Date
17/10/11

Information Theory & Coding

AARUPADAI VEEDU INSTITUTE OF TECHNOLOGY Sheet No.

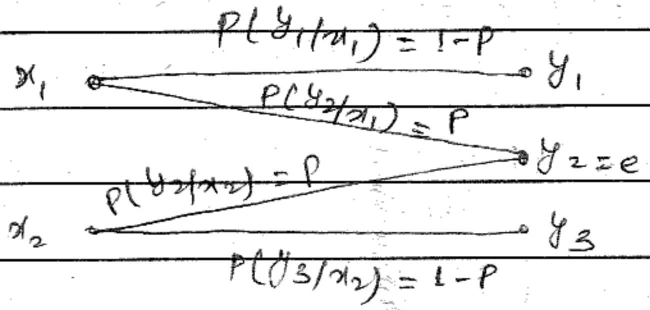
UNIT – I

6. (i) A channel has the following matrix.

$$P(Y/X) = \begin{bmatrix} 1-P & P & 0 \\ 0 & P & 1-P \end{bmatrix}$$

$$P(Y/X) = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) & P(y_3/x_1) \\ P(y_1/x_2) & P(y_2/x_2) & P(y_3/x_2) \end{bmatrix}$$

(a) The channel diagram is –



There are two inputs & three o/ps and y_2 shows error and called Binary erasure channel

(b) If the source has equally likely outputs –

i.e.

$x_1 = x_2 = 0.5$ {∵ Total prob. = 1}

and, $P = 0.2$ (given)

(b) To calculate $P(y_1)$, $P(y_2)$ & $P(y_3)$:

$$P(x_1) = P(x_2) = 0.5 = \frac{1}{2}$$

The output probabilities are –

$$\begin{bmatrix} P[y_1] \\ P[y_2] \\ P[y_3] \end{bmatrix} = [P(x_1) \; P(x_2)] \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

$$= [P(x_1) \; P(x_2)] \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

$$= [(0.5) \; (0.5)] \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 \\ 0.1+0.1 \\ 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}$$

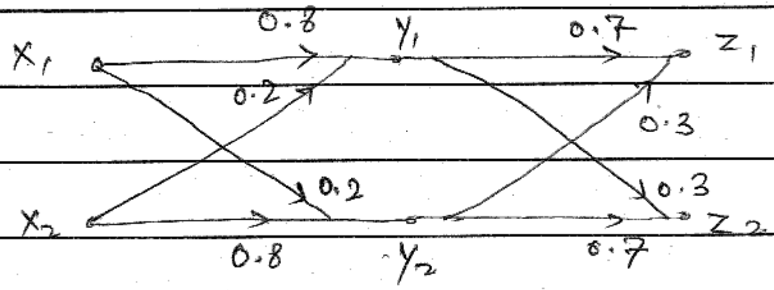
Thus,

$P(y_1) = 0.4$

$P(y_2) = 0.2$

$P(y_3) = 0.4$

(ii) Two BSC's are connected in cascade -



① channel matrix:

$$P(Y/X) = \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P(Z/Y) = \begin{bmatrix} P(z_1/y_1) & P(z_2/y_1) \\ P(z_1/y_2) & P(z_2/y_2) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Hence, the resultant channel is -

$$P(Z/X) = \begin{bmatrix} P(z_1/x_1) & P(z_2/x_1) \\ P(z_1/x_2) & P(z_2/x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

$$P(Z/X) = \begin{bmatrix} 0.56+0.06 & 0.14+0.24 \\ 0.14+0.24 & 0.56+0.06 \end{bmatrix}$$

$$P(Z/X) = \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

(ii) find $P(Z_1)$ and $P(Z_2)$ if $P(X_1) = 0.6$ and $P(X_2) = 0.4$

$$\because \quad P(Z) = P(X)\, P(Z/X)$$

$$\begin{bmatrix} P(Z_1) \\ P(Z_2) \end{bmatrix} = \begin{bmatrix} P(X_1) & P(X_2) \end{bmatrix} \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.62 & 0.38 \\ 0.38 & 0.62 \end{bmatrix}$$

$$= \begin{bmatrix} 0.372 + .152 \\ 0.228 + 0.248 \end{bmatrix}$$

$$= \begin{bmatrix} 0.524 \\ 0.476 \end{bmatrix}$$

$P(Z_1) = 0.524$

$P(Z_2) = 0.476$ Ans

7.] Prove that the mutual information of the channel is symmetric i.e.

$$I(X;Y) = I(Y;X)$$

Soln:

~~Here, H(X/Y) is the Conditional entropy~~ and it is ~~given as~~ –

$$\cancel{H(X/Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i y_j) \log_2 \frac{1}{P(x_i/y_j)}}$$

Let us consider some standard relationships from probability theory, these are as follows –

$$P(x_i y_j) = P(x_i/y_j)\, P(y_j) \quad \text{— (I)}$$

$$P(x_i y_j) = P(y_j/x_i)\, P(x_i) \quad \text{— (II)}$$

From eqn (I) & (II), we can write –

$$P(x_i/y_j)\, P(y_j) = P(y_j/x_i)\, P(x_i) \quad \text{— (III)}$$

$$\therefore \frac{P(x_i/y_j)}{P(x_i)} = \frac{P(y_j/x_i)}{P(y_j)} \quad \text{— (IV)}$$

The average mutual information is given as –

$$I(X,Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} \quad \text{— (V)}$$

and, $$I(Y,X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left[\frac{P(y_j/x_i)}{P(y_j)} \right] \quad \text{(VI)}$$

From eqn (VI) and (IV) –

$$I(Y,X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left[\frac{P(x_i/y_j)}{P(y_j)} \right]$$

$$= I[X,Y]$$

Proved

where,

→ $P(x_i, y_j)$ is the joint Probability that x_i is transmitted and y_j is received.

→ $P(x_i / y_j)$ is the conditional probability of that x_i is transmitted and y_j is received.

→ $P(y_j / x_i)$ is the conditional probability of that y_j is ~~transmit~~ received and x_i is transmitted.

→ $P(x_i)$ is the Probability of symbol x_i for transmission.

→ $P(y_j)$ is the Probability for symbol y_j ~~for~~ is received.

(ii) Prove that the mutual information is always positive.

$$I(X,Y) \geq 0.$$

Soln:

mutual information is given as –

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left(\frac{P(x_i/y_j)}{P(x_i)} \right) \quad \text{—(I)}$$

As we know that –

$$P(x_i/y_j) = \frac{P(x_i y_j)}{P(y_j)} \quad \text{—(II)}$$

from (I) & (II) –

$$I(X;Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i y_j)}{P(x_i)\, P(y_j)}$$

As we know that $\log_2 \frac{x}{y}$ can be written as $-\log_2 \frac{y}{x}$.

$$\therefore -I(X,Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i)\, P(y_j)}{P(x_i y_j)} \quad \text{—(III)}$$

and, As we know that –

$$\sum_{k=1}^{m} p_k \log_2 \left[\frac{q_k}{p_k} \right] \leq 0 \quad \text{—(IV)}$$

From (III) & (IV), we get -

$$-I(X;Y) \leq 0$$

$$I(X,Y) \geq 0 \quad \underline{\text{Proved}}$$

8.(a) $I[X,Y] = H(X) - H(X/Y)$

Here, $H(X/Y)$ is the conditional entropy and it is given as -

$$H(X/Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i/y_j)} \quad \text{— (I)}$$

$H(X/Y)$ is the information lost in the noisy channel. It is the average conditional self information.

