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Decoding :-

→ To achieve reliability, we must determine the I/P message with high degree of accuracy. Thus we are looking for the best decoding scheme.

→ Suppose channel has I/P alphabets $x_1, x_2, \ldots x_m$ & O/P alphabets $y_1, y_2 \ldots y_n$. & Channel matrix $P[(y_j/x_i)]$.

→ A decoder or decision scheme is an assignment to every O/P symbol (y_j) of an I/P symbol (x_j^*). from the alphabets

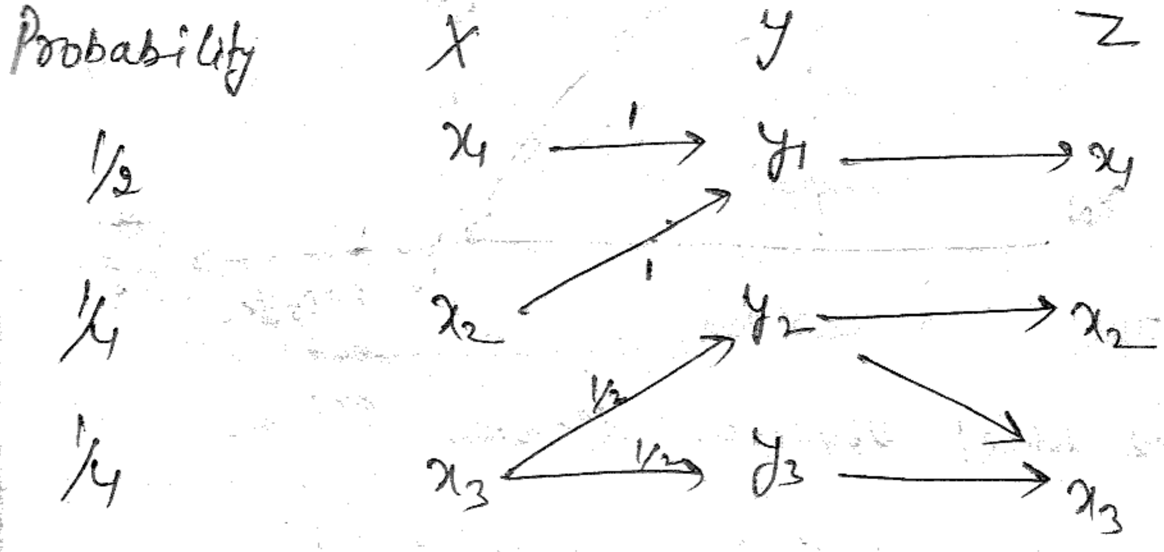
$x_1, x_2, \ldots x_m$.

⇒ If y_j is received it will decoded as (x_j^*).

⇒ If Z is the O/P of decoder, we may express $\boxed{Z = g(y)}$

Example of Channel & Decoder

Example of channel & decoder



For an a given i/p x_i, consider the decision scheme that minimises the overall probability of error. Such a decision scheme is called <u>ideal observer</u>.

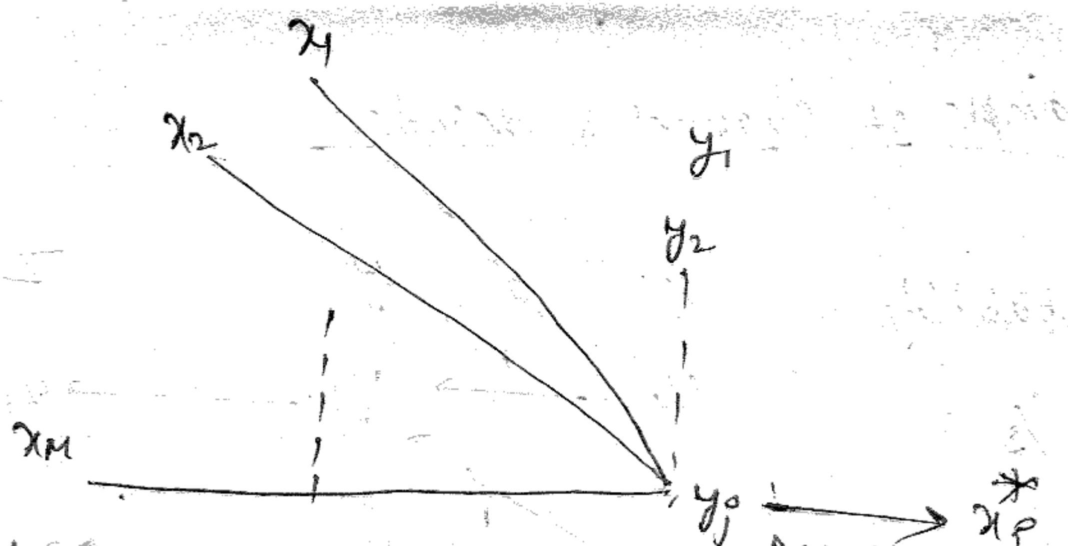
Let $p(e) \rightarrow$ possibility of error.

