$$\lambda_3 = -\frac{1}{2} - \frac{1}{2} i\sqrt{3} = e^{-i\frac{2\pi}{3}}$$

Thus $N(k) = A\,2^K + B\cos\left(\frac{2\pi}{3}\right)k + C\sin\left(\frac{2\pi}{3}\right)k,\ k = 0, 1, \ldots$

The boundary conditions are

$$N(0) = 1 = A + B$$

$$N(1) = 1 = 2A - \frac{1}{2}B + \frac{1}{2}\sqrt{3}\,C$$

$$N(2) = 2 = 4A - \frac{1}{2}B - \frac{1}{2}\sqrt{3}\,C.$$

These equations have the solution $A = \frac{4}{7}, B = \frac{3}{7}, C = \frac{\sqrt{3}}{21}$

$$N(k) = \left(\frac{4}{7}\right)2^K + \left(\frac{3}{7}\right)\cos\left(\frac{2\pi}{3}\right)k + \left(\frac{\sqrt{3}}{21}\right)\sin\left(\frac{2\pi}{3}\right)k \ , k = 0, 1, \ldots.$$

Example 1.26

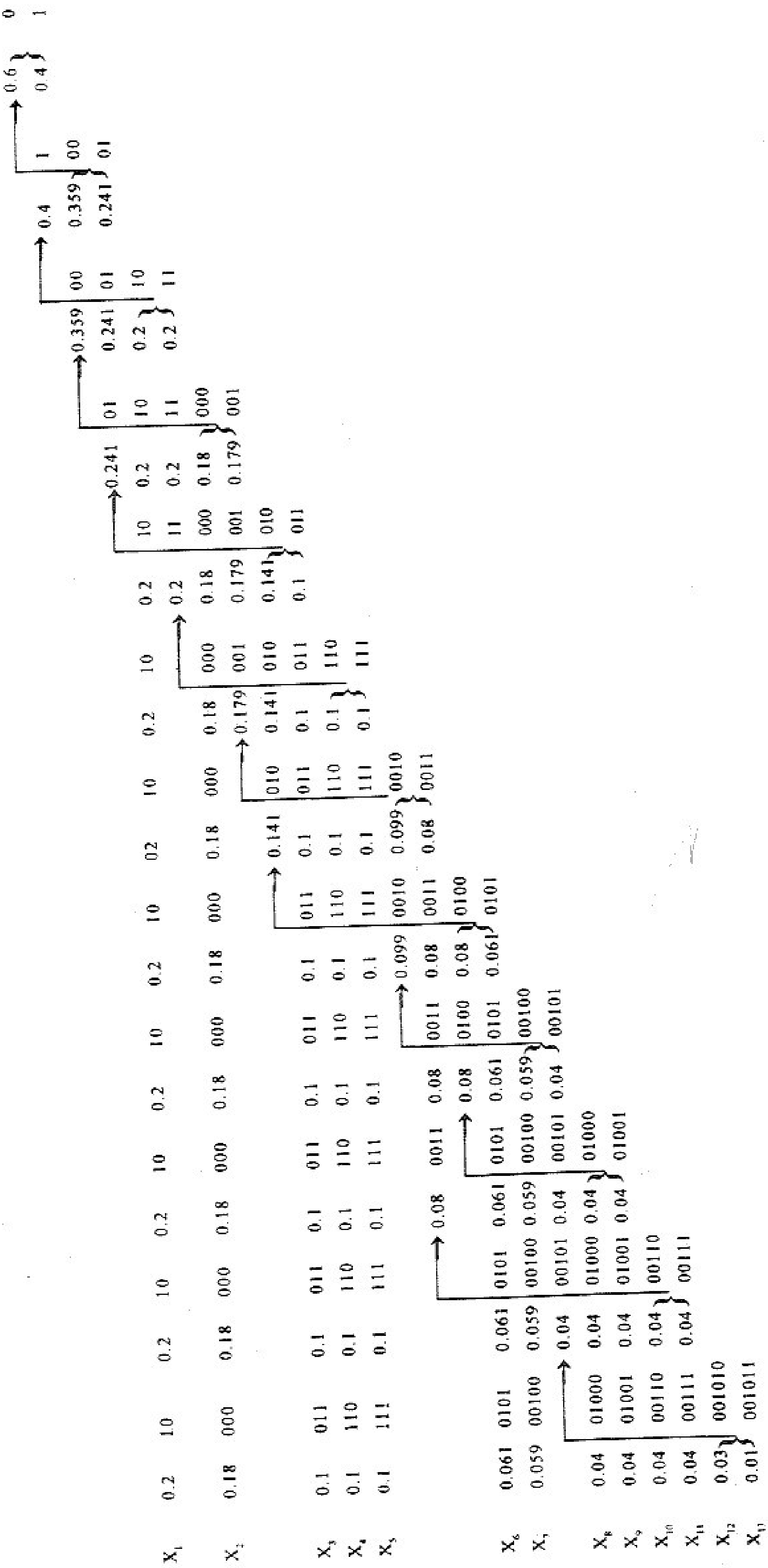
Construct a Huffmann code for the symbols below. Compare the average code-word length with the uncertainity H(X).

