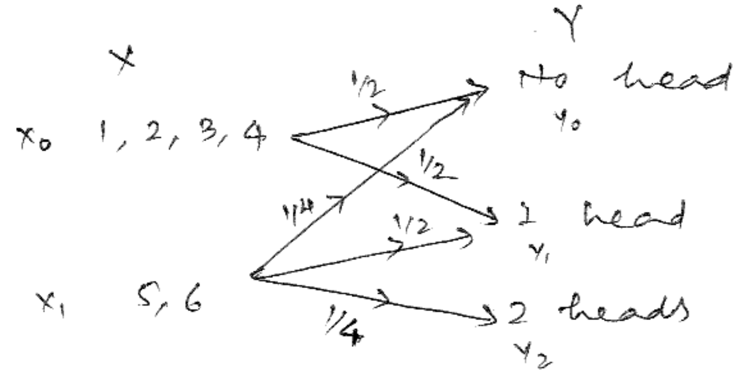
A single die is tossed once. If the face of the die is 1, 2, 3, 4 then the coin is tossed once. If the face of the die is 5 or 6, the coin is tossed twice. Find the information conveyed about the face of the die by the no. of its obtained.

<u>Solution</u>:



$$P(Y/X) = \begin{array}{c|ccc} & y_0 & y_1 & y_2 \\ \hline x_0 & 1/2 & 1/2 & 0 \\ x_1 & 1/4 & 1/2 & 1/4 \end{array}$$

$P(x_0) = 1/2$, $P(x_1) = 1/2$

$P(X,Y) = P(X) \cdot P(Y/X)$

$P(X) = [1/2 , 1/2]$

$$P(X,Y) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \begin{array}{ccc} y_0 & y_1 & y_2 \\ \end{array} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$= \begin{array}{ccc} y_0 & y_1 & y_2 \end{array} \begin{bmatrix} 1/4 & 1/4 & 0 \\ 1/8 & 1/4 & 1/8 \end{bmatrix}$$

$P(Y) =$ sum of columns

$Y_0 = 1/4 + 1/8 = \frac{2+1}{8} = 3/8$

$Y_1 = 1/4 + 1/4 = \frac{2}{4} = 1/2$

$Y_2 = 1/8$

$$P(X/Y) = \frac{P(X,Y)}{P(Y)}$$

$$= \frac{\begin{bmatrix} 1/4 & 1/4 & 0 \\ 1/8 & 1/4 & 1/8 \end{bmatrix}}{P(Y)}$$

$$= \begin{bmatrix} 2/3 & 1/2 & 0 \\ 1/3 & 1/2 & 1 \end{bmatrix}$$

$$H(X) = \frac{-1}{\log_{10} 2}\left[1/2 \log 1/2 + 1/2 \log 1/2\right]$$

$$= \frac{-1}{0.3}(-0.15 - 0.15)$$

$$= \frac{0.3}{0.3} = 1.$$

$$\therefore \boxed{H(X) = 1}$$

$$H(Y) = \frac{-1}{\log_{10} 2}\left[3/8 \log 3/8 + 1/2 \log 1/2 + 1/8 \log 1/8\right]$$

$$= \frac{-1}{0.3}\left[-0.159 - 0.15 - 0.112\right]$$

$$= \frac{0.421}{0.3} = 1.4.$$

$$\therefore \boxed{H(Y) = 1.4}$$

For $H(X,Y)$,

$$P(X,Y) = \begin{bmatrix} 1/4 & 1/4 & 0 \\ 1/8 & 1/4 & 1/8 \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 & 0 \\ 1/8 & 1/4 & 1/8 \end{bmatrix}$$

we know,

$$H(X,Y) = \frac{-1}{\log_{10} 2}\left[\tfrac{1}{4}\log \tfrac{1}{4} + \tfrac{1}{8}\log \tfrac{1}{8} + \tfrac{1}{4}\log \tfrac{1}{4} + \tfrac{1}{4}\log \tfrac{1}{4} + \tfrac{1}{8}\log \tfrac{1}{8}\right]$$