Let,

$$I[X;Y] = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{P(x_i/y_j)}{P(x_i)} \quad \text{— (II)}$$

Let us write the above eqn as -

$$I[X,Y] = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i)}\right] - \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \left(\frac{1}{P(x_i/y_j)}\right)$$

From eqn (I) -

$$I[X;Y] = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(x_i)} - H(X/Y) \quad \text{— (III)}$$

Here, let us use the standard probability relation which is given as -

$$\sum_{j=1}^{m} P(x_i, y_j) = P(x_i)$$

Date: $I[X;Y]$: The average information transferred per symbol across the channel.

so, eqn (2) will be -

$$I[X,Y] = \sum_{i=1}^{n} P(x_i) \log_2 \frac{1}{P(x_i)} - H(X/Y) \quad \text{--- (IV)}$$

$$\therefore \quad H(X) = \sum_{i=1}^{n} P(x_i) \log_2 \frac{1}{P(x_i)} \quad \text{(shows entropy)} \quad \text{--- (V)}$$

$\therefore$ The eqn (3) will become -

$$\boxed{I[X,Y] = H(X) - H[X/Y]}$$ Proved

(b.) $I[X;Y] = H[Y] - H[Y/X]$

Consider the average mutual information -

$$I[Y;X] = \sum_{i=1}^{n}\sum_{j=1}^{m} P[x_i, y_j] \log_2 \frac{P(y_j/x_i)}{P(y_j)}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} P[x_i, y_j] \log_2 \frac{1}{P(y_j)} - \sum_{i=1}^{n}\sum_{j=1}^{m} P[x_i, y_j] \log_2 \frac{1}{P(y_j/x_i)} \quad \text{--- (I)}$$

The Conditional entropy $H(Y/X)$ is given as -

$$H(Y/X) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(y_j/x_i)} \quad \text{--- (II)}$$

from eqn (I) & (II) -

$$I[Y;X] = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(y_j)} - H(Y/X) \quad \text{--- (III)}$$

Here, let us use the standard probability eqⁿ -

$$\sum_{i=1}^{n} P(x_i, y_j) = P(y_j) \quad \text{--- (IV)}$$

from eqⁿ (III) & (IV) -

$$I[X;Y) = \sum_{j=1}^{m} P(y_j) \log_2 \left[\frac{1}{P(y_j)}\right] - H(Y/X)$$

First term of above represents entropy -

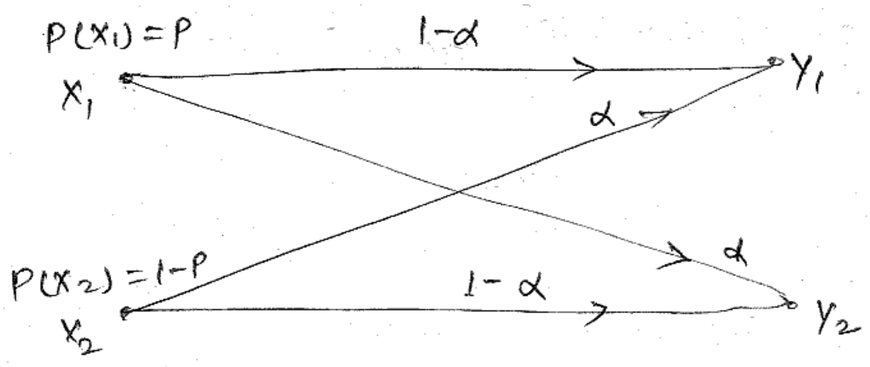
$$H(Y) = \sum_{j=1}^{m} P(y_j) \log_2 \frac{1}{P(y_j)} \quad \text{--- (V)}$$

∴ eqⁿ (3) becomes -

$$\boxed{I[X;Y] = H(Y) - H(Y/X)}$$

Proved

(9.) The Given Binary Symmetric channel is -



Here, for BSC, the various probabilities are as follows -

$P(x_1) = P$ and $P(x_2) = 1-P$

$P(y_1/x_1) = 1-\alpha$

$P(y_2/x_1) = \alpha$

$P(y_1/x_2) = \alpha$

$P(y_2/x_2) = 1-\alpha$

(a) To obtain $H(X)$:-

Here, 'X' has two symbols i.e, x_1 and x_2. Their probabilities are (P) and $(1-P)$.

The entropy for X is -

$$\Omega(P) \text{ (or) } H(X) = P \log_2\left(\frac{1}{P}\right) + (1-P) \log_2\left(\frac{1}{1-P}\right) \quad \text{—— (1)}$$

This is also called Horseshoe function.

(b) To obtain $H(Y)$:

The standard probability relation states that -

$$P(y_j) = \sum_{i=1}^{n} [P(x_i, y_j)] \quad \text{—— (2)}$$

As we know that -

$$P(x_i\, y_j) = P(y_j/x_i)\; P(x_i) \quad \text{—— (3)}$$

from ② & ③ :-

$$P(y_j) = \sum_{i=1}^{n} P(y_j/x_i)\, P(x_i)$$

$$P(y_1) = \sum_{i=1}^{2} P(y_1/x_i)\, P(x_i)$$

$$P(y_1) = P(y_1/x_1)\, P(x_1) + P(y_1/x_2)\, P(x_2) \quad \text{— ④}$$

Similarly ;

$$P(y_2) = P(y_2/x_1)\, P(x_1) + P(y_2/x_2)\, P(x_2) \quad \text{— ⑤}$$

Putting values in eqn ④ :-

∴

$$P(y_1) = (1-\alpha)P + \alpha(1-P)$$

$$= P - P\alpha + \alpha - P\alpha$$

$$P(y_1) = P + \alpha - 2P\alpha \quad \text{— ⑥}$$

Similarly :

$$P(y_2) = \alpha(P) + (1-\alpha)(1-P)$$

$$\Rightarrow P(y_2) = \alpha P + 1 - P - \alpha + \alpha P$$

$$\Rightarrow P(y_2) = 2\alpha P + 1 - P - \alpha$$

$$= 1 - (P + \alpha - 2P\alpha)$$

$$P(y_2) = 1 - P(y_1)$$

$$\therefore H(Y) = \Omega[P(y_1)]$$

$$= \Omega[P + \alpha - 2P\alpha] \quad \text{— ⑦}$$

(c) To obtain $H(Y/X)$:-

$$H(Y/X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(y_j/x_i)}$$

As we know that -

$$P(x_i, y_j) = P(y_j/x_i)\, P(x_i)$$

From above eq^n -

$$H(Y/X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(y_j/x_i) P(x_i) \log_2 \frac{1}{P(y_j/x_i)}$$

$$= \sum_{i=1}^{2} P(x_i) \left[\sum_{j=1}^{2} P(y_j/x_i) \log_2 \frac{1}{P(y_j/x_i)} \right]$$

$$= P(x_1) \left[P(y_1/x_1) \log_2 \frac{1}{P(y_1/x_1)} \right] + P(x_2) \left[P(y_2/x_2) \log_2 \frac{1}{P(y_2/x_2)} \right]$$

$H(Y/X) = \Omega(\alpha)$

(d) To obtain $I(X,Y)$:-

$$I[X,Y] = H(Y) - H(Y/X)$$

$$= \Omega[P(x_i)] - \Omega(\alpha)$$

$$= \Omega[\alpha + p - 2\alpha p] - \Omega(\alpha)$$ Ans

(ii) $I[X;Y] = H(X) + H(Y) - H(X,Y)$

As we know that, —

$$H(X,Y) = H(X/Y) + H(Y)$$

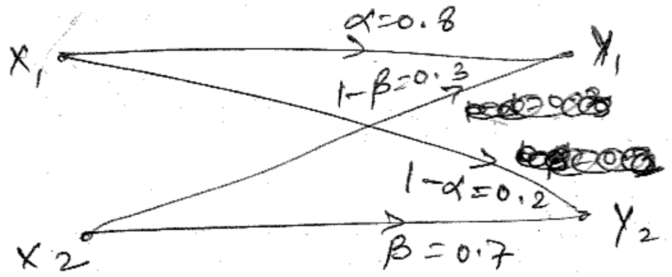
$$H(X/Y) = H(X,Y) - H(Y)$$

Now, mutual information is given as —

$$I[X;Y] = H[X] - H[X/Y]$$

$$I[X,Y] = H[X] + H(Y) - H[X,Y]$$ Proved

(10) (i) The Given channel is



Given :

$$P(X_1) = 0.6$$

$$P(X_2) = 0.4$$

(1) To obtain H(X):

let $P(x_1) = P$ and $P(x_2) = 1-P$

Here, $P = 0.6$

$$\therefore H(X) = P \log_2\left(\frac{1}{P}\right) + (1-P)\log_2\left(\frac{1}{1-P}\right)$$

$$= (0.6)\log_2\left(\frac{1}{0.6}\right) + (0.4)\log_2\left(\frac{1}{0.4}\right)$$

$$= 0.442 + 0.528$$

$$= 0.97 \text{ bits/symbol.}$$

(2) To obtain H(Y):-

$$\begin{bmatrix} P(y_1) \\ P(y_2) \end{bmatrix} = \begin{bmatrix} P(x_1) & P(x_2) \end{bmatrix} \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.48 + 0.12 \\ 0.12 + 0.28 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