$p(e') \rightarrow$ overall probability of correct transmission.

Probability of correct transmission is:-

$$P(e') = \sum_{j=1}^{N} P(y_j)\, P(e'/y_j)$$

$$= \sum_{j=1}^{N} P(y_j)\, P(X = x_j^{*}/y_j)$$



<u>27/7/11</u>

⇒ The ideal observer decision scheme associates with each o/p symbol y_j the i/p symbol X that maximises ~~P(~~

$P(x/y_j)$.

⇒ If more than 1 i/p symbol ~~&~~ yields, maximum, any one of the maximum can be chosen. The probability of error will not be affected

⇒ The decision scheme minimising the overall probability of error for the given i/p distribution is given by:-

$$P(\alpha_1, \alpha_2, \ldots \alpha_N / \beta_1, \beta_2, \ldots \beta_N)$$
$$= \frac{P(\alpha_1, \alpha_2, \ldots \alpha_N) \prod_{k=1}^{n} P\left(\frac{\beta_k}{\alpha_k}\right)}{P(\beta_1, \beta_2, \ldots \beta_N)}$$

Where,

$\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_N \rightarrow$ i/p sequence.

$\beta_1, \beta_2, \ldots \beta_N \rightarrow$ o/p "

If the symbol are equally likely,

$$\boxed{P(x_i/y) = \frac{1}{MP(y)} P(y/x_i)}$$

Thus when all the i/ps are equally likely the ideal observer selects the i/p x_i, for which $P(y/x_i)$ is max^m. The resulting Decoder is called <u>maximum likelyhood decision scheme</u>.

<u>Disadvantages</u> :-

(i) It is defined only for a particular i/p distribution. i.e when the i/p probability is changed, the decision scheme also changes.

ii) It is possible that certain i/ps are not received correctly. It is more desirable to have a decision scheme with uniform error bound.

FANO'S INEQUALITY THEOREM

→ Given an arbitrary code (s, n) consisting of words $x^{(1)}, x^{(2)}, \ldots x^{(s)}$.

Let $X = (x_1, x_2, \ldots x_n)$, with a probability $P(x^{(i)})$, $i = 1, 2, \ldots s$, where

$$\sum_{i=1}^{S} P(x^{(i)}) = 1$$

Let $Y = (Y_1, Y_2, \ldots Y_N)$ be the corresponding o/p sequence.

If $P(e)$ = Probability of error

then

$$\boxed{H(X/Y) \leqslant H[P(e), 1-P(e)] + P(e)\log(S-1)}$$

SHANNON'S FUNDAMENTAL THEOREM

or,

Noisy Coding Theorem

It states that :- " Consider a discrete memoryless channel with non zero capacity C, fixed two numbers H & e such that

$0 < H < C$ and $e > 0$ "

→ Let us transmit 'm' messages $u_1, u_2 \ldots u_m$ by codewords each of length 'n', then 'n' may chosen such that $m \geq 2^{nH}$

CAPACITY OF BANDLIMITED GAUSSIAN CHANNEL :-

⇒ The rate of transmission for a continuous channel is given by

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y) \log\left[\frac{P(x,y)}{p(x)\cdot p(y)}\right] dx\cdot dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(x,y)\left[\log P(x,y) - \log P(x)\cdot P(y)\right] dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(x,y)\, dx\, dy$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \left[\log p(x) + \log p(y)\right] dx\, dy$$

$$= -H(x,y) + H(x) + H(y)$$

$$\Rightarrow \boxed{I(X;Y) = H(X) + H(Y) - H(X,Y)}$$

$$\because H(x) = \int_{-\infty}^{\infty} p(x,y) \log \frac{1}{p(x)}\, dx$$

$$H(y) = \int_{-\infty}^{\infty} p(x,y) \left(\log \frac{1}{p(y)}\right) dy$$

Also,

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log \left[\frac{p(x/y)}{p(x)}\right] dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(x/y)\, dx\, dy$$

$$- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(x)\, dx\, dy.$$

$$= -H(X/Y) + H(X) = H(X) - H(X/Y)$$

$$\because -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(x)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} - p(x,y) \log p(x)\, dx\, dy$$

$$\boxed{I(X,Y) = H(X) - H(X/Y)}$$

Silly

$$I(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(y/x)\, dx\, dy$$

$$\boxed{I(X;Y) = H(Y) - H(Y/X)}$$

⇒ Let ~~y~~ y be a random variable and $H(y)$ is maximum when y has ~~max~~ ~~distr~~ gaussian distribution.

$$H(y)_{max} = \ln \sqrt{2\pi e}\ \sigma_y$$

where,
$\sigma_y \rightarrow$ variance of y.

Channel capacity, $\boxed{C = \ln \sqrt{2\pi e \sigma_y^2} - \sqrt{2\pi e \sigma_n^2}}$

Note

$\sigma_n \rightarrow$ variance of noise power.