$$= \frac{-1}{0.3}\left[-0.15 - 0.112 - 0.15 - 0.15 - 0.112\right]$$

$$= \frac{0.674}{0.3} = 2.24$$

$$\therefore \boxed{H(X,Y) = 2.24}$$

$$P(X,Y) = \begin{bmatrix} 1/4 & 1/4 & 0 \\ 1/8 & 1/4 & 1/8 \end{bmatrix} \overset{P(Y/X)}{\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}}$$

$$H(Y/X) = \sum_{i=1}^{K-1}\sum_{j=1}^{K-1} P(X,Y)\log P(Y/X)$$

$$= \frac{-1}{\log_{10} 2}\left[\tfrac{1}{4}\log \tfrac{1}{2} + \tfrac{1}{8}\log \tfrac{1}{4} + \tfrac{1}{4}\log \tfrac{1}{2} + \tfrac{1}{4}\log \tfrac{1}{2} + \tfrac{1}{8}\log \tfrac{1}{4}\right]$$

$$= \frac{-1}{0.3}\left[-0.075 - 0.075 - 0.075 - 0.075 - 0.075\right]$$

$$= \frac{0.376}{0.3}$$

$$= 1.25.$$

$$\therefore \boxed{H(Y/X) = 1.25}$$

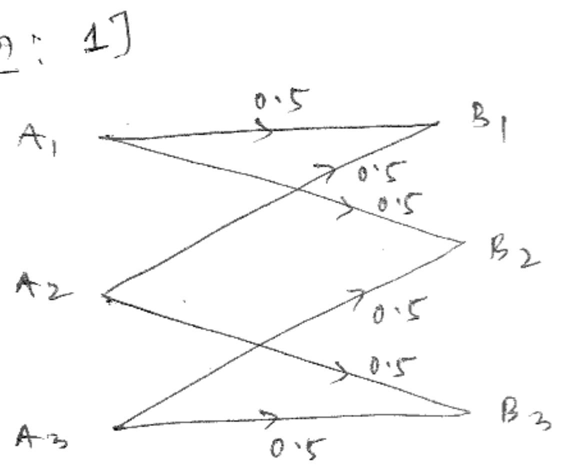
$$\therefore I(X,Y) = H(Y) - H(Y/X)$$
$$= 1.4 - 1.25$$
$$= 0.15 \text{ bits/symbol.}$$

Hence, $\boxed{I(X,Y) = 0.15 \text{ bits/symbol.}}$ //

Q: Given below, the noise characteristics. Determine the rate of transmission through this channel.

~~Solution~~: 1]

~~Here,~~



$p(A_1) = 0.6$

$p(A_2) = 0.3$

$p(A_3) = 0.1$

Solution:

From the question, the noise characteristics is given. i.e. ~~P(Y/X)~~ $P(B/A)$

$$\therefore P(B/A) = \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 0.5 & 0.5 & 0 \\ A_2 & 0.5 & 0 & 0.5 \\ A_3 & 0 & 0.5 & 0.5 \end{array}$$

From the question, we are having the values of $P(A_1)$, $P(A_2)$ and $P(A_3)$. So, substituting these values in $P(B/A)$. Then,

$$P(A,B) = P(A) \cdot P(B/A)$$

$$= \begin{array}{c} \\ 0.6 \\ 0.3 \\ 0.1 \end{array} \begin{array}{ccc} B_1 & B_2 & B_3 \\ \end{array}$$

$$= \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline 0.6 & 0.5 & 0.5 & 0 \\ 0.3 & 0.5 & 0 & 0.5 \\ 0.1 & 0 & 0.5 & 0.5 \end{array}$$

$$\therefore P(A,B) = \begin{array}{ccc} B_1 & B_2 & B_3 \\ \end{array} \begin{bmatrix} 0.3 & 0.3 & 0 \\ 0.15 & 0 & 0.15 \\ 0 & 0.05 & 0.05 \end{bmatrix}$$