Symbols	*Probabilities*	*Symbols*	*Probabilities*
x_1	0.2	x_8	0.04
x_2	0.18	x_9	0.04
x_3	0.1	x_{10}	0.04
x_4	0.1	x_{11}	0.04
x_5	0.1	x_{12}	0.03
x_6	0.061	x_{13}	0.01
x_7	0.059		

Solution

The process of constructing the code is indicated schematically in the following figure.

Huffman's Code table for problem 1.26



Symbols	Code word
x_1	10
x_2	000
x_3	011
x_4	110
x_5	111
x_6	0101
x_7	00100
x_8	01000
x_9	01001
x_{10}	00110
x_{11}	00111
x_{12}	001010
x_{13}	001011

The average code-word length $\overline{L} = \sum_{i=1}^{13} p_i n_i$

$$\overline{L} = 0.4 + 0.54 + 0.3 + 0.3 + 0.3 + 0.244 + 0.295 + 0.2 + 0.2 + 0.2 + 0.18 + 0.06$$

$$= 3.419$$

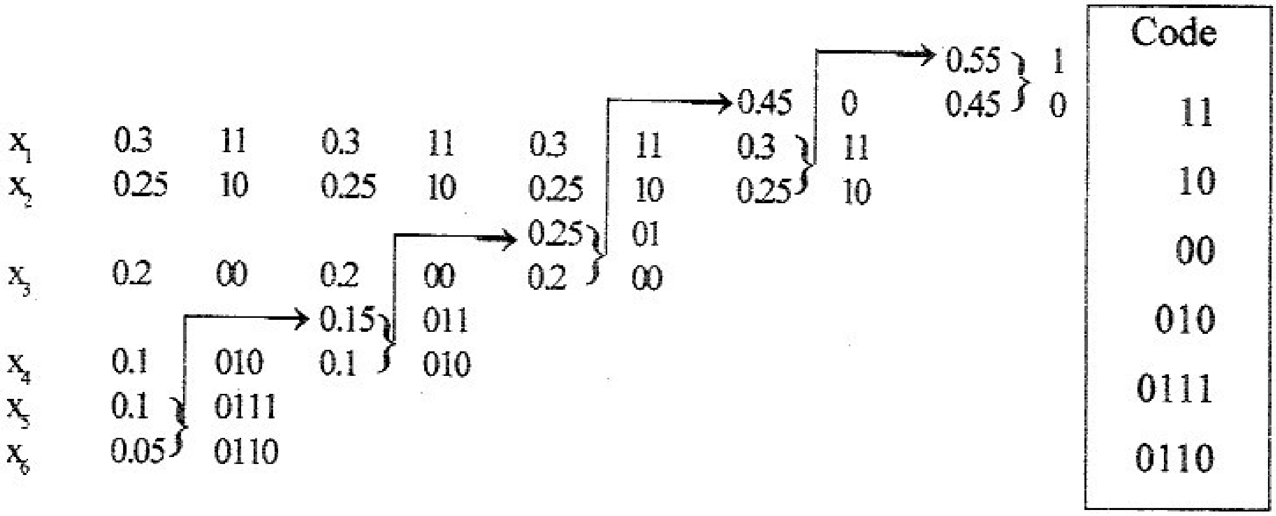
The uncertainity $H(X) = -\sum p_i \log p_i = 3.355$

Example 1.27

Construct a Huffmann code for the symbols below.

x_1	0.3
x_2	0.25
x_3	0.2
x_4	0.1
x_5	0.1
x_6	0.05

Solution



Example 1.28

For a signal x(t) which is known to have a uniform density function in the range $0 \le x \le 4$ find H(X). If the same signal is amplified by a factor of 8 determine then H(X).