I/P + noise = O/P

i.e. $\sigma_y^2 = \sigma_x^2 + \sigma_n^2$

$\Rightarrow \sigma_x^2 = \sigma_y^2 - \sigma_n^2$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \frac{1}{\log p(x)}\, dx\, dy = H(x)$$

$$\Rightarrow C = \ln\left[\frac{\sigma_y^2}{\sigma_n^2}\right]^{1/2}$$

$$= \ln\left[\frac{\sigma_x^2+\sigma_n^2}{\sigma_n^2}\right]^{1/2}$$

$$= \ln\left[\frac{\sigma_x^2}{\sigma_n^2}+1\right]^{1/2}$$

$$= \ln\left[1+\frac{S}{N}\right]^{1/2} = \frac{1}{2}\ln\left[1+\frac{S}{N}\right]$$

$$\boxed{C = \frac{1}{2}\ln\left[1+\frac{S}{N}\right]}$$

For bandlimited Gaussian channel

we have $2W$ samples / sec.

$$C = \cancel{2}W \times \frac{1}{\cancel{2}}\ln\left[1+\frac{S}{N}\right]\log_2 e \text{ bits/sec}$$

$$= \frac{W \times \log_2\left(1+\frac{S}{N}\right)}{\cancel{\log_2 e}} \cdot \cancel{\log_2 e}$$

$$\boxed{C = W\log_2\left(1+\frac{S}{N}\right)} \text{ bit/sec}$$

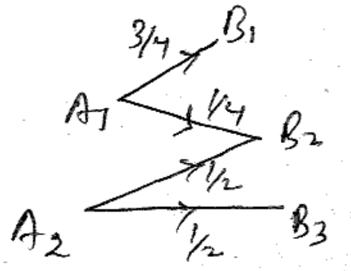
← Shennon Hartly

28/7/14 IMPORTANT Q

Q) Given the noise characteristics of communication channel. Evaluate the rate of Transmission through the channel

(Pg 57)

$P(A_1) = P(A_2) = 1/2$. Ⓐ



Sol:- $P(y/x) = \begin{array}{c c} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3/4 & \frac{1}{4} & 0 \\ 0 & 1/2 & \frac{1}{2} \end{bmatrix} \end{array}$

$P(x,y) = P(y/x) \cdot P(x)$

$= \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 3/8 & \frac{1}{8} & 0 \\ 0 & \frac{1}{4} & 1/4 \end{bmatrix} \end{array}$

$\because P(x) = P(A_1) = P(A_2) = 1/2$

$P(y_1) = \frac{3}{8} + 0 = \frac{3}{8}$

$P(y_2) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$P(y_3) = 0 + \frac{1}{4} = \frac{1}{4}$.

$H(X) = \sum_{i=1}^{M} P_k \log_2 \frac{1}{P_k}$

$= \sum_{i=1}^{M} P_k \log_2 (P_k)^{-1}$

Hastly

$$H(Y) = \frac{-3}{8}\log_2\left(\frac{3}{8}\right) - \frac{3}{8}\log_2(3/8) - \frac{1}{4}\log_2\left(\frac{1}{4}\right)$$

$$= \frac{-3}{8}\frac{\log\frac{3}{8}}{\log 2} - \frac{3}{8}\frac{\log\left(\frac{3}{8}\right)}{\log 2} - \frac{1}{4}\frac{\log \frac{1}{4}}{\log 2}$$

$$= \frac{-3}{8}\left(\frac{-0.425}{0.301}\right) - \frac{3}{8}\left(\frac{-0.425}{0.301}\right) - \frac{1}{4}\left(\frac{-0.602}{0.301}\right)$$

$= 1.56$ bits/symbol.

$H(Y) = 1.56$ bits/symbol.

$$H(Y/X) = -\sum_{i=1}^{K} P(x,y)\log P(y/x)$$

$$= \sum_{i=} \frac{-3}{8}\log_2\left(\frac{3}{4}\right) - \frac{1}{8}\log_2\left(\frac{1}{4}\right) - \frac{1}{4}\log_2\left(\frac{1}{2}\right) - \frac{1}{4}\log_2\left(\frac{1}{2}\right)$$

$$= \frac{-3}{8}\frac{\log\frac{3}{4}}{\log 2} - \frac{1}{8}\frac{\log\frac{1}{4}}{\log 2} - \frac{1}{4}\frac{\log \frac{1}{2}}{\log 2} - \frac{1}{4}\frac{\log \frac{1}{2}}{\log 2}$$

$$= \frac{-3}{8}\left(\frac{-0.124}{0.301}\right) - \frac{1}{8}\left(\frac{-0.602}{0.301}\right) - \frac{1}{4}\left(\frac{-0.301}{0.301}\right) - \frac{1}{4}\left(\frac{-0.301}{0.301}\right)$$