Now, $H(Y) = 0.6 \log_2\left(\frac{1}{0.6}\right) + 0.4 \log_2\left(\frac{1}{0.4}\right)$

$$= 0.442 + 0.528$$

$$= 0.97 \text{ bits/symbol}$$

Thus, $\boxed{H(X) = H(Y)}$

To obtain, $H(Y/X)$:-

$$H(Y/X) = \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) \log_2 \frac{1}{P(y_j/x_i)}$$

$$H(Y/X) = P(x_1, y_1) \log_2 \frac{1}{P(x_1, y_1)} + P(x_1, y_2) \log_2 \frac{1}{P(x_1, y_2)} +$$
$$P(x_2 y_1) \log_2 \frac{1}{P(x_2 y_1)} + P(x_2 y_2) \log_2 \frac{1}{P(x_2 y_2)}$$

As we know that -

$$P(x_i y_i) = P(y_j / x_i) \, P(x_i)$$
$$P(x_1, y_1) = P(y_1/x_1) P(x_1) = \alpha P$$
$$P(x_1, y_2) = P(y_2/x_1) P(x_1) = (1-\alpha) P$$
$$P(x_2 y_1) = P(y_1/x_2) \, P(x_2) = (1-\beta)(1-P)$$
$$P(x_2 y_2) = P(y_2/x_2) \, P(x_2) = \beta(1-P)$$

$$\therefore H(Y/X) = \alpha P \log_2 \frac{1}{P} + (1-\alpha) P \log_2 \frac{1}{P} +$$
$$(1-\beta)(1-P) \log_2 \frac{1}{(1-P)} + \beta(1-P) \log_2 \frac{1}{(1-P)}$$

$$= 0.48 \log_2 \frac{1}{0.6} + 0.12 \log_2 \frac{1}{0.83} + 0.12 \log_2 \frac{1}{0.12} +$$
$$0.28 \log_2 \frac{1}{0.27}$$

$$= 0.508 + 0.367 + 0.367 + 0.514$$

$$= 1.756$$

$$H(Y/X) = \alpha p \log_2 \frac{1}{\alpha} + (1-\alpha) p \log_2 \frac{1}{1-\alpha} + (1-\alpha)(1-p) \log_2 \frac{1}{1-\beta}$$
$$+ \beta (1-p) \log_2 \frac{1}{\beta}$$

$$= 0.48 \log_2 \frac{1}{0.8} + 0.12 \log_2 \frac{1}{0.2} + 0.12 \log_2 \frac{1}{0.3}$$
$$+ 0.28 \log_2 \frac{1}{0.7}$$

$$= 0.1545 + 0.2786 + 0.2084 + 0.1441 = 0.7855 \text{ bits/sym}$$

(4) To obtain $I[X;Y]$:-

$$I[X;Y] = H(Y) - H(Y/X)$$
$$= 0.9709 - 0.7855 = 0.1853 \text{ bits/symbol}$$

(ii) A binary channel matrix is given as :-

$$\begin{bmatrix} 2/3 & 1/3 \\ 1/10 & 9/10 \end{bmatrix}$$

Here, $p(x_1) = \frac{2}{3}$

and $p(x_2) = \frac{1}{3}$

(1) To obtain $H(X)$:-

$$H(X) = \frac{2}{3} \log_2 \frac{3}{2} + \frac{1}{3} \log_2 (3)$$
$$= 0.3899 + 0.5283$$
$$= 0.9182 \text{ bits/symbol}$$

② To obtain $H(Y)$ =

$$\begin{bmatrix} P(y_1) \\ P(y_2) \end{bmatrix} = \begin{bmatrix} P(x_1) & P(x_2) \end{bmatrix} \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/10 & 9/10 \end{bmatrix}$$

$$= [0.2222 + 0.06666 \quad 0.1111 + 0.6]$$

$$= [0.2888 \quad 0.7111]$$

$$H(Y) = 0.2888 \log_2 \frac{1}{0.2888} + 0.7111 \log_2 \frac{1}{0.7111}$$

$$= 0.5174 + 0.3497$$

$$= 0.8671 \text{ bits/Symbol}$$

③ To obtain $H(X/Y)$:-

$$P(x_i y_j) = P(y_j/x_i)\, P(x_i)$$

$$P(x_1 y_1) = P(y_1/x_1)\, P(x_1) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$$

$$P(x_1 y_2) = P(y_2/x_1)\,(x_1) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

$$P(x_2 y_1) = P(y_1/x_2)\,(x_2) = \frac{1}{10} \times \frac{2}{3} = \frac{2}{30}$$

$$P(x_2 y_2) = P(y_2/x_2)\,(x_2) = \frac{9}{10} \times \frac{2}{3} = \frac{18}{30}$$

$$H(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i y_j) \log_2 \frac{1}{P(x_i y_j)}$$

$$= P(x_1 y_1) \log_2 \frac{1}{P(x_1 y_1)} + P(x_1 y_2) \log_2 \frac{1}{P(x_1 y_2)} + P(x_2 y_1) \log_2 \frac{1}{P(x_2 y_1)}$$

$$+ P(x_2 y_2) \log_2 \frac{1}{P(x_2 y_2)}$$

$$= \frac{2}{9} \log_2 \frac{9}{2} + \frac{1}{9} \log_2 9 + \frac{2}{30} \log_2 \frac{30}{2} + \frac{18}{30} \log_2 \left(\frac{30}{18}\right)$$

$= 1.5365$ bits/symbol.

(5) To obtain $H(X/Y)$ & $H(Y/X)$:-

$$H(Y/X) = H(X,Y) - H(X)$$

$$= 1.5365 - 0.9182$$

$= 0.6183$ bits/symbol.

$$H(X/Y) = \cancel{H(X/Y)}\ H(X,Y) - H(Y)$$

$$= 1.5365 - 0.8672$$

$= 0.6692$ bits/symbol

(6) To obtain $I[X,Y]$:

$$I[X,Y] = H(X) - H(X/Y)$$

$$= 0.9182 - 0.6692$$

$= 0.249$ bits/symbol.

⑤ $H(X,Y) = H(X/Y) + H(Y)$

$= H(Y/X) + H(X)$

Solⁿ :-

$$H(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 \left[\frac{1}{P(x_i, y_j)}\right] \quad \text{—①}$$

$$= -\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 [P(x_i y_j)]$$

From Probability Theorem;

$P(AB) = P(A/B)\,P(B)$

$\therefore \quad P(x_i, y_j) = P(x_i/y_j)\,P(y_j)$

From eqⁿ ① —

$$H[X,Y] = -\sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 P(x_i/y_j)$$

$$-\sum_{i=1}^{m} \sum_{j=1}^{m} P(x_i, y_j) \log_2 P(y_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 \frac{1}{P(x_i/y_j)} - \sum_{j=1}^{m} \left\{ \sum_{j=1}^{n} P(x_i y_j) \log_2 P \right.$$

Now, from the standard probability theory:—

$$\sum_{i=1}^{m} P(x_i, y_j) = P(y_j)$$

Hence, $H(X,Y) = H(X/Y) - \sum_{j=1}^{m} P(y_j) \log_2 P(y_j)$

$$= H(X/Y) + \sum_{i=1}^{m} P(y_j) \log_2 P\left(\frac{1}{y_j}\right)$$

$\therefore$ $\boxed{H(X,Y) = H(X/Y) + H(Y)}$ proved

Similarly:-

$$H(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 [P(y_j/x_i) \cdot P(x_i)]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 P(y_j/x_i)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j) \log_2 [P(x_i)]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i y_j) \log_2 \frac{1}{P(y_j/x_i)} - \sum_{i=1}^{m} \sum_{j=1}^{n} P(x_i, y_j)$$

$$\log_2 [P(x_i)]$$

From standard probability theorem:-

$$\sum_{j=1}^{n} P(x_i, y_j) = P(x_i)$$

$\therefore$ $\boxed{H(X,Y) = H(Y/X) - H(X)}$

Any

UNIT-II

5) ~~[illegible]~~

(b) Determine the capacity of a ternary channel with the stochastic matrix.

