$$C = \log_2\left[1+\exp\left(\frac{\alpha-h}{1-\alpha}\right)\right]$$

**Example 2.23**

*If $\pi_1$ and $\pi_2$ are channel matrices of discrete memoryless channels $k_1$ and $k_2$ respectively, the sum of $k_1$ and $k_2$ is defined as the channel whose matrix is*

$$\begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix}$$

*Thus the sum channel may be operated by choosing one of the individual channels and transmitting a digit through it. The input or output symbol always identifies the particular channel used. If $C_i$ is the capacity of $K_i$, $i = 1, 2$ and $C$ is the capacity of the sum show that*

$$2C = 2^{c_1} + 2^{c_2}.$$

***Solution***

Let the input alphabet of $k_1$ and $k_2$ be $x_1, \ldots . x_r$ and $x_{r+1}, \ldots . x_M$ respectively; let the output alphabets of $k_1$ and $k_2$ be $y_1, \ldots y_s$ and $y_{s+1}, \ldots$ $y_N$. If $p(x)$ is any input distribution, and $P = \sum_{i=1}^{r} p(x_i)$, then

$$H(X) = -\sum_{i=1}^{M} p(x_i)\log p(x_i)$$

$$= -p\log p - (1-p)\log(1-p) + pH_1(X) + (1-p)H_2(X)$$

(By grouping axiom)

where $$H_1(X) = -\sum_{i=1}^{r}\frac{p(x_i)}{p}\log\frac{p(x_i)}{p}$$

is the input uncertainity of $k_1$ under the distribution

$$\{p(x_i)/p, i = 1, \ldots r\}$$

and $$H_2(X) = -\sum_{i=r+1}^{M} \frac{p(x_i)}{1-p} \log \frac{p(x_i)}{1-p}$$

is the input uncertainity of $k_2$ under the distribution

$$\{p(x_i)/(1-p),\ i = r+1, \ldots M\}.$$

[Alternately, $$H(X) = -\sum_{i=1}^{r} p \frac{p(x_i)}{p} \log \left[ \left( \frac{p(x_i)}{p} \right) p \right]$$

$$-\sum_{i=r+1}^{M} \frac{(1-p)p(x_i)}{(1-p)} \log \left[ \frac{p(x_i)}{(1-p)} (1-p) \right]$$

leading to the same expression as above.]

Now, $$H(X/Y) = -\sum_{i=1}^{M} \sum_{j=1}^{N} p(x_i, y_j) \log p(x_i / y_j)$$

$$= -\sum_{i=1}^{r} \sum_{j=1}^{s} p \frac{p(x_i)}{p} p(y_j / x_i) \log p(x_i / y_j)$$

$$-\sum_{i=r+1}^{M} \sum_{j=s+1}^{N} (1-p) \left[ \frac{p(x_i)}{1-p} \right] p(y_j / x_i) \log p(x_i / y_j)$$

Now if $1 \le i \le r,\ 1 \le j \le s$, the conditional probability.

$$p(x_i/y_j) = \frac{p(x_i)p(y_j / x_i)}{p(y_j)} = \frac{p[p(x_i)/p]p(y_j / x_i)}{p\sum_{i=1}^{M} \frac{p(x_i)}{p} . p(y_j / x_i)}$$

$$= p\{X_1 = x_i / Y_1 = y_j\}$$

that is, the conditional probability that the input of $k_1$ is $x_i$ given that the output of $k_1$ is $y_j$, under the distribution $\{p(x_i)/p,\ i = 1, 2, \ldots, r\}$.

Thus

$$H(X/Y) = pH_1(X/Y) + (1-p)H_2(X/Y)$$

and

$$I(X ; Y) = H(X) - H(X/Y).$$

Therefore, $H(X/Y) = H(p, 1 - p) + p\, I_1(X ; Y) + (1 - p)\, I_2(X ; Y)$

where the subscript $i$ denotes that the indicated quantity is calculated for $k_i$ under the appropriate input distribution. For a given p, we are completely free to choose the probabilities $p(x_i)/p$, $i = 1, \ldots r$ and $p(x_i)/(1 - p)$, $i = r+1, \ldots M$.