$\log_2\left(\frac{1}{4}\right)$ $= 0.9056$

$I(X,Y) = H(Y) - H(Y/X)$

$= 1.56 - 0.9056$

$= 0.6557$ Ans

$\frac{1}{4}$

$\left(\frac{-0.602}{0.301}\right)$

Q: Calculate the channel matrix whose noise ~~margin~~ ~~margin~~ matrix given below:- Pg 136

Sol:-

$$\begin{bmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/4 & 0 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log\frac{1}{4} + 0 + \frac{1}{4}\log\frac{1}{4} \\ 1\log 1 \\ 1\log 1 \\ \frac{1}{4}\log\frac{1}{4} + 0 + \frac{1}{4}\log\frac{1}{4} + \frac{1}{2}\log\frac{1}{2} \end{bmatrix}$$

$0.5\log(0.5) + 2\times 0.25\log 0.25$ / $\log 2$

$= -1.5$

$$= \begin{bmatrix} -1.5 \\ 0 \\ 0 \\ -1.5 \end{bmatrix} \quad \text{--- (2)}$$

$$\begin{bmatrix} \frac{1}{2}Q_1 + \frac{1}{4}Q_2 + \frac{1}{4}Q_4 \\ Q_2 \\ Q_3 \\ \frac{1}{4}Q_1 + \frac{1}{4}Q_3 + \frac{1}{2}Q_4 \end{bmatrix} \quad -(3)$$

Comp (2) & (3),

$\therefore Q_2 = 0$ & $Q_3 = 0$.

$$\frac{1}{2}Q_1 = -1.5$$

$$\boxed{Q_1 = -3.}$$

Now,

$$\frac{1}{4}(-3) + \frac{1}{2}(Q_4) = -1.5$$

$$\Rightarrow -0.75 + \frac{1}{2}Q_4 = -1.5$$

$$\Rightarrow \frac{1}{2}Q_4 = -1.5 + 0.75$$

$$= -0.75$$

$$Q_4 = -(2 \times 0.75)$$

$$\boxed{Q_4 = -1.5}$$

$\frac{1}{2}Q_1 + \frac{1}{4}Q_2 + \frac{1}{4}Q_4 = -1.5$

$\because Q_2 = 0$

$\Rightarrow \frac{1}{2}Q_1 + \frac{1}{4}Q_4 = -1.5$ — (1)

(3)

$\frac{1}{4}Q_1 + \frac{1}{4}Q_3 + \frac{1}{2}Q_4$

$\Rightarrow \frac{1}{4}Q_1 + \frac{1}{2}Q_4 = -1.5 \quad \times 2$

$\Rightarrow \frac{Q_1}{2} + Q_4 = -3$ — (2)'

$-\left(\frac{Q_1}{2} + \frac{Q_4}{4} = -1.5\right)$

$Q_4 - \frac{Q_4}{4} = -3 + 1.5 = -1.5$

$\frac{3Q_4}{4} = -1.5$

$Q_4 = \frac{4(-1.5)}{3} = \frac{-6}{3} = -2$

$\boxed{Q_4 = -2}$

$\boxed{Q_1 = -2}$

$\frac{1}{2}Q_1 + \frac{1}{4}(-2) = -1.5$

$\Rightarrow \frac{1}{2}Q_1 - \frac{1}{2} = -1.5$

$\frac{1}{2}Q = -1.5 + 1.5$

$$C = \log_2 \left[2^{Q_1} + 2^{Q_2} + 2^{Q_3} + 2^{Q_4} \right]$$

$$\Rightarrow \log_2 \left[2^{-2} + 2^{0} + 2^{0} + 2^{-2} \right]$$

$$\Rightarrow \log_2 \left[0.25 + 1 + 1 + 0.25 \right]$$

$$\Rightarrow \log_2 (2.5)$$

$$= \frac{\log 2.5}{\log 2}$$

$$\Rightarrow 1.32.$$

$\boxed{C = 1.32}$ bits

Channel Capacity.

Q3:- given the messages $m_1, m_2 \ldots m_6$
Pg 58 with the probabilities $\frac{1}{3}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{12}, \frac{1}{12}$
respectively.
Find Shannon's fano code and also calculate the coding ~~rid~~ ridenty.

Sol:-

	Probability	I	II	III	Code	length
m_1	1/3	0	0	–	00	2
m_2	1/4	0	1	–	01	2
m_3	1/8	1	0	0	100	3
m_4	1/8	1	0	1	101	3
m_5	1/12	1	1	0	110	3
m_6	1/12	1	1	1	111	3

$R_c = 1 - \eta_c$

η_c, Effieceny $= \dfrac{H(x)}{L}$

$H(x) = \sum_{i=1}^{k} P_k \log_2 \frac{1}{P_k}$

$= -\sum_{i=1}^{k} P_k \log_2 P_k$

$= P(x_1) \log_2 P(x_1)$

$\bar{L} = \sum P(x_i) \cdot L$

$= \frac{1}{3} \times 2 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{8} \times 3 + \frac{1}{12} \times 3 + \frac{1}{12} \times 3$

$= 2.4$ / symbols.