We know,

$P(B)$ = Adding the columns B_1, B_2 and B_3 individually, then,

$$P(B_1) = 0.3 + 0.15 = \cancel{0.18}\ 0.45$$

$$P(B_2) = 0.3 + 0.05 = 0.35$$

$$P(B_3) = 0.15 + 0.05 = 0.20$$

We know,

Rate of transmission is,

$$\boxed{I(A,B) = H(B) - H(B/A)} \quad \text{— (1)}$$

We know,

$$H(B) = -\sum_{i=1}^{n} P(b_j) \log_2 P(b_j)$$

$$= -\left[0.45\log_2 0.45 + 0.35\log_2 0.35 + 0.20\log_2 0.20\right]$$

$$= \frac{-1}{0.3}\left[-0.156 - 0.259 - 0.139\right]$$

$$= \frac{0.454}{0.3}$$

$$= 1.51 \text{ bits/symbol.}$$

$$\therefore \boxed{H(B) = 1.51 \text{ bits/symbol.}}$$

Now, for ~~H Y~~ $H(B/A)$.

$$P(A,B) = P(A,B) \cdot P(B/A)$$

$$= \begin{bmatrix} 0.30 & 0.30 & .0 \\ 0.15 & 0 & 0.15 \\ 0 & 0.05 & 0.05 \end{bmatrix} \begin{array}{c} P(B/A) \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

we know,

$$\cancel{H Y} \; H(B/A) = -\sum_{j=1}^{n}\sum_{i=1}^{m} \log P(a_i, b_j) \log P(b_j/a_i)$$

$$= -\Big[0.3 \log 0.5 + 0.15 \log 0.5 + 0.3 \log 0.5 + 0.05 \log 0.5 + 0.15 \log 0.5 + 0.05 \log 0.5\Big]$$

$$= -\frac{1}{0.3}\Big[-0.09 - 0.045 - 0.09 - 0.025 - 0.045 - 0.025\Big]$$

$$= \frac{0.3}{0.3}$$

$$= 1 \text{ bits/symbol}$$

$$\therefore \boxed{H(B/A) = 1 \text{ bits/symbol}}$$

Hence,
Now substituting the values of $H(B)$ and $H(B/A)$ in (1) then,

The rate of transmission is given by,

$$I(A,B) = H(B) - H(B/A)$$

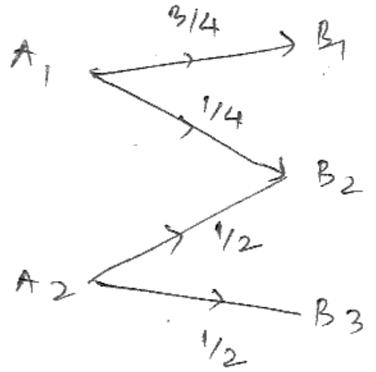
$$= 1.51 - 1$$

$$= 0.51 \text{ bits/symbol.}$$

Hence,

$$\boxed{I(A,B) = 0.51 \text{ bits/symbol.}}$$

2]



<u>solution</u>:
From above figure, the noise matrix can be written as,

$$P(Y/X) = \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \end{array}$$

Being only two ~~but~~ probability of chances, both $P(A_1)$ and $P(A_2)$ have equal chances.
i.e. $P(A) = P(B) = 1/2$

$$P(X,Y) = P(A) \cdot P(Y/X)$$

$$= \begin{matrix} 1/2 \\ 1/2 \end{matrix} \begin{bmatrix} 3/4 & 1/4 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P(X,Y) = \begin{bmatrix} 3/8 & 1/8 & 0 \\ 0 & 1/4 & 1/4 \end{bmatrix}$$

$P(Y) =$ sum of columns.

$$P(B_1) = 3/8 \quad , P(B_2) = 3/8 \; , P(B_3) = 1/4$$

$$H(Y) = -\sum_{j=1}^{n} P(y_j) \log P(y_j)$$

$$= -\left[3/8 \log_2 3/8 + 3/8 \log_2 3/8 + 1/4 \log_2 1/4\right]$$

$$= -\frac{1}{0.3}\left[3/4 \log_{10} 3/8 + \cancel{3/8 \log_{10} 3/8} + 1/4 \log_{10} 1/4\right]$$

$$= -\frac{1}{0.3}\left[-0.319 - 0.150\right]$$

$$= \frac{4.695}{0.3}$$

$= 1.565$ bits/symbol.