$$[P] = \begin{bmatrix} \alpha & 1-\alpha & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1-\alpha & \alpha \end{bmatrix}, \quad 0 \le \alpha \le 1$$

Soln:

Since, the channel matrix is a square matrix;

$$[P]\,[Q] = -[H]$$

$$[Q] = -[P]^{-1}[H]$$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = -\begin{bmatrix} \frac{1}{2\alpha} & 1 & -\frac{1}{2\alpha} \\ \frac{1}{2(1-\alpha)} & \frac{\alpha}{1-\alpha} & \frac{1}{2(1-\alpha)} \\ \frac{-1}{2\alpha} & 1 & \frac{1}{2\alpha} \end{bmatrix} \begin{bmatrix} h \\ 1 \\ h \end{bmatrix}$$

where, $h = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha)$

$$Q = \begin{bmatrix} -1 \\ -\left(\frac{h-\alpha}{1-\alpha}\right) \\ -1 \end{bmatrix}$$

and,

$$C = \log_2\left[2^{Q_1} + 2^{Q_2} + 2^{Q_3}\right]$$

$$= \log_2\left[2^{-1} + 2^{-\left(\frac{h-\alpha}{1-\alpha}\right)} + 2^{-1}\right]$$

$$C = \log_2\left[1 + \exp\left(\frac{\alpha - h}{1+\alpha}\right)\right]$$

8] Find the capacity of the following three Binary channels, given below —

a] $P_{11} = P_{22} = 1$

b] $P_{11} = P_{12} = P_{21} = P_{22} = 1/2$

c] $P_{11} = P_{12} = 1/2$; $P_{21} = 1/4$; $P_{22} = 3/4$.

Soln:

a] $P_{11} = P_{22} = 1$

$P_{21} = P_{12} = 0$

$Q_1 = \log P_{11} = \log 1 = 0$

$Q_2 = \log P_{22} = \log 1 = 0$.

Channel capacity, $C = \log_2\left[2^{Q_1} + 2^{Q_2}\right]$

$= \log_2\left[2^0 + 2^0\right]$

$= \log_2 2$

$= 1$ bit

b] $P_{11} = P_{12} = P_{21} = P_{22} = \frac{1}{2}$

$Q_1 = \log P_{11} = \log_2 \frac{1}{2} = -1$

$Q_2 = \log P_{22} = \log_2 \frac{1}{2} = -1$

channel capacity, $c = \log_2 [2^{-1}+2^{-1}]$

$= \log_2 [1]$

$\boxed{C = 0}$

c] $P_{11} = P_{12} = \frac{1}{2}$

$P_{21} = \frac{1}{4}$

$P_{22} = \frac{3}{4}$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 3/4 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2} \\ \frac{1}{4}\log_2\frac{1}{4} + \frac{3}{4}\log_2\frac{3}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}Q_1 + \frac{1}{2}Q_2 \\ \frac{1}{4}Q_1 + \frac{3}{4}Q_2 \end{bmatrix} = \begin{bmatrix} -0.5 - 0.5 \\ -0.5 - 0.3112 \end{bmatrix}$$

$\frac{1}{2}Q_1 + \frac{1}{2}Q_2 = -1$ — (1)

$\frac{1}{4}Q_1 + \frac{3}{4}Q_2 = -0.8112$ — (2)

From eqn (1) —

$Q_1 + Q_2 = -2$ — (3)

From eqn (2) —

$Q_1 + 3Q_2 = -3.2448$ — (4)

from eqn (3) & (4) —

$$\varphi_1 + \varphi_2 = -2$$

$$\varphi_1 + 3\varphi_2 = -3.2448$$

$$- \quad - \quad +$$

$$-2\varphi_2 = 1.2448$$

$$\boxed{\varphi_2 = -0.6224}$$

from (3) —

$$\varphi_1 - 0.6224 = -2$$

$$\varphi_1 = -2 + 0.6224$$

$$\boxed{\varphi_1 = -1.3776}$$

Now,

$$\text{channel capacity}, C = \log_2 \left[2^{-1.38} + 2^{-0.62}\right]$$

$$= \log_2 [0.3842 + 0.6506]$$

$$= \log_2 [1.0348)$$

$$= 0.0493 \text{ bits}$$

9.] (i) Find the Capacity of the channel with noise matrix –

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Soln :-

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\log\frac{1}{2} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4} \\ 1\log 1 \\ 1\log 1 \\ \frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{2}\log\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}Q_1 + \frac{1}{4}Q_2 + \frac{1}{4}Q_4 \\ Q_2 \\ Q_3 \\ \frac{1}{4}Q_1 + \frac{1}{4}Q_3 + \frac{1}{2}Q_4 \end{bmatrix} = \begin{bmatrix} -0.5 - 0.5 - 0.5 \\ 0 \\ 0 \\ -0.5 - 0.5 - 0.5 \end{bmatrix}$$

$$\frac{1}{2}Q_1 + \frac{1}{4}Q_2 + \frac{1}{4}Q_4 = -1.5 \qquad \text{—(I)}$$

$$Q_2 = 0 \qquad \text{—(II)}$$

$$Q_3 = 0 \qquad \text{—(III)}$$

$$\frac{1}{4}Q_1 + \frac{1}{4}Q_3 + \frac{1}{2}Q_4 \qquad \text{—(IV)}$$

Putting $\varphi_2 = \varphi_3 = 0$ in eqn (I) & (IV), we get –

$$\frac{1}{2}\varphi_1 + \frac{1}{4}\varphi_4 = -1.5 \quad \text{— (A)} \quad \times \frac{1}{4}$$

$$\frac{1}{4}\varphi_1 + \frac{1}{2}\varphi_4 = -1.5 \quad \text{— (B)} \quad \times \frac{1}{2}$$

$$\frac{1}{8}\varphi_1 + \frac{1}{16}\varphi_4 = -0.375$$

$$\frac{1}{8}\varphi_1 + \frac{1}{4}\varphi_4 = -0.75$$

$$\frac{1}{16}\varphi_4 - \frac{1}{4}\varphi_4 = 0.375$$

$$\frac{4-1}{16}\varphi_4 = 0.375$$

$$\frac{3}{16}\varphi_4 = 0.375$$

$$\varphi_4 = \frac{0.375}{0.1875}$$

$$\boxed{\varphi_4 = 2}$$

$$\frac{1}{4}\varphi_1 + \frac{1}{8}\varphi_4 = -0.75$$

$$\frac{1}{16}\varphi_1 + \frac{1}{8}\varphi_4 = -0.375$$

$$\frac{1}{4}\varphi_1 - \frac{1}{16}\varphi_1 = -0.375$$

$$\frac{3}{16}\varphi_1 = -0.375$$

$$\boxed{\varphi_1 = -2}$$

$\therefore$ From eqn (A) –

$$\frac{1}{2}\times -2 + \frac{1}{4}\varphi_4 = -1.5$$

$$\Rightarrow \frac{1}{4}\varphi_4 = -1.5 + 1$$

$$\Rightarrow \frac{1}{4}\varphi_4 = -0.5$$

$$\Rightarrow \varphi_4 = -0.5 \times 4$$

$$\Rightarrow \boxed{\varphi_4 = -2}$$

From eqn (A) –

$$\frac{1}{2}\varphi_1 + \frac{1}{4}\times 2 = -1.5$$

$$\Rightarrow \frac{1}{2}\varphi_1 = -1.5 - 0.5$$

$$\Rightarrow \varphi_1 = -2 \times 2 = -4$$

$$\boxed{\varphi_1 = -4}$$

$\therefore$ channel capacity, $c = \log_2\left[2^{Q_1} + 2^{Q_2} + 2^{Q_3} + 2^{Q_4}\right]$