$H(X) = P_k \log \frac{1}{P_k}$

$\cancel{H(x) = \sum_{i=1}^{n} P(x) \log \frac{1}{\log P(x)}}$

$H(X) = -\left[\frac{1}{3} \log_2 \left(\frac{1}{3}\right) + \frac{1}{4} \log_2 \left(\frac{1}{4}\right) + \frac{1}{8} \log_2 \left(\frac{1}{8}\right) + \frac{1}{8} \log_2 \left(\frac{1}{8}\right) + \frac{1}{12} \log_2 \left(\frac{1}{12}\right) + \frac{1}{12} \log_2 \left(\frac{1}{12}\right) \right]$

$= 2.3758$

$$\eta_c = \frac{2.3758}{2.41} = 0.985$$

$$R_c = 1 - 0.985 = 0.015.$$

$$\boxed{R_c = 0.015}$$

Q.4 Calculate $I(X;Y)$ given ~~P(X)~~

$P(X) = [\,1/4 \;\; 2/5 \;\; \frac{3}{20} \;\; 3/20 \;\; 1/20\,]$ Pg 126

and noise matrix

$$P(Y/X) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & \frac{3}{4} & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(2 marks) Q Calculate $H(X)$ for a discrete memoryless channel which has the probabilities $P(x_1) = 0.4$ $p(x_2) = 0.3$ $p(x_3) = 0.2$, & $P(x_4) = 0.4$.

$$H(X) = -\sum_{k=1}^{K} P_k \log_2 P_k$$

Q5 Calculate the capacity of Low pass Channel with Bandwidth of 3000 Hz. and S/N (signal to noise) ratio = 1000. at the channel o/p. Assume the channel to be gaussian and wide. pg 120

$C = W\log_2(1+\frac{S}{N})$

126 Q6 State and prove Kraft's Inequality. (Pg 32)

Q7 Derive the channel capacity of Band limited gaussian channel. Pg 116.

Q8 Explain in detail about decoding schemes and ideal observer. 110,

9 State and prove any three properties of mutual information. → Pg 88

Q10 Derive the capacity of unsymmetric channel Pg 107

$\frac{8}{11}$

$I(X;Y) = H(X) - H(X/Y)$

$H(Y) - H(Y/X)$ $\frac{2}{5} \times \frac{1}{4}$ $\frac{1}{10}$ $\frac{2}{4}, \frac{4}{10}$ $2, 5$

2, $P(X/Y)$ $H(X) + H(Y) - H(X,Y)$ $\frac{3}{10}$ $\frac{1}{4} + \frac{1}{10}$

$\frac{1}{4} + 1$

$P(X,Y) = P(X/Y) \cdot P(Y)$

$P(Y/X) \cdot P(X)$

$\frac{5+2}{20} =$ $\frac{20}{9} \times$

$\frac{2}{20} \times \frac{9}{7}$ $\frac{7}{20}$

$H \frac{1}{4}$ $\frac{5}{4}$

Capacity of unsymmetric Binary Channel.

Consider an unsymmetric binary channel

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Let $P(x_1) = p_1 \quad P(x_2) = P_2$ & $P(y_1) = P_1'$

$$P(y_2) = P_2'$$

then

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} P_{11}\log_2 P_{11} + P_{12}\log_2 P_{12} \\ P_{21}\log_2 P_{21} + P_{22}\log_2 P_{22} \end{bmatrix}$$

$$I(X,Y) = H(Y) - H(Y|X)$$

$$= -(P_1'\log_2 P_1' + P_2'\log_2 P_2') + P_1(P_{11}Q_1 + P_{12}Q_2) + P_2(P_{21}Q_1 + P_{22}Q_2)$$

$$= -(P_1'\log P_1' + P_2'\log P_2') + P_1'Q_1 + P_2'Q_2$$

where,

$$P_1' = P_1P_{11} + P_2P_{21}$$

$$P_2' = P_1P_{12} + P_2P_{22}$$

Apply Lagranges method

$$F = I(X,Y) + \lambda(P_1' + P_2') \quad -①$$

Pd ① w.r to P_1' & P_2'

$$\frac{\partial f}{\partial p_1'} = -(\log_2 e + \log_2 p_1') + Q_1 + \lambda = 0$$

$$\frac{\partial f}{\partial p_2'} = -(\log_2 e + \log_2 p_2') + Q_2 + \lambda = 0$$

$$\Rightarrow \lambda = -Q_1 + (\log_2 e + \log_2 p_1') = 0$$

$$\lambda = -Q_2 + (\log_2 e + \log_2 p_2') = 0$$

$$Q_1 = Q_2 + \log p_1' - \log p_2' \quad — (1).$$

$$I(X,Y) = -p_1' \cancel{\log p_1'} - p_2' \log p_2' + p_1' Q_2 + p_1' \cancel{\log p_1'} - p_1' \log p_2' + p_2' Q_2$$

$$= Q_2 - \log p_2'$$

$$= Q_1 - \log p_1'$$

also,

$$p_1' = \frac{e^{Q_1}}{e^c} \quad \& \quad p_2' = \frac{e^{Q_2}}{c}.$$

$$p_1' + p_2' \Rightarrow \frac{e^{Q_1} + e^{Q_2}}{e^c} = 1$$

$\Rightarrow$ Channel capacity, $C = \log_2 [2^{Q_1} + 2^{Q_2}]$

Capacity of unsymmetric Channel

Consider an unsymmetric binary channel

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and $P(x_1) = P_1$ & $P(x_2) = P_2$

