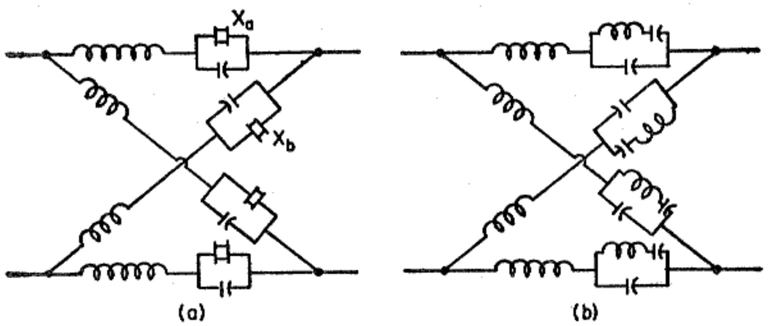


Fig. 4-31. (a) Circuit of a lattice crystal filter with series inductors and parallel capacitors; (b) the electrical equivalent of (a).

elements will then be found to extend from the lowest crystal resonant frequency to the highest crystal antiresonant frequency, or a width of pass band equal to twice the separation of the resonant and antiresonant frequencies of one crystal. This range will result in a pass band a fraction of 1 per cent wide. The band width can be reduced by putting adjustable capacitors in parallel with the crystal, furnishing a means of adjustment of the width of the pass band.

By the addition of coils in series with the crystals the pass bands may be widened. Since the added coils have Q values very much below those of the crystals, there will be some loss in sharpness at cutoff. A circuit including series coils is shown in Fig. 4-31(a), with its equivalent drawn at (b). The reactance curves for the A and B portions of this circuit are drawn in Fig. 4-32(a), which shows how the resonances and antiresonances are arranged. The presence of the series coil adds an additional resonance, and the pass band exists from the lowest resonance of one crystal to the highest resonance of the other. If f_1 and f_2 are the frequencies of resonance of one of the

circuits and f_R is that of the antiresonance, then

$$f_{1,2} = f_R\sqrt{1 \mp \frac{C_s}{C_p}}$$

The separation of f_1 and f_2 represents two-thirds of the pass band and is seen to depend on the $\sqrt{C_s/C_p}$ ratio. Since C_s/C_p may be of the order of 0.01, it can be seen that the separation of f_1 and f_2 may be of

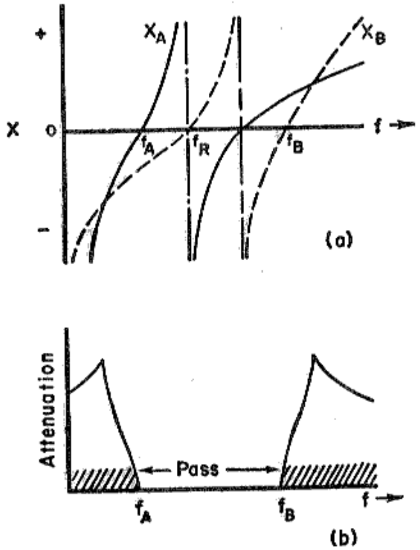


Fig. 4-32. (a) Reactance curves for the circuit of Fig. 4-31(a); (b) attenuation curves for that circuit.

the order of 0.10 f_R, or 10 per cent of the resonant frequency. By placing coils in series with the crystals, it has been possible to widen the pass band considerably. By adjustment of C_p it is then possible to narrow the band to any desired amount.

Thus the use of coils permits the bands to be widened to pass speech frequencies, and crystal filters are quite generally used to separate the various channels in carrier telephone circuits, in the range above 50 kilocycles.

PROBLEMS

4-1. (a) A generator with output of 0.0025 w supplies power to an amplifier which in turn supplies a 600-ohm load. If the power level in the load is to be +16 db above 0.001 w reference, what power gain in decibels is required of the amplifier?

(b) If a transformer of 65 per cent efficiency is placed between amplifier and load, what amplifier power gain will be needed, in decibels, to maintain the same load power?

4-2. A radio receiver has an input impedance that is 200 ohms resistive. The signal picked up and applied to this input is 400 μv. The electric output to the loudspeaker is to be at +32 db level (6 mw reference). Find

(a) Input power level in decibels.

(b) Decibels gain in the receiver.

(c) Output power in watts.

4-3. The output of a certain vacuum tube is 5.2 w.

(a) Find the decibel level (0.001 w reference).

(b) How many decibels will be added to the output level if the power is made 4 times greater?

(c) A transformer of 70 per cent efficiency is used between the output and the load. What is the decibel loss in the transformer?

4-4. The output of a certain amplifier is at a level of +37 db (0.006 w reference). It is applied to a telephone line of 60 per cent efficiency, followed by a transformer with losses measured at 2.4 w.

(a) Find the power delivered by the transformer, in watts and decibels.

(b) Find the line loss in decibels and nepers.

(c) Find the decibels transformer loss.

(d) Find the current in the load if it is 20 ohms, resistive.

4-5. A T section having $Z_1/2 = j125$ ohms, $Z_2 = -j600$ ohms, is used between a 10-v generator of Z_0 impedance and a Z_0 load.

(a) Find the power transferred to the load.

(b) Find the values of γ, α, and β.

(c) If Z_2 is changed to $+j600$ ohms, repeat (a) and (b).

4-6. A π section has the series arm made up of a 100-mh coil, each shunt arm consisting of a 0.15-μf, capacitor. Plot the magnitude and angle of Z_0, from zero frequency to 2500 c.

4-7. If the inductor in Prob. 4-6 has a resistance of 50 ohms, and neglecting the resistance of the capacitors, calculate by means of open- and short-circuit impedances, the Z_0 at 0, 500, 1000, 1500, and 2000 c.

