1) Write short notes on fano's Inequality :-

→ Given, an arbitrary code (s, n) consisting of words, $x^{(1)} \ldots x^{(s)}$. Let $X = (X_1 \ldots X_n) \rightarrow$ random vector that equals $x^{(i)}$ with Prob $P(x^{(i)})$, $i = 1, 2 \ldots s$, Where,

$$\sum_{i=1}^{P} P(x^{(i)}) = 1 \quad \text{——} (1)$$

Let, $Y = (Y_1, \ldots Y_n)$ be the corresponding o/p sequence.

If $P(e) \rightarrow$ Prob. of error of code, compute for the given i/p distribution, then,

$$H(X|Y) \leq H[P(e), 1 - P(e)] + P(e) \log(s-1) \quad \text{——} (2)$$

Proof :-

Consider an o/p sequence y,

Let, $g(y) \rightarrow$ i/p selected by decoder.

thus, if y is received an error will occur if and only if the fixed sequence is not equal to $g(y)$.

Now, let us divide the set of values

of into 2 groups. one group consisting of all other code words,

from the basic axioms of Entropy.

$$H(X/y, y) = H(q, 1-q) + q\, H(1) + (1-q)\, H(q_1, \ldots q_{s-1}) \quad \text{--- (3)}$$

where, $q = P[x = g(y)/y = y]$ & $q_1, \ldots q_{s-1}$

are of the form,

$$\frac{P(X/y)}{\sum_{x \neq g(y)} P(X/y)} \quad \text{--- (4)}$$

with x ranging over all the code words except g(y). we have a theorem,

$H(P_1, P_2, \ldots P_m) \leq \log m$, with equality if and only if,

$P_i = \frac{1}{m}$, thus by this theorem.

$$H(q_1, \ldots q_{s-1}) \leq \log (s-1) \quad \text{--- (5)}$$

We obtain,

$$H(X/y : y) \leq H[P(e/y, 1 - P(e/y)] + P(e/y) \log (s-1) \quad — (6)$$

By convexity of H,

$$H[P(e), 1 - P(e)] = H\left(\sum_y P(y) P(e/y), 1 - \sum_y P(y) P(e/y)\right]$$

$$= H\left[\sum_y P(y) P(e/y), \sum_y P(y) [1 - P(e/y)]\right]$$

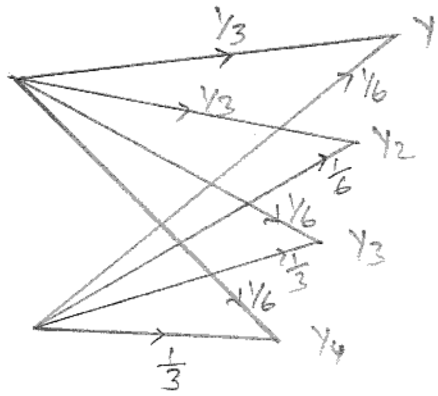
$$\geq \sum_y P(y) H[P(e/y), 1 - P(e/y)] \longrightarrow (7)$$

If we multiply (6) by $P(y)$ and sum over all y. we find using (7) that,

$$H(X/y) \leq H[P(e), 1 - P(e)] + P(e) \log (S-1) \longrightarrow (8)$$

Hence, the fano's inequality has been proved.

2) Find the capacity of the channel given below,



SOLUTION) →

For a symmetric channel,

$$C = \log_2 m - h \qquad \text{where } h = H(Y/X)$$

$$C = \log_2 4 - \frac{2}{3}\log_2 3 - \frac{1}{3}\log_2 6$$

$$C = \frac{5}{3} - \log_2 3 \text{ bits.}$$

3) Find the capacity of the channel whose noise matrix is,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

SOLUTION :-

$$\begin{bmatrix} \frac{1}{2} & 1/4 & 0 & 1/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\log\frac{1}{2} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4} \\ 1\log 1 \\ 1\log 1 \\ \frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{2}\log\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2}Q_1 + \frac{1}{4}Q_2 + \frac{1}{4}Q_4 \\ Q_2 \\ Q_3 \\ \frac{1}{4}Q_1 + \frac{1}{4}Q_3 + \frac{1}{2}Q_4 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 0 \\ 0 \\ -1.5 \end{bmatrix}$$