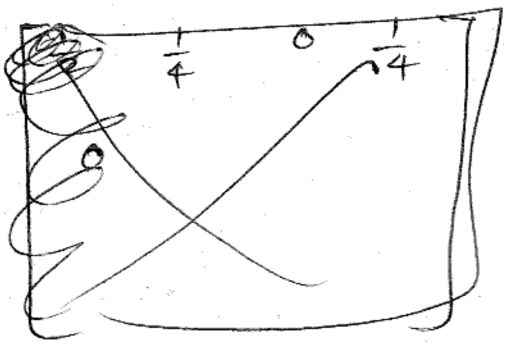
$$c = \log_2\left[2^{-2} + 2^{0} + 2^{0} + 2^{-2}\right]$$

$$= \log_2\left[0.25 + 1 + 1 + 0.25\right]$$

$$= \log_2\left[2.5\right]$$

$$\boxed{c = 1.322 \text{ bits}}$$

(ii) Derive the expression for channel capacity of a symmetric noise characteristics channel.



$$\begin{bmatrix} p & 1-p & 0 & 0 \\ 0 & p & 1-p & 0 \\ 0 & 1-p & p & 0 \\ 0 & 0 & 1-p & p \end{bmatrix}$$

As we know that -

The expression for channel capacity of the symmetric noise characteristics channel is -

$$c = \log m - h$$

where $h = H(Y/X)$

and, here $\boxed{m = 2}$

$\therefore \quad H(Y/X) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{(1-p)}$

$= -[p \log_2 p + (1-p) \log_2 (1-p)]$

$\therefore \quad C = \log m - h$

$\Rightarrow \boxed{C = \log_2 2 + p \log p + (1-p) \log_2 (1-p)}$ Ans

(10) (i) Evaluate the channel capacity of the channel whose matrix is given to be

$$\begin{bmatrix} \frac{1-p}{2} & \frac{1-p}{2} & \frac{p}{2} & \frac{p}{2} \\ \frac{p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} \end{bmatrix}$$

This is a symmetric channel.

$\therefore$ channel capacity, $C = \log m - h$

where, $h = H(Y/X)$

and $\boxed{m = 4}$ Here.

$\therefore \quad H(Y/X) = -\left[\frac{1-p}{2} \log_2 \left(\frac{1-p}{2}\right) + \frac{p}{2} \log_2 \left(\frac{p}{2}\right)\right]$

$\therefore \quad C = \log_2 2 + \left[\frac{1-p}{2} \log_2 \left(\frac{1-p}{2}\right) + \frac{p}{2} \log_2 \left(\frac{p}{2}\right)\right]$

$\boxed{C = 1 + \left[\frac{1-p}{2} \log_2 \left(\frac{1-p}{2}\right) + \frac{p}{2} \log_2 \left(\frac{p}{2}\right)\right.}$

(ii) (a) Evaluate the capacity of a channel whose matrix is

$$\left[\begin{array}{cc|c} 1-\beta & \beta & 0 \\ \beta & 1-\beta & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

we have;

$$\pi_1 = \begin{bmatrix} 1-\beta & \beta \\ \beta & 1-\beta \end{bmatrix}$$

$$\pi_2 = [1]$$

As we know that;

$$\boxed{2^C = 2^{C_1} + 2^{C_2}}$$

$$C_1 = \log m - h$$

$$\Rightarrow C_1 = \log_2 2 + (1-\beta)\log_2(1-\beta) + \beta\log_2(\beta)$$

$$\Rightarrow C_1 = 1 - H(\beta, 1-\beta)$$

and, $\boxed{C_2 = 0}$

$$\therefore \quad 2^C = [2^{1-H(\beta,1-\beta)} + 2^0]$$

$$\therefore \quad C = [\log_2(2^{C_1} + 2^{C_2})]$$

$$C = [\log_2[2^{1-H(\beta,1-\beta)} + 1]]$$ Ans

(b) Determine the capacity of a ternary channel with the stochastic matrix;

$$[P] = \begin{bmatrix} \alpha & 1-\alpha & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1-\alpha & \alpha \end{bmatrix}, \quad 0 \le \alpha \le 1$$

Since, the channel is a square matrix;

$$[P][Q] = -[H]$$

$$[Q] = -[P]^{-1}[H]$$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = - \begin{bmatrix} \frac{1}{2\alpha} & 1 & -\frac{1}{2\alpha} \\ \frac{1}{2(1-\alpha)} & \frac{\alpha}{1-\alpha} & \frac{1}{2(1-\alpha)} \\ \frac{-1}{2\alpha} & 1 & \frac{1}{2\alpha} \end{bmatrix} \begin{bmatrix} h \\ 1 \\ h \end{bmatrix}$$

where, $h = -\alpha \log \alpha - (1-\alpha) \log (1-\alpha)$

$$Q = \begin{bmatrix} -1 \\ -\left(\frac{h-\alpha}{1-\alpha}\right) \\ -1 \end{bmatrix}$$

$$C = \log_2 \left[2^{Q_1} + 2^{Q_2} + 2^{Q_3}\right]$$

$$c = \log_2 \left[1 + \exp\left(\frac{\alpha - h}{1-\alpha} \right) \right]$$

(7.) A zero memory source contains $X = \{x_1, x_2, x_3, x_4\}$

with $D(X) = \{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8} \}$

(a.) To obtain $H(X)$:-

$$H(X) = \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{8}\log_2 8.$$

$$= 0.5 + 0.5 + 0.375 + 0.375$$

$$= 1.75 \text{ bits/message}$$

(b) To show $H(X^2) = 2H(X)$:

$L.H.S = H(X)^2 = (1.75)^2 = 3.0625$

$$\boxed{H(X^2) = 3.0625}$$

Now,

$R.H.S = 2H(X) = 2 \times 1.75$

$= 3.5$

(ii) To obtain second order ~~expression~~ extension of the source.

The source alphabet X contains four symbols. Hence its second order extension will contain sixteen symbols. These symbols, their probabilities and entropy calculation are shown in the given table -

S.No.	Second order extension symbol (σ_i)	Probability of symbol (σ_i) $P(\sigma_i)$	$P(\sigma_i)\log_2 \frac{1}{P(\sigma_i)}$
1	x_1x_1	$\frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$	$\frac{1}{4}\log 4 = 0.5$
2	x_1x_2	$\frac{1}{2}\times\frac{1}{4} = 1/8$	$\frac{1}{8}\log 8 = 0.375$
3	x_1x_3	$\frac{1}{2}\times\frac{1}{8} = 1/16$	$\frac{1}{16}\log 16 = 0.25$
4	x_1x_4	$\frac{1}{2}\times\frac{1}{8} = \frac{1}{16}$	$\frac{1}{16}\log 16 = 0.25$
5	x_2x_1	$\frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$	$\frac{1}{8}\log 8 = 0.375$
6	x_2x_2	$\frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$	$\frac{1}{16}\log 16 = 0.25$
7	x_2x_3	$\frac{1}{4}\times\frac{1}{8} = \frac{1}{32}$	$\frac{1}{32}\log 32 = 0.1562$
8	x_2x_4	$\frac{1}{4}\times\frac{1}{8} = \frac{1}{32}$	$\frac{1}{32}\log 32 = 0.1562$
9	x_3x_1	$\frac{1}{8}\times\frac{1}{2} = \frac{1}{16}$	$\frac{1}{16}\log 16 = 0.25$
10	x_3x_2	$\frac{1}{8}\times\frac{1}{4} = \frac{1}{32}$	$\frac{1}{32}\log 32 = 0.1562$
11	x_3x_3	$\frac{1}{8}\times\frac{1}{8} = \frac{1}{64}$	$\frac{1}{64}\log 64 = 0.09375$
12	x_3x_4	$\frac{1}{8}\times\frac{1}{8} = \frac{1}{64}$	$\frac{1}{64}\log 64 = 0.09375$
13	x_4x_1	$\frac{1}{8}\times\frac{1}{2} = \frac{1}{16}$	$\frac{1}{16}\log 16 = 0.25$
14	x_4x_2	$\frac{1}{8}\times\frac{1}{4} = \frac{1}{32}$	$\frac{1}{32}\log 32 = 0.1562$
15	x_4x_3	$\frac{1}{8}\times\frac{1}{8} = \frac{1}{64}$	$\frac{1}{64}\log 64 = 0.09375$
16	x_4x_4	$\frac{1}{8}\times\frac{1}{8} = \frac{1}{64}$	$\frac{1}{64}\log 64 = 0.09375$