$\therefore$ $\boxed{H(Y) = 1.565 \text{ bits/symbol.}}$

For ~~$H(X,Y)$~~ $H(Y/X) = -\sum_{j=1}^{n} \sum_{i=1}^{m} P(x_i, y_j) \log P(y_j/x_i)$

$$= -\left[3/8 \log_2 3/4 + 1/8 \log_2 1/4 + \frac{1}{4} \log_2 1/2 + \frac{1}{4} \log_2 1/2\right]$$

$$= -\frac{1}{0.3}\left[3/8 \log_{10} 3/4 + 1/8 \log_{10} 1/4 + 1/2 \log_{10} 1/2\right]$$

$$= -\frac{1}{0.3}\left[-0.0468 - 0.075 - 0.150\right]$$

$$= \frac{0.2723}{0.3}$$

$$= 0.9077 \text{ bits / symbol.}$$

$$\therefore \boxed{H(Y/X) = 0.9077 \text{ bits / symbol}}$$

Hence,
The rate of transmission through the channel is given by,

$$I(X,Y) = H(Y) - H(Y/X)$$

$$= 1.585 - 0.9077$$

$$\boxed{I(X,Y) = 0.6573 \text{ bits / symbol.}}$$ //.

~~Hence~~

3] $$P(X,Y) = \begin{bmatrix} 3/40 & 1/40 & 1/40 \\ 1/20 & 3/20 & 1/20 \\ 1/8 & 1/8 & 3/8 \end{bmatrix}$$

Solution,

we know,

$$P(X,Y) = P(X) \cdot P(Y/X) \quad — ①$$

& from given,

$$P(X,Y) = \begin{bmatrix} 3/40 & 1/40 & 1/40 \\ \frac{1}{20} & \frac{3}{20} & 1/20 \\ 1/8 & 1/8 & 3/8 \end{bmatrix}$$

$$P(Y/X)$$

$$p(X,Y) = \begin{array}{c|c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} \frac{1}{40} \\[2ex] \frac{1}{20} \\[2ex] \frac{1}{8} \end{array} & \begin{bmatrix} 3 & 1 & 1 \\[2ex] 1 & 3 & 1 \\[2ex] 1 & 1 & 3 \end{bmatrix} \end{array} \quad - (2)$$

comparing (1) & (2)

$$P(x_i) = 1/40 \,, \quad P(x_2) = 1/20 \,, \quad P(x_3) = 1/8$$

$$P(Y) = P(y_1) + P(y_2) + P(y_3)$$

$$P(y_1) = 3+1+1 = 5$$

$$P(y_2) = 5 \,, \quad P(y_3) = 5$$

$$H(Y) = -\sum_{j=1}^{n} P(y_j) \log P(y_j)$$

$$= -\left[5 \log_2 5 + 5 \log_2 5 + 5 \log_2 5 \right]$$

$$= - \; 15 \log_2 5$$

$$= - \; \frac{15 \log_{10} 5}{\log 2} = \frac{-10.48}{0.3} = -34.82$$

$$H(Y) = -34.82$$

For $H(Y/X)$,

$$\cancel{p(X,Y)} \begin{bmatrix} 3/40 & \frac{1}{40} & \frac{1}{40} \\ 1/20 & 3/20 & 1/20 \\ 1/8 & 1/8 & 3/8 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

~~H(X,Y) =~~

$$H(Y/X) = -\sum_{i=1}^{n}\sum_{j=0}^{m} p(x_i, y_j)\log p(y_j/x_i)$$

$$= -\Big[\tfrac{3}{40}\log_2 3 + \tfrac{1}{20}\log_2 1 + \tfrac{1}{8}\log_2 1 + \tfrac{1}{40}\log_2 1 + \tfrac{3}{20}\log_2 3$$
$$+ \tfrac{1}{8}\log_2 1 + \tfrac{1}{40}\log_2 1 + \tfrac{1}{20}\log_2 1 + \tfrac{3}{8}\log_2 3\Big]$$