$\frac{1}{2} Q_1 + \frac{1}{4} Q_4 = -1.5$

$\frac{1}{4} \cdot Q_1 + \frac{1}{2} Q_4 = -1.5$

$Q_1 = -2.$ $\qquad Q_2 = 0$

$Q_3 = 0$ $\qquad Q_4 = -2.$

channel capacity $C = \log_2[2^{Q_1} + 2^{Q_2} + 2^{Q_3} + 2^{Q_4}]$

$= \log_2 [2^{-2} + 2^0 + 2^0 + 2^{-2}]$

$\boxed{\cancel{C = 1.322 \text{ bits}}}$

$= \log_2 \left[\frac{1}{4} + 1 + 1 + \frac{1}{4}\right]$

$= \log_2 \left[\frac{5}{2}\right]$

$= \frac{\log(5/2)}{\log 2} = 1.32$

$\boxed{C = 1.32 \text{ bits}}$

4) An ideal communication system with average Power limitation and Perturbed by While gaussian noise has a bandwidth of 1 MHz and a signal to noise ratio of 10.

a) Determine the channel capacity in bits/second.

b) If the signal to noise ratio drops to 5, what Bandwidth is required for the same channel capacity?

c) If the Bandwidth is decreased to 0.5 MHz, what S/N ratio is required to maintain the same channel capacity?

SOLUTION :-

a) hannel capacity $c = W \log_2\left[1 + \frac{S}{N}\right]$

$$= (1 \times 10^6) \log_2 [1 + 10]$$

$$C = 3.45 \times 10^6 \text{ bits/second.}$$

b) $W \log_2 [1+5] = C$

$$W = \frac{C}{\log_2 6} = \frac{3.45 \times 10^6}{\log_2 6} = 1.33 \times 10^6$$

c.) $W \log_2 \left[1 + \frac{S}{N}\right] = C$

$$\left[1 + \frac{S}{N}\right] = 2^{C/W}$$

$$\Rightarrow \frac{S}{N} = 2^{C/W} - 1$$

$$\Rightarrow \frac{S}{N} = 2^{6.9} - 1$$

$$\Rightarrow \boxed{\frac{S}{N} = 118.42}$$

//.

5) Apply Shannon - Fano encoding Procedure to the following massage.

$$[X] = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$$

$$[P] = [0.49, 0.14, 0.14, 0.07, 0.07, 0.04, 0.02, 0.02, 0.01].$$

SOLUTION :-

Massage	Probability	Codeword	length
m_1	0.49	0	1
m_2	0.14	1 0 0	3
m_3	0.14	1 0 1	3
m_4	0.07	1 1 0 0	4
m_5	0.07	1 1 0 1	4
m_6	0.04	1 1 1 0	4
m_7	0.02	1 1 1 1 0	5
m_8	0.02	1 1 1 1 1 0	6
m_9	0.01	1 1 1 1 1 1	6

$$\bar{L} = 0.49 \times 1 + 0.14 \times 3 + 0.14 \times 3 + 0.07 \times 4 + 0.07 \times 4$$

$$+ 0.04 \times 4 + 0.02 \times 5 + 0.02 \times 6 + 0.01 \times 6$$

$$= 0.49 + 0.42 + 0.42 + 0.28 + 0.28 + 0.16$$

$$+ 0.1 + 0.12 + 0.06$$

$$\boxed{\bar{L} = 2.33}$$

$$H = -\Big[0.49 \log_2(0.49) + 0.14 \log_2(0.14) + 0.14 \log_2(0.14)$$

$$+ 0.07 \log_2(0.07) + 0.07 \log_2(0.07) + 0.04 \log_2(0.04)$$

$$+ 0.02 \log_2(0.02) + 0.02 \log_2(0.02) + 0.01 \log_2(0.01)\Big]$$

$$= -\frac{1}{\log 2}\Big[0.49 \log(0.49) + 0.14 \log(0.14) + 0.14 \log(0.14)$$

$$+ 0.07 \log(0.07) + 0.07 \log(0.07) + 0.04 \log(0.04)$$

$$+ 0.02 \log(0.02) + 0.02 \log(0.02)$$

$$+ 0.01 \log(0.01)\Big]$$

$$= -3.32\Big[-0.15 - 0.11 - 0.11 - 0.08 - 0.08 - 0.05$$

$$- 0.03 - 0.03 - 0.02\Big]$$

$H = -3.32\,(-0.66)$

$\boxed{H = 2.19}$ bits/sample

Efficiency $\eta = \frac{H}{L}$

$= \frac{2.19}{2.33} = 0.939\,\%$

$\boxed{\eta = 0.939\,\%}$

$Red = 1 - \eta$

$= 1 - 0.939$

$\boxed{Red = 0.061}$ ~~bits~~