The entropy of second order extension of the source can be obtained as –

$$H(X^2) = \sum_{i=1}^{16} P(\sigma_i) \log_2 \frac{1}{P(\sigma_i)}$$

$\therefore H(X^2) = 0.5 + 0.375 + 0.25 + 0.25 + 0.375 + 0.25 + 0.1562 + 0.1562 +$
$0.25 + 0.1562 + 0.09375 + 0.09375 + 0.25 + 0.1562 + 0.09375$
0.09375

$= 3.4$ bits/symbol (extended)

and, $2H(X) = 2 \times 1.75$
$= 3.5$ bits/symbol

$\therefore$ $\boxed{2H(X) = H(X^2)}$ Proved

5. (a) Fano's Inequality:

An arbitary code (S, n) consisting of words $x^{(1)}, \ldots, x^{(S)}$

UNIT-III

(4) Given data :-

channel Bandwidth, $B = 3.4\ KHz = 3400\ Hz$

$$\left(\frac{S}{N}\right)_{dB} = 30\ dB$$

As we know that -

$$\left(\frac{S}{N}\right)_{dB} = 10 \log_{10} \frac{S}{N}$$

$$3\cancel{0} = 1\cancel{0} \log_{10} \frac{S}{N} \Rightarrow \frac{S}{N} = 10^3 = 1000$$

$$\boxed{\frac{S}{N} = 1000}$$

(a) To calculate the capacity of the channel:

capacity of the channel is given as -

$$C = B \log_2 \left(1 + \frac{S}{N}\right)$$

$$= 3400 \log_2 (1 + 1000)$$

$$= 3400 \log_2 1001$$

$$= 3400 (9.9672)$$

$$\boxed{C = 33.888\ \text{kbps}}$$

(b) To obtain minimum $\left(\frac{S}{N}\right)$ for 9.6 KBPS data -

Here the data rate is 9.6 KBPS, from channel coding theorem,

$$\boxed{R \leq C}$$

$$C = B \log_2 \left(1 + \frac{S}{N}\right)$$

$$9.6 \leq 3400 \log_2 \left(1 + \frac{S}{N}\right)$$

$$9600 \leq 3400 \log_2 \left(1 + \frac{S}{N}\right)$$

$$\log_2 \left(1 + \frac{S}{N}\right) \geq 2.823$$

~~$\frac{\log 1 + \log \frac{S}{N}}{\log 2} \geq 2.823$~~

~~$\log \frac{S}{N} \geq 2.823 \times$~~

~~$\frac{S}{N} \geq e^{0.849}$~~

~~$\left(\frac{S}{N}\right)_{min} \geq 2.33$ to transmit data at the rate of 9.6 KBPS~~

$$\Rightarrow \frac{\log_{10}\left(1 + \frac{S}{N}\right)}{\log_2} \geq 2.823$$

$$\Rightarrow \log_{10}\left(1 + \frac{S}{N}\right) \geq 0.849$$

$$\Rightarrow 1 + \frac{S}{N} \geq 10^{0.849}$$

$$\Rightarrow 1 + S/N \geq 7.063$$

$$(S/N)_{min} \geq 6.063$$

$\left(\frac{S}{N}\right)_{min} = 6.063$ to transmit data at 9.6 KbPS.

(5) Given:

Bandwidth, $B = 4\ KHz = 4000\ Hz$

$\therefore$ Nyquist rate $= 2B = 2 \times 4000$

$= 8000\ Hz$.

Hence, sampling rate, $r = 1.25 \times$ Nyquist rate

$= 1.25 \times 8000$

$= 10{,}000$

Since the message samples are quantized into 256 equally likely level, there will be,

$M = 256$ sample.

Each sample will have probability of occurance,

$p = \frac{1}{M} = \frac{1}{256}$.

$\therefore$ Entropy for such message,

$H = \log_2 M = \log_2 256$

$= 8$ bits/sample

(a.) Information rate of this source,

$R = rH = 10{,}000 \times 8 = 80{,}000$ bits/second.

(6.) To check the error free transmission of

$B = 10$ KHZ, $\frac{S}{N} = 20$ dB.

$$\left(\frac{S}{N}\right)_{dB} = 20 = 10 \log_{10}\left(\frac{S}{N}\right)$$

$$\log_{10}\frac{S}{N} = 2$$

$$\frac{S}{N} = 10^2 = 100$$

$$\therefore \left(\frac{S}{N}\right)_{dB} = 100$$

$$C = B\log_2\left(1+\frac{S}{N}\right)$$

$$= 1000 \log_2 (1+100)$$

$$= 1000 \log_2 101$$

$$= 1000 \times 6.6582$$

$$= 6658.2$$

$$= 66.582 \text{ KbPS}$$

if $R \leq C$, then error free transmission is possible.

Here, $R = 80,000$ and $C = 66.582$ KbPS

Hence, $R > C$. Therefore error free transmission is not possible.

(c) Bandwidth Required for error free transmission:-

As we know that -

$R \leq C$

Also, $\left(\frac{S}{N}\right)_{dB} = 25\,dB$ (Given)

$25 = 10 \log\left(\frac{S}{N}\right)$

$2.5 = \log\left(\frac{S}{N}\right)$

$\frac{S}{N} = 10^{2.5}$

$\boxed{\frac{S}{N} = 316.22}$

$\therefore \quad 80{,}000 \leq B \log_2\left(1+\frac{S}{N}\right)$

$\Rightarrow 80{,}000 \leq B \log_2 (1+316.22)$

$\Rightarrow 80{,}000 \leq B \log_2 317.22$

$\Rightarrow 80{,}000 \leq B\,(8.31)$

$\Rightarrow B \geq \frac{80{,}000}{8.31}$

$B \geq 9626.95$ Hz

$\boxed{B \geq 9.6\,\text{KHz}}$

(3) <u>Given</u>:

Bandwidth, $B = 3.4\ KHz = 3400\ Hz$

$\left(\frac{S}{N}\right)_{dB} = 20\ dB$

$\left(\frac{S}{N}\right)_{dB} = 10 \log_{10} \frac{S}{N}$

$\cancel{20} = 1\cancel{0} \log_{10} \frac{S}{N}$

$\frac{S}{N} = 10^2 = 100$

$\therefore \boxed{\frac{S}{N} = 100}$

$C = B \log_2 \left(1 + \frac{S}{N}\right)$

$= 3400 \log_2 (1 + 100)$

$= 3400 (6.658)$

$= 22637.2$

$\boxed{C = 22.637\ kbps}$

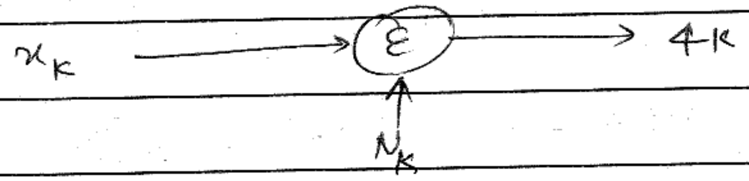
(b) To calculate the max^m symbol rate for which error free transmission over the channel is possible:-

(1) ## Information Capacity Theorem:

Consider a zero mean stationary process $x(t)$ that is band limited to B Hz. let, X_K, $K=1,2,3 \ldots K$. denotes the continous ~~___~~ random variable obtained by uniform sampling of the process $x(t)$ at the nyquist rate 2B samples per second. These samples are transmitted over the noisy channel, also ~~___~~ band limited to B hz in T sec.