We can do no better than to choose the probabilities so that

$$I_1(X ;Y) = C_1, \quad I_2(X ; Y) = C_2.$$

Thus it remains to maximize

$$H(p, 1 - p) + pC_1 + (1 - p)C_2,$$

Differentiating we obtain,

$$-1 - \log p + 1 + \log(1 - p) + C_1 - C_2 = 0$$

$$p = \frac{2^{C_1}}{2^{C_1} + 2^{C_2}}$$

Hence, $$C = H\left(\frac{2^{C_1}}{2^{C_1} + 2^{C_2}}, \frac{2^{C_1}}{2^{C_1} + 2^{C_2}}\right) + \frac{C_1 2^{C_1} + C_2 2^{C_2}}{2^{C_1} + 2^{C_2}}$$

$$= \frac{2^{C_1}}{2^{C_1} + 2^{C_2}} \log\left(2^{C_1} + 2^{C_2}\right) + \frac{2^{C_2}}{2^{C_1} + 2^{C_2}} \log\left(2^{C_1} + 2^{C_2}\right)$$

Therefore $C = \log\left(2^{c_1} + 2^{c_2}\right)$

**Example 2.24**

*Evaluate the capacity of the channel whose matrix is given as*

$$\begin{bmatrix} 1-\beta & \beta & 0 \\ \beta & 1-\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

***Solution***

The given channel matrix is

$$\left[\begin{array}{cc|c} 1-\beta & \beta & 0 \\ \beta & 1-\beta & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

where $$\pi_1 = \begin{bmatrix} 1-\beta & \beta \\ \beta & 1-\beta \end{bmatrix}$$

$$\pi_2 = [1],$$

referring to the previous problem.

We have, $$2^C = 2^{c_1} + 2^{c_2}$$

Here, $$C_1 = \log m - h$$

$$= \log_2 2 - H(\beta, 1-\beta)$$

$$C_1 = 1 - H(\beta, 1-\beta)$$

$$C_2 = 0$$

Therefore, $$2^C = 2^{1-H(\beta, 1-\beta)} + 2^0$$

$$C = \log\left[1 + 2^{1-H(\beta, 1-\beta)}\right]$$

**Example 2.25**

*Evaluate the channel capacity of the channel whose matrix is given to be*

$$\begin{bmatrix} \frac{1-p}{2} & \frac{1-p}{2} & \frac{p}{2} & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} \end{bmatrix}$$

***Solution***

This is a symmetric channel.

Channel capacity $C = \log_2 4 + p \log \frac{1}{2} p + (1-p) \log \frac{1}{2}(1-p)$

or $C = 1 - H(p, 1-p)$

= capacity of a binary symmetric channel with error probability p.

**Example 2.26**

*Evaluate the channel capacity of the channel whose matrix is*

$$\begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & erase & 1 \end{array} \\ \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix} \end{array}$$

***Solution***

This is a Binary erasure channel (BEC).

If $p(X = 0) = \alpha$,

then $H(X) = H(\alpha, 1-\alpha)$

$$H(X/Y) = p(\text{erase})\, H(X/Y = \text{erase})$$

$$= p.\ H(\alpha, 1-\alpha)$$

Thus $I(X ; Y) = (1-p)\, H(\alpha, 1-\alpha)$,

which is a maximum for $\alpha = ½$.

Therefore $C = 1 - p.$

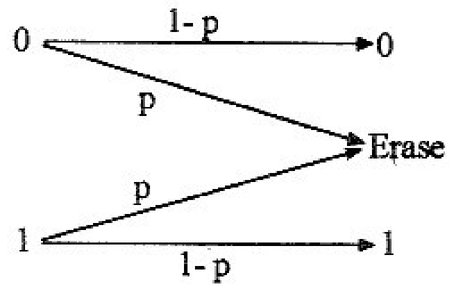


**Fig. 2.16** Illustration of channels for problem 2.26

## Example 2.27

*A signal is bandlimited to 5 KHz, sampled at a rate of 10, 000 samples per second, and quantized to 4 levels such that the sampled value can take values 0, 1, 2, 3 with probability p, p, (1/2) – p, (1/2)– p respectively and the channel is characterized by the transition probability matrix*

$$P = \{P_{ij}\} = P(X = x_i / Y = y_j]$$

$$Y = j$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

*Find the capacity of the channel.*

***Solution***

The channel model is shown below.