Solution:

$$H(X) = -\int_0^4 \frac{1}{4}\log\frac{1}{4}\,dx = 2$$

If the same signal is amplified by a factor of 8,

$$H(X) = -\int_0^{32} \frac{1}{32}\log\frac{1}{32}\,dx = 5.$$

Example 1.29

Show that for a Huffmann binary code $H \le \overline{L} \le H + 1$

Solution

We have established the theorem

$$\frac{H(X)}{\log D} \le \overline{L} \le \frac{H(X)}{\log D} + 1.$$

for a random variable X with uncertainity H(X). D is the base (alphabet) of the instantaneous code whose average code-word length is $\overline{L}$.

For a binary code, D = 2

$$\text{Hence,} \quad \frac{H(X)}{\log 2} \le \overline{L} \le \frac{H(X)}{\log 2} + 1$$

$$H(X) \le \overline{L} \le H(X) + 1$$

Example 1.30

A source is transmitting two symbols A and B with p(A) = 1/16 and p(B) = 15/16, design the code which would provide a transmission efficiency of around 70 percent.

Solution

For A = 0, B = 1, we have the simplest code

$$H = p(A)\log\frac{1}{p(A)} + p(B)\log\frac{1}{p(B)}$$

$$= -\left[\frac{1}{16}\log\frac{1}{16} + \frac{15}{16}\log\frac{15}{16}\right]$$

$$= \frac{52}{16} \text{ bits/symbol}$$

Since average number of digits is 1, C = 1 bit/symbol

$$\eta_c = \frac{H}{C} = \frac{52}{16} = 32 \text{ percent}$$

This is the efficiency without coding.
To achieve higher efficiency, we try paired coding

Symbol	*Probability*
AA	$\left(\frac{1}{16}\right)\left(\frac{1}{16}\right) = \frac{1}{256}$
BB	$\left(\frac{1}{16}\right)\left(\frac{15}{16}\right) = \frac{15}{256}$
BA	$\left(\frac{15}{16}\right)\left(\frac{1}{16}\right) = \frac{15}{256}$
BB	$\left(\frac{15}{16}\right)\left(\frac{15}{16}\right) = \frac{225}{256}$.

Now taking the maximum probability, i.e., BB., we have the following code for the paired symbols:

BB	0
BA	10
AB	110
AA	111

Now, $$C = \sum \text{probability} \times \text{Number of digits}$$

$$= \left(\frac{225}{256}\right)(1) + \left(\frac{15}{256}\right)(2) + \left(\frac{15}{256}\right)(3) + \left(\frac{1}{256}\right)(3)$$

$$= \frac{303}{256} \text{ bits/pair of symbols}$$

$$= \frac{1}{2} \times \frac{303}{256} = \frac{303}{512} \text{ bits/symbol}$$

Therefore $\eta = \frac{H}{C} = \frac{5.2}{16} \times \frac{512}{303} = 54$ percent

Now since η is not 70 percent, we undertake 3-bit coding:

AAA $$\left(\frac{1}{16}\right)\left(\frac{1}{16}\right)\left(\frac{1}{16}\right) = \left(\frac{1}{4096}\right)$$

AAB $$\left(\frac{1}{16}\right)\left(\frac{1}{16}\right)\left(\frac{15}{16}\right) = \left(\frac{15}{4096}\right)$$

ABA $$\left(\frac{1}{16}\right)\left(\frac{15}{16}\right)\left(\frac{1}{16}\right) = \left(\frac{15}{4096}\right)$$

ABB $$\left(\frac{1}{16}\right)\left(\frac{15}{16}\right)\left(\frac{15}{16}\right) = \left(\frac{225}{4096}\right)$$

BAA $$\left(\frac{15}{16}\right)\left(\frac{1}{16}\right)\left(\frac{1}{16}\right) = \left(\frac{15}{4096}\right)$$

BAB $$\left(\frac{15}{16}\right)\left(\frac{1}{16}\right)\left(\frac{15}{16}\right) = \left(\frac{225}{4096}\right)$$