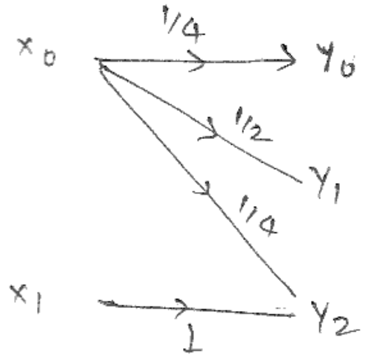
$$= -\frac{1}{0.3}\left[0.035 + 0.071 + 0.1789\right]$$

$$= -0.949$$

$$I(X,Y) = H(Y) - H(Y/X)$$
$$= -34.82 + 0.949$$

$$\therefore \boxed{I(X,Y) = -33.87 \text{ bits/symbol.}}$$

④



prove : $I(X,Y) = I(Y,X)$

Solution

From the above Fig. we can write

$p(x_0) = p(x_1) = 1/2$

And, the characteristic noise matrix is,

$$P(Y/X) = \begin{array}{c|ccc} & y_0 & y_1 & y_2 \\ \hline x_1 & 1/4 & 1/2 & 1/4 \\ x_2 & 0 & 0 & 1 \end{array}$$

$$P(X,Y) = \begin{array}{c} 1/2 \\ 1/2 \end{array} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P(X,Y) = \begin{bmatrix} 1/8 & 1/4 & 1/8 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$P(Y) = P(y_0) + P(y_1) + P(y_2)$$

$$P(y_0) = 1/8 \quad , \quad P(y_1) = \frac{1}{4} \quad , \quad P(y_2) = 5/8$$

$$H(Y) = -\sum_{j=0}^{K-1} P(y_j) \log_2 P(y_j)$$

$$= -\left[1/8 \log_2 1/8 + 1/4 \log_2 1/4 + 5/8 \log_2 5/8 \right]$$

$$= -\frac{1}{0.3} \left[-0.11 - 0.15 - 0.1275 \right]$$

$$= \frac{0.387}{0.3}$$

$$\boxed{H(Y) = 1.29 \text{ bits/symbol.}}$$

So $H(Y/X) = -\sum_{j=0}^{K-1}\sum_{i=0}^{K-1} P(x_i, y_j) \log_2 P(y_j/x_i)$

$= -\left[\frac{1}{8}\log_2 \frac{1}{4} + \frac{1}{4}\log_2 \frac{1}{2} + \frac{1}{8}\log_2 \frac{1}{4} + \frac{1}{2}\log_2 1\right]$

$= \frac{-1}{0.3}\left[-0.075 - 0.075 - 0.075 + 0\right]$

$= \frac{0.225}{0.3}$

$H(Y/X) = 0.75$ bits/ system.

Hence,

$I(X,Y) = H(Y) - H(Y/X)$

$= 1.29 - 0.75$

$\boxed{I(X,Y) = 0.54 \text{ bits /symbol.}}$ — (1)

similarly we ~~know~~, can write,

$I(Y,X) = H(X) - H(X/Y)$

$H[X;Y] = -\sum_{i=0}^{K-1} P(x_i) \log P(x_i)$

$= -\left[\frac{1}{2}\log_2 \frac{1}{2} + \frac{1}{2}\log_2 \frac{1}{2}\right]$

$= -\frac{1}{0.3}\left(\log \frac{1}{2}\right)$

$= \frac{0.3}{0.3}$

$= 1$

$\boxed{H(X) = 1 \text{ bits/symbol.}}$

$$P(X/Y) = \frac{P(X,Y)}{P(Y)}$$

$$\therefore P(X/Y) = \begin{bmatrix} 1 & 1 & 1/5 \\ 0 & 0 & 4/5 \end{bmatrix}$$

$$H(X/Y) = P(X,Y) \cdot P(X/Y)$$

$$= -\sum_{j=0}^{K-1} \sum_{i=0}^{K-1} p(x_i, y_j) \log P(x_i/y_j)$$

$$= -\left[1/8 \log_2 1 + \frac{1}{4} \log_2 1 + 1/8 \log_2 1/5 + 1/2 \log_2 4/5 \right]$$

$$= \frac{-1}{0.3} \left[-0.0870 - 0.0484 \right]$$

$$= \frac{0.135}{0.3}$$

$$= 0.45$$

$\therefore H(X/Y) = 0.45$ bits/symbol.

Hence,

$$I(Y,X) = H(X) - H(X/Y)$$

$$= 1 - 0.45$$

$$\boxed{I(Y,X) = 0.55 \text{ bits/symbol.}} \quad \text{--- } (2)$$

Here, from eqns (1) & (2), we can say that

$$\boxed{I(X,Y) = I(Y,X)}$$

Hence, proved.