$P(y_1) = P_1'$ & $P(y_2) = P_2'$

then
$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} P_{11}\log P_{11} + P_{12}\log P_{12} \\ P_{21}\log P_{21} + P_{22}\log P_{22} \end{bmatrix}$$

$$I(X;Y) = H(Y) - H(Y/X)$$

$$= -[P_1'\log P_1' + P_2'\log P_2'] + P_1(P_{11}Q_1 + P_{12}Q_2) + P_2(P_{21}Q_1 + P_{22}Q_2)$$

$$= -[P_1'\log P_1' + P_2'\log P_2'] + P_1'Q_1 + P_2'Q_2 \quad \text{--- (1)}$$

where,

$P_1' = P_1P_{11} + P_2P_{21}$

$P_2' = P_1P_{12} + P_2P_{22}$.

Now, Langranges multiplier method,

$F = I(X;Y) + \lambda(P_1' + P_2')$

P.d above eqⁿ w.r.to P_1' & P_2'

$$\frac{\partial F}{\partial p_1'} = -(\log_2 e + \log p_1') + Q_1 + \lambda = 0$$

$$\frac{\partial F}{\partial p_2'} = -(\log_2 e + \log p_2') + Q_2 + \lambda = 0$$

$$\Rightarrow \lambda = -Q_1 + (\log_2 e + \log p_1')$$

$$\lambda = -Q_2 + (\log_2 e + \log p_2')$$

equating both eq^ns we get

$$Q_1 = Q_2 + \log p_1' - \log p_2' \quad -②$$

Sub in ①

$$I(X;Y) = \cancel{-p_1'\log p_1'} - p_2'\log p_2' + p_1'Q_2 + p_1'\cancel{\log p_1'} - p_1'\log p_2' + p_2'Q_2$$

$$= Q_2 - \log p_2'$$

$$= Q_1 - \log p_1'$$

also,

$$p_1' = \frac{e^{Q_1}}{e^c} \quad \& \quad p_2' = \frac{e^{Q_2}}{e^c}$$

$$p_1' + p_2' = \frac{e^{Q_1} + e^{Q_2}}{e^c} = 1$$

$\Rightarrow$ Channel cap $\Rightarrow$ $\boxed{C = \log_2 (2^{Q_1} + 2^{Q_2})}$

11/8/11

UNIT 3

CHANNEL CODING

(Simon Haykin's)

INFORMATION CAPACITY THEOREM :-

Consider a zero mean stationary process $X(t)$, i.e band limited to 'B' hertz

X_k, where $k = 1, 2, \dots K$; denote continous random variable obtained by uniform sampling of the process, $X(t)$ at nyquist rate (i.e $2 \times$ Bandwidth) '2B' samples/sec; This samples are transmitted in 'T' seconds over noisychannel, also bandlimited to 'B' hertz. The no. of samples K is given by ~~K = 2BT~~

$$K = 2BT \quad \text{— (1) eq}^n.$$

Let X_k be the samples of transmitted signal,

$Y_k \rightarrow$ samples of Received signal

The channel o/p is pertubed (i.e some amount of noise is added) by white gaussian noise of zero mean and power spectral density "$\frac{N_0}{2}$".

$$\boxed{Y_k = X_k + N_k} \quad -② $$

where, N_k = white gaussian noise

Variance of N_k is given by:- $\sigma^2 = \frac{N_0}{2} \times 2B$

$$\boxed{\sigma^2 = N_0 B.} \quad -③ \qquad \Rightarrow \boxed{\sigma^2 = N_0 \cdot B}$$

The channel described by eqⁿ ② & ③, is called discrete time memoryless ~~the~~ gaussian channel.

- The Information Capacity is defined as maximum of mutual information between S_k and Y_k over all the distributions on S_k.