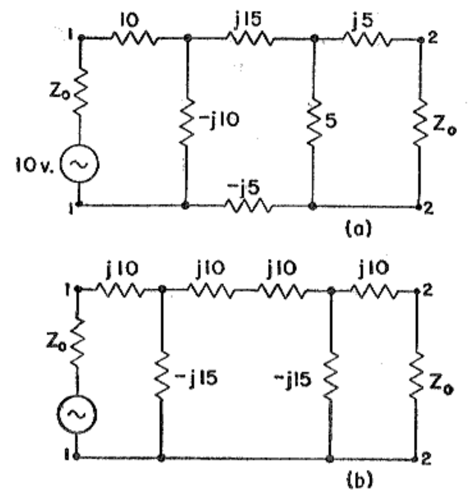


Fig. 4-33.

Fig. 4-34.

4-8. In Fig. 4-33(a), find the values of α and β, the power delivered to the Z_0 load, and the nepers and decibels loss in the network, between 1,1 and 2,2 terminals.

4-9. Repeat Prob. 4-8, using Fig. 4-33(b). $E = 10$ v.

4-10. Design a low-pass T section with cutoff at 2500 c to work in an audio amplifier between a vacuum tube of 10,000 ohms plate resistance and a 10,000-ohm load.

(a) Compute the attenuation in nepers and decibels at 3000 c, and at the second harmonic of the cutoff, 5000 c.

(b) Find the phase shift at 500, 1000, 1500, and 2000 c.

4-11. Find the values of circuit elements needed for a high-pass T section filter with cutoff of 1500 c, to work into a 1000-ohm load. Draw the circuit diagram.

4-12. For the networks of Fig. 4-34, determine Z_{1i} and Z_{2i}.

4-13. Compute the elements of a constant-k low-pass network having $f_c = 2000$ c and load of 5000 ohms. Compute α and β for the network over the range of 0 to 6000 c, and plot the results.

4-14. The series arm Z_1 of a filter consists of a 0.5-μf capacitor in series with an inductor of 0.35 h. If $R_k = 500$ ohms, determine the elements for the shunt arm and their manner of connection.

4-15. A filter on the input to a telephone line is to attenuate all frequencies above 1500 c, with particularly large attenuation at 2000 c. The input resistance of the telephone line is 550 ohms. Design and draw the resultant circuit diagram, assuming a reasonably constant Z_0 is desired. Plot the attenuation characteristic in the stop band.

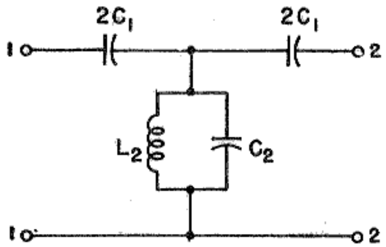


Fig. 4-35.

4-16. Determine the propagation characteristics of the circuit shown in Fig. 4-35 by use of reactance curves.

4-17. A power-supply filter for a radio receiver consists of a π section with a 20-h series arm and 20-μf capacitors as shunt arms.

(a) Calculate the cutoff frequency.
(b) Find the decibels attenuation at 60 and 120 c.
(c) If a voltage

$$e = 400 + 320 \sin 2\pi 60t + 160 \sin 2\pi 120t$$

is applied to this filter, find the current of each frequency flowing in a 4000-ohm resistive load.

(d) Find the ratio of 60-c power to d-c power in the load, in decibels.

4-18. A composite low-pass filter has two T prototype sections

and an m-derived section with $m = 0.4$. For the filter $R_k = 600$ ohms, and $f_c = 796$ c, plot α and β for the range from 0 to 5000 c.

4-19. Design a low-pass filter to work into 1250 ohms, with cutoff at 600 c and with high attenuation at 900 and 1200 c. Terminate it properly and draw the complete circuit. Plot the attenuation characteristic over the attenuation band, in decibels.

4-20. A filter to pass the band between 1250 and 2000 c, with high attenuation at 2500 c, is desired to work into a 4000-ohm load. Design the circuit and find the second frequency of high attenuation. Use proper terminating sections.

4-21. An antenna filter in a 3-megacycle radio transmitter is needed to suppress the second harmonic of the output frequency. Choose and design a section to be used, keeping economy of equipment in mind. The load may be considered as 100 ohms.

4-22. Set up the basic circuit for the T type m-derived band-elimination filter.

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Chapter 5

TRANSMISSION-LINE PARAMETERS

The networks which have so far been discussed are called circuits of *lumped parameters*, wherein the resistance, inductance, and capacitance are individually concentrated or lumped at discrete points in the circuit, and can be identified definitely as representing a particular parameter. The electric line used for transmission of telephone messages or for the transmission of power is a common example of an electric circuit with *distributed parameters*. This term implies that the resistance, inductance, and capacitance are distributed along the circuit, each elemental length of the circuit having its own values, and concentration of the individual parameters is not possible.

The first few sections of this chapter will treat the inductance and capacitance of two special forms of line, namely, the open-wire and the coaxial line. The later sections will develop methods for calculation of the parameters of more general forms of multiconductor lines.

5-1. Line parameters

A common form of transmission line is known as the *open-wire line* because of its construction. The ordinary telephone line, strung on cross arms on poles, or the power transmission line on towers, are examples of the open line. The conductors of such lines may be considered parallel and separated by air dielectric.

Another form of line construction is the *cable*. For telephone use this consists of hundreds of individually paper-insulated conductors, twisted in pairs, and combined inside a protective lead or plastic sheath. Power transmission cables will employ only two or three large conductors, insulated with oil impregnated paper or other solid dielectric, inside the protective sheath. In either case the conductors may be considered again as parallel, with a solid dielectric.

A different form of construction is employed with the *coaxial*