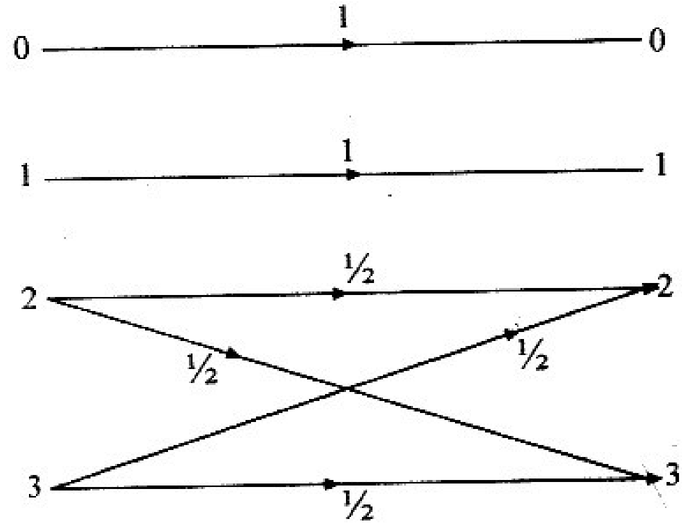


**Fig. 2.17** Illustration of channels for problem 2.27

$$P(X = 0) = p$$

$$P(X = 1) = p$$

$$P(X = 2) = ½ - p$$

$$P(X = 3) = ½ - p$$

$$H(X) = -\sum_{i=0}^{3} P(X = x_i)\log_2[p(X = x_i)]$$

$$= -2p\log_2(p) + 2\left(\frac{1}{2} - p\right)\log_2\left(\frac{1}{2} - p\right)$$

$$P(y = 0) = P(Y = 0 / X = 0)\,P(x = 0) + P(Y = 0 / X = 1)\,P(X = 1)$$
$$+ P(Y = 0 / X = 2)\,P(X = 2) + P(Y = 0 / X = 3)\,P(X = 3)$$

$$= 1(p) + 0\left(\frac{1}{2} - p\right) + 0\left(\frac{1}{2} - p\right) + 0(p) \quad = p$$

Similarly,

$P(Y = 1) = p$

$P(Y = 2) = P(Y = 2 / X = 0)\,P(X = 0) + P(Y = 2 / X = 1)\,P(X = 1)$

$$+ P(Y = 2 / X = 2)\, P(X = 2) + P(Y = 2 / X = 3)\, P(X = 3)$$

$$= 0.p + 0\left(\frac{1}{2} - p\right) + \frac{1}{2}\left(\frac{1}{2} - p\right) + \left(\frac{1}{2} - p\right)\frac{1}{2}$$

$$= \frac{1}{2} - p$$

Similarly,

$$P(Y = 3) = \left(\frac{1}{2} - p\right)$$

$$P(X = 0, Y = 0) = P(X = 0 / Y = 0)\, P(Y = 0)$$

$$= 1(p) = p$$

$$P(X = 0, Y = 1) = P(X = 0 / Y = 1)\, P(Y = 1)$$

$$= 0(p) = 0$$

$$P(X = 0, Y = 2) = P(X = 0 / Y = 0)\, P(Y = 2)$$

$$= 0\left(\frac{1}{2} - p\right) = 0$$

$$P(X = 0, Y = 3) = 0$$

$$P(X=1, Y=0)=0$$

$$P(X = 1, Y = 1) = P(X = 1 / Y = 1)\, P(Y = 1) = 1.p = p$$

$$P(X = 1, Y = 2) = 0$$

$$P(X = 1, Y = 3) = 0$$

$$P(X = 2, Y = 0) = 0$$

$$P(X = 2, Y = 1) = 0$$

$$P(X = 2, Y = 2) = 0 = \frac{1}{2}\left(\frac{1}{2} - p\right)$$

$$P(X = 2, Y = 3) = 0 = \frac{1}{2}\left(\frac{1}{2} - p\right)$$

$$P(X = 3, Y = 0) = 0$$

$$P(X = 3, Y = 1) = 0$$

$$P(X = 3, Y = 2) = \frac{1}{2}\left(\frac{1}{2} - p\right)$$

$$P(X = 3, Y = 3) = \frac{1}{2}\left(\frac{1}{2} - p\right)$$

Therefore

$$H(X/Y) = -p\log_2(1) - p\log_2(1) - \frac{1}{2}\left(\frac{1}{2} - p\right)\log_2\left(\frac{1}{2}\right)$$