- Let $I(X_k ; Y_k)$ denoted the average mutual information between S_k and Y_k. thus the information capacity of the channel is given by

$$C = \max_{f_{X_k}(x)} \left\{ I(X_k ; Y_k) : E[X_k^2] = P \right\} \quad -④ \text{ eq}^n$$

where,

$f_{X_k}(x) \rightarrow$ Probability distribution function of $X(k)$

where,

$E[X_k^2]$ = Cost of informat^n ~~data~~ transmitted

P = transmitted power,

The avg mutual information $I(X_k; Y_k)$ is given by:-

$$I(X_k; Y_k) = h(Y_k) - h(Y_k/X_k) \quad (5)$$

now, $h(Y_k/X_k) = h(N_k) \quad (6)$

$$\therefore I(X_k; Y_k) = h(Y_k) - h(N_k) \quad (7)$$

Since $h(N_k)$ is independent of $X(k)$, maximising $I(X_k; Y_k)$ requires maximising of $h(Y_k)$

for $h(Y_k)$ to be maximum Y_k has to be a gaussian random variable, this implies that X_k also must be gaussian, Thus the maximizat^n in eq^n (4) is attained by choosing samples of transmitted signal from a noise like process of average power P.

⇒ To evaluate the information capacity, we proceed in three stages :-

1) $$h(Y_k) = \frac{1}{2}\log_2\left[2\pi e\,(P+\sigma^2)\right] \quad \text{—(8)}$$

2) $$h(N_k) = \frac{1}{2}\log_2\left[2\pi e\,\sigma^2\right] \quad \text{—(9)}$$

3) Channel capacity, $$\boxed{C = \frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \text{ bits/transm}^n} \quad \text{—(10)}$$

$$C = \cancel{2}B \times \frac{1}{\cancel{2}}\log_2\left(1+\frac{P}{N_0 B}\right) \qquad \because \sigma^2 = N_0 \cdot B$$

$$\boxed{C = B\log_2\left(1+\frac{P}{N_0 B}\right) \text{ bits/sec}} \quad \text{—(11)}$$

17-8-11

IMPLICATIONS OF INFORMATION CAPACITY THEOREM

⇒ An ideal system is defined as one that transmits data at a bit rate R_b equal to the information capacity C.

i.e, $\boxed{R_b = C}$

⇒ Average Transmitted power is given by

$$P = E_b\, C \quad -①$$

where, E_b = Transmitted ~~signal~~ energy / bit.