BBA	$\left(\frac{15}{16}\right)\left(\frac{15}{16}\right)\left(\frac{1}{16}\right) = \left(\frac{225}{4096}\right)$
BBB	$\left(\frac{15}{16}\right)\left(\frac{15}{16}\right)\left(\frac{15}{16}\right) = \left(\frac{3375}{4096}\right)$ (maximum)
BBB	0
BBA	10
BAB	110
ABB	1110
BAA	11110
ABA	111110
AAB	1111110
AAA	1111111

$$\text{Therefore, } C = \left(\frac{3375}{4096}\right)(1) + \left(\frac{225}{4096}\right)(2) + \left(\frac{225}{4096}\right)(3) + \left(\frac{225}{4096}\right)(4)$$

$$+\left(\frac{15}{4096}\right)(5) + \left(\frac{15}{4096}\right)(6) + \left(\frac{15}{4096}\right)(7) + \left(\frac{1}{4096}\right)(7)$$

$$= \frac{5677}{4096} \text{ bits/tri-symbol}$$

$$\text{Hence capacity/symbol} = \frac{1}{3}\left(\frac{5677}{4096}\right)$$

$$\eta_c = \frac{H}{C} = \frac{5.2}{16} \times \frac{3 \times 4096}{5677} \times 100$$

$$\therefore \eta_c = 70.5 \text{ percent}$$

Hence the efficiency can be 70 percent by using three symbol coding.

Example 1.31

Device the Huffmann code for the following message probabilities with three alphabets in coding (D = 3).

$P(m_1) = 0.3$; $P(m_2) = 0.2$; $P(m_3) = 0.2$; $P(m_4) = 0.15$; $P(m_5) = 0.15$

Solution

				1.0
			0.5	0
m_1	0.3	1	0.3	1
m_2	0.2	2	0.2	2
m_3	0.2	00		
m_4	0.15	01		
m_5	0.15	02		

Message	Code word
m_1	1
m_2	2
m_3	00
m_4	01
m_5	02

Example 1.32

A source emits six messages with probabilities 0.30, 0.25, 0.15, 0.12, 0.1 and 0.08 respectively. Find

(a) The 4- ary Huffmann code

(b) Average word length, code efficiency and redundancy.

Solution

m_1	0.30	0	0.3	0
			0.3	1
m_2	0.25	2	0.25	2
m_3	0.15	3	0.15	3
m_4	0.12	10		
m_5	0.1	11		
m_6	0.08	12		
m_7	0.00	13		

The procedure to draw the γ - ary code is that

(i) The last γ messages are combined each time until the final set reduces to γ messages.

(ii) The last γ messages are assigned one of the γ numbers 0, 1, 2, $\ldots \gamma - 1$.

Since each reduction decreases the number of messages by $\gamma - 1$, there will be exactly γ messages left in the last reduced set, if and only if, the total nymber of original messages is equal to $\gamma + K(\gamma - 1)$, where K is an integer and is equal to the number of reductions made by combinations.

In case the original messages do not satisfy this condition, some dummy messages with zero probability are added in order to satisfy this condition.

This procedure has been applied in this problem. Here the number of messages is 6 and we want to develop 4 -ary code, then $\gamma = 4$, and we must add one dummy message with zero probability so that the total number of messages is equal to [4 + (4 –1)] or 7.

symbol	code word
m_1	0
m_2	2
m_3	3
m_4	10
m_5	11
m_6	12
m_7	13

(b) The average word length is

$$\bar{L} = \sum_{i=1}^{6} p_i n_i$$

$$= (0.3)(1) + (0.25)(1) + (0.15)(1) + (0.12)(2) + (0.1)(2) + (0.08)(2)$$

$$= 1.3 \quad \text{4- ary units.}$$

$$H_4(m) = -\sum_{i=1}^{6} p_i \log_4 p_i$$

$$= 1.209 \quad \text{4-ary units}$$

$$\eta_i = \frac{H_4(m)}{\bar{L}} = \frac{1.209}{1.3} = 0.93 \qquad \text{Redundancy} = 0.07.$$

Example 1.33

A random variable has the density function shown in fig. 1.20. Find the corresponding entropy function.