Hence, the number of samples K is given by -

$$K = 2BT \quad \text{——} (1)$$



let, the continous random variable Y_K, $K = 1,2,3 \ldots K$ denotes the samples of received signal.

$$Y_K = X_K + N_K \quad \text{——} (2)$$

Here, N_K is Gaussian noise sample with zero mean and variance given by -

$$\sigma^2 = N_0 B \quad \text{——} (3)$$

Also, the average transmitted power

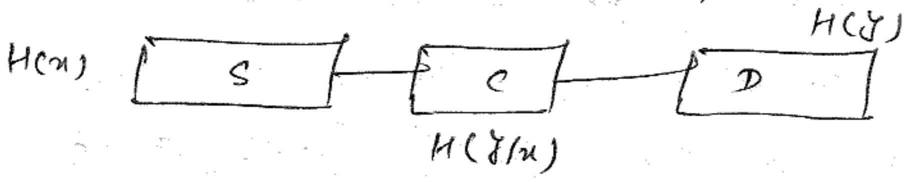
$$E[X_K^2] = P \quad \text{——} (4)$$

The information capacity of the channel is defined as the maximum of mutual information between the channel input X_K and channel output Y_K. Overall distribution on the channel i/p

Therefore,

$$C = \max \{ I(X_K ; Y_K) : E[X_K^2] = P \} \quad \text{--- (5)}$$

where, maximization is performed w.r.to $f_{X_K}(x)$, the probability density function of X_K.



Here,
$H(x)$ = source entropy
$H(y)$ = destination entropy.
$H(y/x)$ = conditional entropy

Let, $I[X_K, Y_K]$ denotes the mutual information between X_K and Y_K.

$I[X_K, Y_K]$ denotes the mutual information between X_K and Y_K.

$$I(X_K, Y_K) = H(Y_K) - H(Y_K/X_K) \quad \text{--- (6)}$$

Here, $H(Y_K/X_K) = H(N_K)$ --- (7)

where, X_K and Y_K are the independent variable

For the evaluation of information capacity, C, we proceed in three stages:—

i] The variance of sample Y_K of the received signal equals $P+\sigma^2$

$$\mathrm{Var}(Y_K) = P + \sigma^2$$

$$\therefore H(Y_K) = \frac{1}{2}\log_2[2\pi e(P+\sigma^2)] \quad \text{--- (8)}$$

② The variance of the noisy sample, N_K is equal –

$$\therefore \quad H(N_K) = \frac{1}{2}\log_2 (2\pi e \sigma^2) \quad \text{——} (9)$$

③ $I(X_K; Y_K) = H(Y_K) - H(N_K)$

$$= \frac{1}{2}\log_2 \left[\frac{2\pi e\, (P+\sigma^2)}{2\pi e\, \sigma^2}\right]$$

$$= \frac{1}{2}\log_2 \left[1 + \frac{P}{\sigma^2}\right]$$

$$\therefore \quad \boxed{C = I[X,Y] = \frac{1}{2}\log_2\left[1 + \frac{P}{N_0 B}\right]} \quad [\because \sigma^2 = N_0 B]$$

Thus, information capacity theorem implies that for a given average transmitted power P and channel bandwidth 'B', we can transmit information at the rate of 'C' bits per second.

Implementation of Information Capacity Theorem: –

An ideal system defined as 'one that transmit the data at a bit rate R_b equal to the information capacity 'C'.

The average transmitted power can be expressed as –

$$P = E_b \cdot C \quad \text{——} (1)$$

where, E_b is the transmitted energy per bit.

Also, the Ideal System is defined as –

$$C = B \log_2 \left(1 + \frac{P}{N_0 B}\right) \quad \text{—— (2)}$$

On substituting (1) in (2), we get –

$$C = B \log_2 \left(1 + \frac{E_b}{N_0} \frac{C}{B}\right) \quad \text{—— (3)}$$

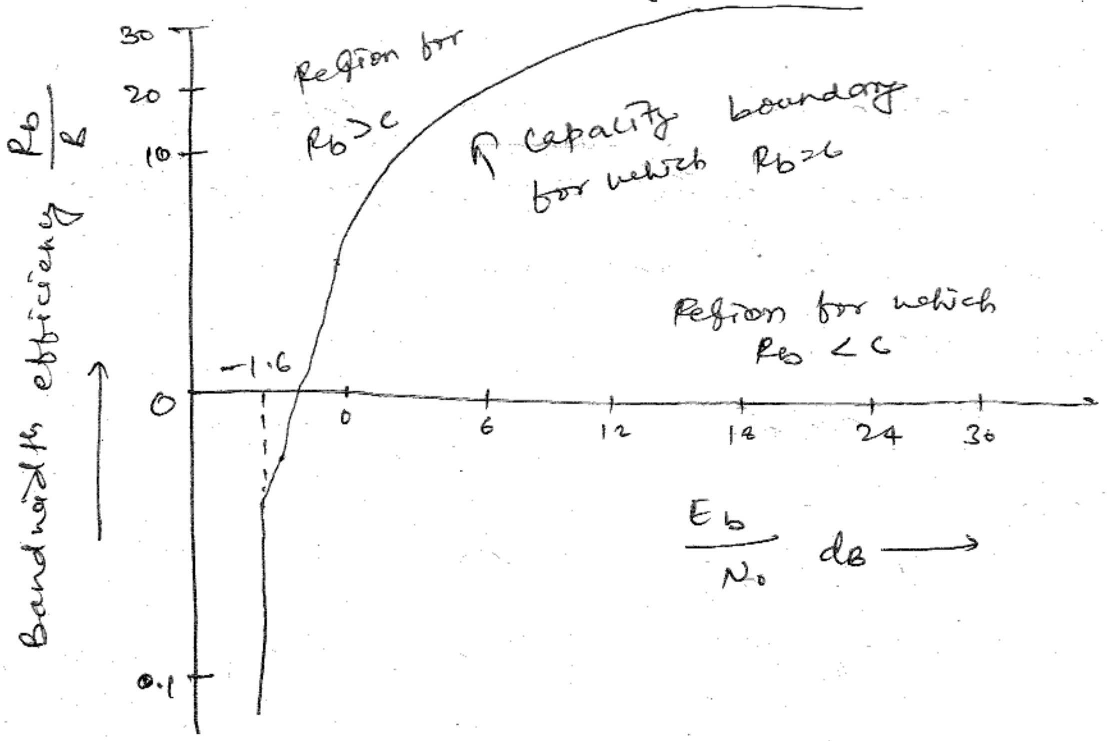
Also,

$$\frac{C}{B} = \log_2 \left(1 + \frac{E_b}{N_0} \frac{C}{B}\right) \quad \text{—— (4)}$$

Also, we may define the signal energy per bit to Noise Power Spectral density $\frac{E_b}{N_0}$ in term of $\frac{C}{B}$ for the ideal system is –

$$\frac{E_b}{N_0} = \frac{2^{C/B} - 1}{C/B} \quad \text{—— (5)}$$

A plot of bandwidth efficiency R_b/B versus E_b/N_0 is called the Bandwidth efficiency diagram.



The curve labeled "capacity boundary" corresponding to the ideal system for which $R_b = C$.

Based on figure, we can make following conclusions –

(a) For infinite bandwidth, the ratio $\frac{E_B}{N_0}$ approaches limiting value –

$$\left(\frac{E_b}{N_0}\right)_\infty = \lim_{B \to \infty} \frac{E_b}{N_0}$$

$$= \log_2 = 0.693$$

This value is called Shannon limit for AWGN channel.

(2) The capacity boundary $R_b = C$ separates the combination of system parameter that have the potential for supporting error free transmission, $R_b < C$ from which error free transmission is not possible.

(3) The diagram highlights the 'trade off' between $\frac{E_b}{N_0}$, R_b and error probability P_e.

UNIT-IV

(1) **Given:**

The generator matrix is -

$$G = \begin{bmatrix} 1 & 0 & 0 & : & 0 & 1 & 1 \\ 0 & 1 & 0 & : & 1 & 0 & 1 \\ 0 & 0 & 1 & : & 1 & 1 & 0 \end{bmatrix}_{(6\times 3)}$$

Here, $n=6$

$k=3$

The code vectors can be obtained from following steps:-

(a) Determine the P submatrix from generator matrix.

(b) Obtain equations for check bits using $C = MP$.

(c) Determine check bits for every message vector.