$$-\frac{1}{2}\left(\frac{1}{2} - p\right)\log_2\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2} - p\right)\log_2\left(\frac{1}{2}\right)$$

$$-\frac{1}{2}\left(\frac{1}{2} - p\right)\log_2\left(\frac{1}{2}\right)$$

$$= 2\left(\frac{1}{2} - p\right)$$

$$f_s = 10 \text{ kHz.}$$

$$C = \max_p\ [H(X) - H(X/Y)].10^4$$

$$= \max_p\left[\left[-2p\log_2 p - 2\left(\frac{1}{2} - p\right)\log\left(\frac{1}{2} - p\right) - 2\left(\frac{1}{2} - p\right)\right].10^4\right]$$

$$\frac{\partial C}{\partial p} = \left\{ -2\log_2(p) - \frac{2p}{p} + 2\log_2\left(\frac{1}{2} - p\right) + \frac{2\left(\frac{1}{2} - p\right)}{\left(\frac{1}{2} - p\right)} + 2 \right\} 10^4 = 0$$

which on simplification gives

$$p = 1/3$$

Therefore,

$$C = \left[ -\frac{2}{3}\log_2\frac{1}{3} - 2\left(\frac{1}{2} - \frac{1}{3}\right)\log_2\left(\frac{1}{2} - \frac{1}{3}\right) - 2\left(\frac{1}{2} - \frac{1}{3}\right) \right].10^4$$

$$C = [\log_2 3]\ 10^4 \text{ bits/sec.}$$

**Example 2.28**

*If the received signal is given by*

$$Y(t) = X(t) + N(t)$$

*and X and Y are jointly Gaussian and the joint p.d.f. is given by*

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} exp\left\{ \frac{-1}{2(1-\rho^2)} \left[ \left(\frac{x}{\sigma_x}\right)^2 - 2\frac{\rho_{xy}}{\sigma_x\sigma_y} + \left(\frac{y}{\sigma_y}\right)^2 \right] \right\}$$

*where*

$$\rho = \frac{E(XY)}{\sigma_x\sigma_y}, \qquad \sigma_x^2 = E(X^2), \quad \sigma_y^2 = E(y^2)$$

*and X(t) and N(t) are independent random processes. Find I(X ; Y) and C.*

***Solution***

The marginal density of X is given by

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy = \frac{1}{\left(2\pi\sigma_x^2\right)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2\sigma_x^{\ 2}}\right).$$

$$f_y(y) = \frac{1}{\left(2\pi\sigma_y^{\ 2}\right)^{\frac{1}{2}}} \exp\left(\frac{-y^2}{2\sigma_y^{\ 2}}\right)$$

$$\frac{f_{xy}(x,y)}{f_x(x)f_y(y)} = \frac{1}{(1-\rho_1^{\ 2})^{1/2}} \exp\left\{\frac{-\rho_1^{\ 2}}{2(1-\rho_1)^2}\left[\left(\frac{x}{\sigma_x^{\ 2}}\right)^2 - \frac{2xy}{\rho_1\sigma_x\sigma_y} + \left(\frac{y}{\sigma_y}\right)^2\right]\right.$$

where $$\rho_1^{\ 2} = \frac{\sigma_x^{\ 2}}{\sigma_y^{\ 2}}$$

$$I(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y) \log \frac{f_x(x)f_x(y)}{f_{xy}(x,y)} dxdy$$

$$= -\frac{1}{2}\log_e\left(1-\rho_1^{\ 2}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\, dxdy$$

$$-\frac{\rho_1^{\ 2}}{2\left(1-\rho_1^2\right)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\left[\left(\frac{x}{\sigma_x}\right)^2 - \frac{2xy}{\rho_1\sigma_x\sigma_y} + \left(\frac{y}{\sigma_y}\right)^2\right] dxdy$$

Therefore, $$I(X, Y) = -\frac{1}{2}\log_e\left(1-\rho_1^{\ 2}\right)$$

Since X and N are independent,

$$\sigma_y^2 = \sigma_x^2 + \sigma_N^2$$

$$\rho_1^{\ 2} = \frac{\sigma_x^{\ 2}}{\sigma_y^{\ 2}} = \frac{\sigma_x^{\ 2}}{\sigma_x^{\ 2} + \sigma_N^{\ 2}}$$