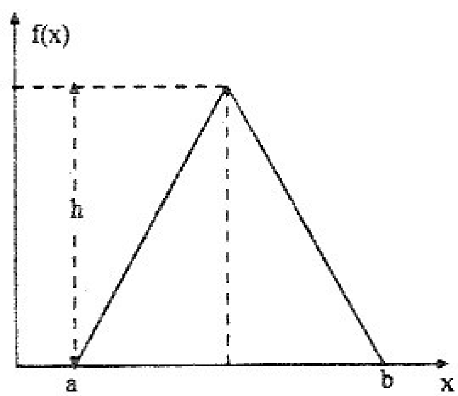


Fig. 1.20 Density Function

Solution

$$f(x) = \frac{2h}{b-a}(x-a) \quad \text{for } a \le x \le \frac{a+b}{2}$$

$$f(x) = \frac{2h}{b-a}(b-x) \quad \text{for } \frac{a+b}{2} \le x \le b$$

$$H(X) = -\int_{a}^{\frac{a+b}{2}} \frac{2h}{b-a}(x-a)\ln\frac{2h}{b-a}(x-a)dx$$

$$-\int_{\frac{a+b}{2}}^{b} \frac{2h}{b-a}(b-x)\ln\frac{2h}{b-a}(b-x)dx$$

The above integrals are evaluated by parts. In doing so, note that

$$\int x \ln \lambda x.dx = \frac{x^2}{2}\ln\lambda x - \frac{x^2}{4}$$

$$\text{Thus, } H(X) = \frac{-h}{b-a}\left[(x-a)^2 \ln\frac{2h}{b-a}(x-a) - \frac{(x-a)^2}{2}\right]_0^{\frac{(a+b)}{2}}$$

$$+\frac{h}{b-a}\left[(b-x)^2 \ln\frac{2h}{b-a}(b-x) - \frac{(b-x)^2}{2}\right]_{\frac{(a+b)}{2}}^{0}$$

$$= \frac{-h}{b-a}\left\{\left[\frac{(b-a)^2}{4}\ln h - \frac{(b-a)^2}{8}\right] + \left[\frac{(b-a)^2}{4}\ln h - \frac{(b-a)^2}{8}\right]\right\}$$

$$= \frac{h(b-a)}{2}\left(-\ln h + \frac{1}{2}\right)$$

and since $\qquad \dfrac{h(b-a)}{2} = \displaystyle\int_{-\infty}^{\infty} f(x)dx = 1$

thus $\qquad H(X) = -\ln h + \dfrac{1}{2}$

The entropy depends on the parameter h, but a translation of the probability curve along the x - axis does not change its value.

It is easy to see that

$$H(x) > 0, \text{ for } h < \sqrt{e}$$

$$H(x) = 0, \text{ for } h = \sqrt{e}$$

$$H(x) < 0, \text{ for } h > \sqrt{e}$$

Example 1.34

Find the entropy of a continuous random variable with the density function as illustrated in fig.1.19

$$f(x) = bx^2 \;,\; 0 \le x \le a$$
$$= 0\,, \quad \text{elsewhere}$$

Determine the entropy $H(x_1)$ when $x_1 = x + d$, $d > 0$.

Determine the entropy $H(X_2)$ when $x_2 = 2x$.

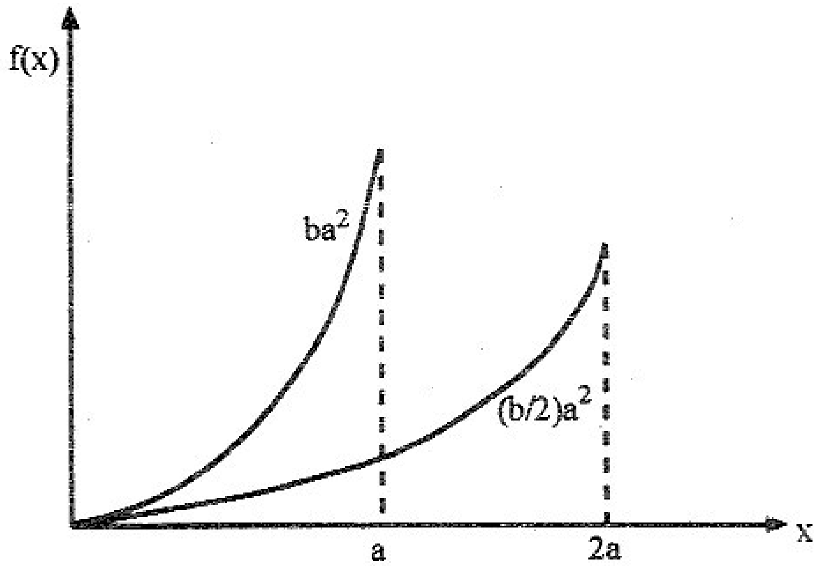


Fig. 1.21 Density Function of Example 1.34

Solution

The value of b which makes the above f(x) a permissable probability density function is given by