(a) **Step - I:**

To determine the P submatrix:-

$$G = \left[I_{K\times K} : P_{K\times q} \right]_{n\times k}$$

$$I_{K\times K} = I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

$$P_{K\times q} = P_{3\times 3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$$

(b) To obtain the eqn for check bits:-

Here, $k=3$, $q=3$ and $n=6$.

$$[C]_{1\times q} = [M]_{1\times k}\,[P]_{k\times q}$$

$$C = MP$$

$$[c_1 \quad c_2 \quad c_3]_{1\times 3} = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}_{1\times 3} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$$

From the above matrix multiplication, we get -

$$c_1 = m_1(0) \oplus m_2 \oplus m_3 \quad \text{——} (1)$$

$$c_2 = m_1 \oplus (0) m_2 \oplus m_3 \quad \text{——} (2)$$

$$c_3 = m_1 \oplus m_2 \oplus (0) m_3 \quad \text{——} (3)$$

So,

$$\left.\begin{aligned} c_1 &= m_2 \oplus m_3 \\ c_2 &= m_1 \oplus m_3 \\ c_3 &= m_1 \oplus m_2 \end{aligned}\right\} \leftarrow (4)$$

(c) To determine check bits and codevectors for every message vector:-

S.No.	Message bits			Check bits			Complete vector code $X = MC$					
	m_1	m_2	m_3	c_1	c_2	c_3	m_1	m_2	m_3	c_1	c_2	c_3
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	1	1	1	0
3	0	1	0	1	0	1	0	1	0	1	0	1
4	0	1	1	0	1	1	0	1	1	0	1	1
5	1	0	0	0	1	1	1	0	0	0	1	1
6	1	0	1	1	0	1	1	0	1	1	0	1
7	1	1	0	1	1	0	1	1	0	1	1	0
8	1	1	1	0	0	0	1	1	1	0	0	0

(3) ~~Res~~ Given:-

$n=6$

$K=3$

$\therefore q=n-K=3$

and, (a) To obtain the generator matrix:

The parity check bits c_4, c_5 and c_6 are given as-

$$\left.\begin{aligned} c_4 &= d_1+d_3 \\ c_5 &= d_1+d_2+d_3 \\ c_6 &= d_1+d_2 \end{aligned}\right\} \leftarrow (A)$$

As we know that -

$$[C]_{1\times q} = [M]_{1\times K}\,[P]_{K\times q}$$

$$\therefore \begin{bmatrix} c_4 & c_5 & c_6 \end{bmatrix}_{1\times 3} = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}_{3\times 3}$$

Therefore,

$$\left.\begin{aligned} c_4 &= d_1 P_{11} + d_2 P_{21} + d_3 P_{31} \\ c_5 &= d_1 P_{12} + d_2 P_{22} + d_3 P_{32} \\ c_6 &= d_1 P_{13} + d_2 P_{23} + d_3 P_{33} \end{aligned}\right\} \leftarrow (B)$$

and, As we know -

$$[G]_{n\times K} = \Big[[I_{K\times K}] : [P]_{K\times q} \Big]_{n\times K}$$

from eq^n (A) & (B); we get -

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$$

$$\therefore \quad [G]_{n\times K} = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 1 & 1 \\ 0 & 1 & 0 & \vdots & 0 & 1 & 1 \\ 0 & 0 & 1 & \vdots & 1 & 1 & 0 \end{bmatrix}_{6\times 3}$$

(b) To obtain the code vectors:

S.No.	Message bits			check bits			all code vector $X = MC$						weight of code vector, $w(X)$
	d_1	d_2	d_3	c_4	c_5	c_6	d_1	d_2	d_3	c_4	c_5	c_6	
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	1	1	1	0	3
3	0	1	0	0	1	1	0	1	0	0	1	1	3
4	0	1	1	1	0	1	0	1	1	1	0	1	4
5	1	0	0	1	1	1	1	0	0	1	1	1	4
6	1	0	1	0	0	1	1	0	1	0	0	1	3
7	1	1	0	1	0	0	1	1	0	1	0	0	3
8	1	1	1	0	1	0	1	1	1	0	1	0	4

(c) To obtain the error correcting capability:-

$$d_{min} = [w(X)]_{min} = 3$$

$$\because \quad d_{min} \geq s+1$$

$$3 \geq s+1$$

$$\boxed{s \leq 2}$$

Thus it will detect two errors.

and,

$$d_{min} \geq 2t+1$$

$$3 \geq 2t+1$$

$$2t \leq 2$$

$$\boxed{t \leq 1}$$ So, it will correct only one error.

(d.) ~~Keep~~ To decode the received word $\{010110\}$.

The Parity check matrix, (H) :-

$$H = [P^T : I_q]_{q \times n}$$

$$H = \begin{bmatrix} 1 & 0 & 1 & : & 1 & 0 & 0 \\ 1 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 1 & 0 & : & 0 & 0 & 1 \end{bmatrix}_{3 \times 6}$$

Now, Taking transpose of Parity check matrix (H) :-

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{6 \times 3}$$

Now,

Syndrome vector, $S = EH^T$

where, E is the received data.

$$E = \{010110\}$$

$$\therefore \quad S = [0\ 1\ 0\ 1\ 1\ 0]_{1 \times 6} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{6 \times 3}$$

$$S = \begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$S = [1\ 0\ 1]$ Ans

(6.) Given:

The generator matrix is –

$$G = \begin{bmatrix} 1 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 1 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 1 & : & 0 & 1 & 1 \end{bmatrix}_{\substack{(k \times n) \\ (3 \times 6)}}$$

Here; $n = 6$

$k = 3$

$\therefore q = n - k = 3$

(a.) To determine the P submatrix from the given generator matrix: –

$$[G]_{(k \times n)} = [I_{(k \times k)} : P_{(k \times q)}]_{(k \times n)}$$

$$[I]_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

$$[P]_{3 \times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3 \times 3}$$

(b) <u>To determine the d_{min}:</u>

First, we have to find eqn for check bits:

Here, $n=6$

$q=3$

$K=3$

$$\therefore [C]_{1\times q} = [K]_{1\times K} [P]_{K\times q}$$

$$[c_1 \; c_2 \; c_3]_{1\times 3} = [m_1 \; m_2 \; m_3]_{1\times 3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{3\times 3}$$

From the above matrix multiplication, we get -

$$c_1 = m_1 \oplus m_2$$

$$c_2 = m_1 \oplus m_2 \oplus m_3$$

$$c_3 = m_1 \oplus m_3$$

Now, we have to determine check bits and all code vectors :-

S.No.	Message bits			Check bits			All code vectors $X = mC$						$W(X)$
	m_1	m_2	m_3	c_1	c_2	c_3	m_1	m_2	m_3	c_1	c_2	c_3	
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	0	1	1	0	0	1	0	1	1	3
3	0	1	0	1	1	0	0	1	0	1	1	0	3
4	0	1	1	1	0	1	0	1	1	1	0	1	4
5	1	0	0	1	1	1	1	0	0	1	1	1	4
6	1	0	1	1	0	0	1	0	1	1	0	0	3
7	1	1	0	0	0	1	1	1	0	0	0	1	3
8	1	1	1	0	1	0	1	1	1	0	1	0	4

As we know that -

$$d_{min} = [w(x)]_{min} = 3$$

$$\because \quad d_{min} \geq s+1$$

$$3 \geq s+1$$

$\boxed{s \leq 2}$ it will detect only two errors.

and,

$$d_{min} \geq 2t+1$$

$$3 \geq 2t+1$$

$\boxed{t \leq 1}$ it will correct only one error.

now,

The parity check matrix (H) is given as -

$$H = [P^T : I_q]_{q \times n}$$

$$= \left[\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]_{3 \times 6}$$

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{6 \times 3}$$

Now, Syndrome vector, $S = YH^T$

where, Y is the received given output.

$$\therefore \quad S = \underset{(1\times 6)}{\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}} \underset{(6\times 3)}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$S = \begin{bmatrix} 1+1 & 1+1 & 1+1+1 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$ Ans

$$\log \frac{S}{N} \geq 2.823 \times 0.301$$

$$\log \frac{S}{N} \geq 0.849$$

$$\frac{S}{N} \geq 2.33$$