Therefore,

$$I(X,Y) = -\frac{1}{2}\log_e\left(\frac{\sigma_N^2}{\sigma_x^2+\sigma_N^2}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{\sigma_x^2+\sigma_N^2}{\sigma_N^2}\right)$$

$$= \frac{1}{2}\log_e\left(1+\frac{\sigma_x^2}{\sigma_N^2}\right)$$

Putting, $\sigma_x^2 = S$, $\sigma_N^2 = N$, noise density, we have

$$I(X, Y) = \frac{1}{2}\log_e\left(1+\frac{S}{N}\right) \text{ nats/sample}$$

$$C = \max_{P(X)} I(X, Y).\, f_s$$

$$= \frac{f_s}{2}\log_e\left(1+\frac{S}{N_0}\right)$$

$$C = B\ \log_e\left(1+\frac{S}{N_0}\right) \qquad \left(f_s = \frac{1}{T_s} = 2B\right)$$

where 2B is the bandwidth of the power spectral density of the random signal X(t) and $N_0 = \eta B$ when the two-sided power spectral density of the noise is $\eta/2$ watts/Hz.

Therefore, $$C = B\log\left(1+\frac{S}{\eta B}\right) \text{ nats/sec.}$$

This equation is known as Shannon-Hartley theorem.

**Example 2.29**

*A binary input-output channel is shown in fig. 2.18.*

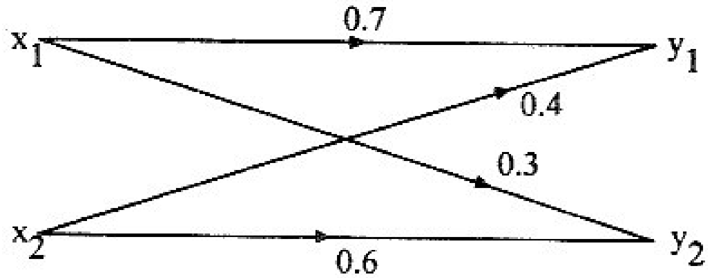


**Fig. 2.18** Binary input-output channel

*Given that the transition probability matrix is*

$$P(Y/X) = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

*and the input probabilities are $p(x_1) = 0.5$, $p(x_2) = 0.5$. Calculate the output probability matrix P(Y) and the joint probability matrix* P(X, Y).

***Solution***

$$[P(Y)] = [P(X)]\ [P(Y/X)]$$

$$[P(Y)] = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.55 & 0.45 \end{bmatrix}$$

If P(X) is written as a diagonal matrix, then

$$[P(X, Y)] = [P(X)]\ [P(Y/X)]$$

$$= \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$[P(X, Y)] = \begin{bmatrix} 0.35 & 0.15 \\ 0.2 & 0.3 \end{bmatrix}$$

## Review Questions

2.1 What do you mean by memoryless channel?

2.2 Define a discrete channel.

2.3 When is a discrete channel said to be "memoryless"?

2.4 Define Mutual information / Transinformation

2.5 Distinguish between noisy reception and perfect reception.

2.6 Name the different types of channels.

2.7 Define channel matrix D.

2.8 What is a lossless channel?

2.9 Define a deterministic channel.

2.10 What is a noiseless channel?

2.11 When is a channel said to be useless?

2.12 Define a symmetric channel.

2.13 What do you understand by BSC and BEC.

2.14 Define channel capacity.

2.15 What is the channel capacity of (i) lossless channel (ii) Deterministic channel (iii) Noiseless channel (iv) Symmetric channel.

2.16 How do you find the channel capacity of unsymmetric channels?

2.17 What is the channel capacity of (i) BEC (ii) BSC.

2.18 Write down Fano's Inequality.

2.19 State the Shannon's Fundamental theorem / Noisy coding Theorem.

2.20 What do you mean by a gaussian channel?

2.21 How do you find the capacity of a gaussian channel?

2.22 State Shannon-Hartely's Law.

2.23. Define information rate of the source.

2.24 Given a channel matrix, how do you find the channel capacity?

2.25 State the applications of coding Techniques.

2.26 State the advantages of coding techniques.

2.27 Distinguish between the different types of channels.

2.28 How do you obtain the channel capacity of a symmetric channel.

2.29 Define a stochastic matrix.

2.30 Distinguish between discrete and continuous type channels.