$$\int_0^a f(x)dx = b\left[\frac{x^3}{3}\right]_0^a = \frac{ba^3}{3} = 1$$

$$H(X) = -\int_0^a bx^2 . \ln bx^2 dx$$

Note that $$\int x^2 . \ln \lambda x dx = \frac{x^3}{3} \ln \lambda x - \frac{x^3}{9}$$

Thus $$H(X) = -2b \left[\frac{x^3}{3} \ln \sqrt{b} x - \frac{x^3}{9} \right]_0^a = - \frac{2ba^3}{3} \left(\ln \sqrt{ba} - \frac{1}{3} \right)$$

$$H(X) = -2 \left(\ln \sqrt{\frac{3}{a}} - \frac{1}{3} \right)$$

$$= \frac{2}{3} + \ln \frac{a}{3}$$

The entropy may be positive, negative, or zero, depending on the parameter a.

$$H(X) > 0 , \ a > 3 \ e^{-\frac{2}{3}}$$

$$H(X) = 0 , \ a = 3 \ e^{-\frac{2}{3}}$$

$$H(X) < 0 , \ a < 3 \ e^{-\frac{2}{3}}.$$

Now consider the simple translation of the vertical axis,

$$X_1 = x + d , \ d > 0$$

The probability density curve will be simply d units shifted to the left. The entropy becomes

$$H(X_1) = \int_d^{d+a} b(x_1 - d)^2 \ln\left[b(x_1 - d)^2\right] dx_1$$

$$H(X_1) = -2b \left[\frac{(x-d)^3}{3} \ln \sqrt{b}(x_1 - d) - \frac{(x_1 - d)^3}{9} \right]_d^{d+a}$$

$$H(X_1) = -2b \frac{a^3}{3} \left(\ln \sqrt{ba} - \frac{1}{3} \right) = \frac{2}{3} + \ln \frac{a}{3} = H(X)$$

For the transformation $x_2 = 2x$, we find

$$p(x_2) = \frac{bx^2}{2} = \frac{b}{8} x_2^2$$

$$H(x_2) = -\int_0^{2a} \frac{b}{8} x_2^2 \log \frac{b}{8} x_2^2 dx_2$$

$$H(X_2) = -\frac{b}{4}\left[\frac{x_2^2}{3}\ln\left(\frac{\sqrt{b}}{2\sqrt{2}}x_2 - \frac{1}{3}\right)\right]_0^{2a} = \ln\frac{2}{3}a + \frac{2}{3}$$

Review Questions

1.1 Define Entropy

1.2 Define self - information

1.3 State the properties of Entropy Function.

1.4 Write down the entropy of three symbols x_1, x_2, x_3 with probabilities p_1, p_2, p_3 respectively.

1.5 State the condition for entropy to be maximum.

1.6 Define Joint and conditional entropies.

1.7 Define Equivocation.

1.8 State the significance of H(Y/X) and H(X/Y).

1.9 Define entropy in the continuous case Function.

1.10 State the properties of continuous Entropy Function.

1.11 What is the purpose of encoding?

1.12 Define the terms 'Encoding' and 'Decoding'.

1.13 What do you mean by uniquely decipherable encoding?

1.14 Define 'efficiency' of coding.

1.15 Define 'Redundancy' of coding.

1.16 What are instantaneous codes?

1.17 Define average length of a code.

1.18 What is Kraft Inequality?

1.19 Write the McMillan's Theorem on Decodability.

1.20 State the Noiseless coding theorem.

1.21 State the different methods of coding.

1.22 What is Shannon's binary coding?

1.23 What is Shannon-Fano Encoding?

1.24 State the necessary condition for an optimal code in Huffman coding.

1.25 What do you understand by 4-ary Huffmann code?