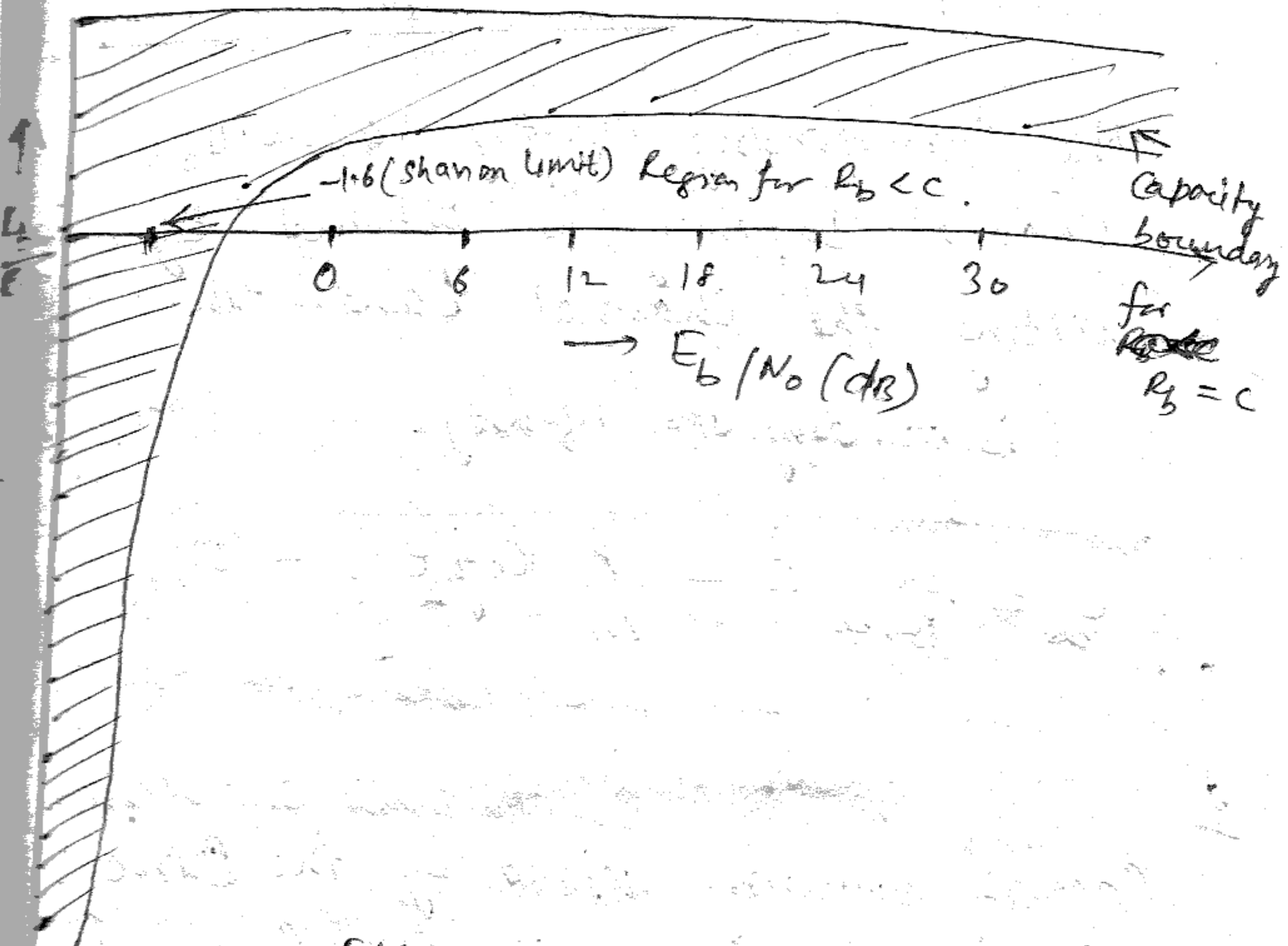
⇒ Ideal system is given by:-

$$\frac{C}{B} = \log_2\left(1 + \frac{E_b}{N_0}\frac{C}{B}\right) \quad -②$$

(above eqn comes from channel capacity eqn of eqn (11)).

⇒ The signal energy / bit to noise power spectral density ratio $\frac{E_b}{N_0}$ in terms of bandwidth efficiency, $\frac{C}{B}$ for ideal system is given by

$$\frac{E_b}{N_0} = \frac{2^{C/B} - 1}{C/B}$$



Bandwidth Efficiency diagram

The plot of R_B/B Vs $\frac{E_b}{N_o}$ graph is called bandwidth efficiency diagram.

→ From the above graph we make the following observations:-

i) for infinite Bandwidth $\frac{E_b}{N_o}$ approaches limiting value i.e

$$\boxed{\left(\frac{E_b}{N_o}\right)_{\infty} = \lim_{B \to \infty} \left(\frac{E_b}{N_o}\right) = \ln 2 = 0.693}$$

This is Shanon's limit

→ In decidel, the value of $\frac{E_b}{N_0}$ is -1.6

The Corresponding limiting value of Channel capacity is obtained by letting the channel bandwidth B tending to infinity.

$$\boxed{C_\infty = \lim_{B \to \infty} C = \frac{P}{N_0} \log_2 e} \quad -(5)$$

(2)

Capacity Boundary defined by the Curve for $R_B = C$, seperate the Combination of System parameter that supports error free transmission, $R_B < C$ i.e from those for which the error free transmission is not possible ($R_B > C$).

$R_B = C$ Ideal system.
$R_B < C$ Error free transmission.
$R_B > C$ transmission with error.

(3) The diagram highlights the potential trade off among $\frac{E_b}{N_0}$, $\frac{R_B}{B}$ and the probability of error, P_e

The Shannon limit exhibited in terms of $\frac{E_b}{N_0}$ required by the ideal system for error free transmission to be possible, is given by :-

$$P_e = \begin{cases} 0, & E_b/N_0 \geq \ln 2 \\ 1, & E_b/N_0 < \ln 2 \end{cases}$$

[5] The Boundary between error free transmission and unreliable transmission with possible errors, defined by Shanon limit Corresponds to the Capacity Boundary.

9/8/11

24/8/4.

Rate Distortion Theory :-

In practical situations there are constraints that force coding to be imperfect, there by resulting in distortion. for ex:- the informat source when have continous amplitude & the requirement is quantised the amplitude of each sample to permit its representation by finite code word length.

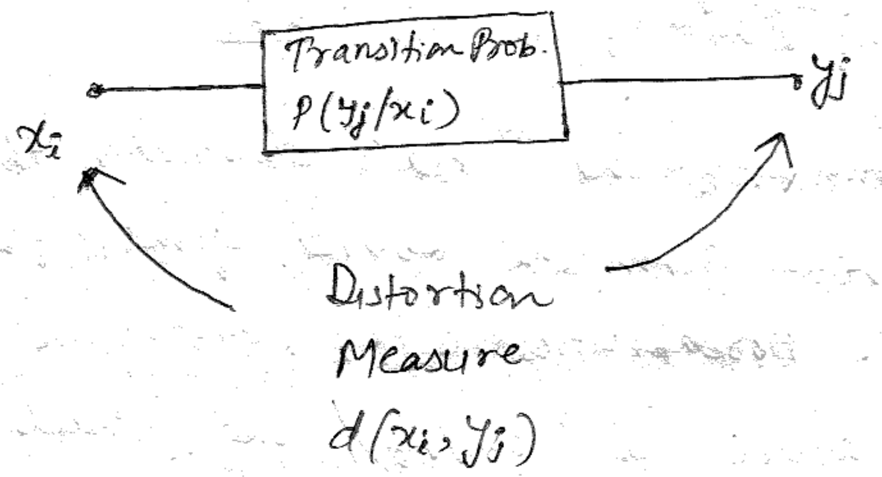
In such cases, the problem is called Source Coding. If fidelity criterion & the branch of information theory that deals with ~~the~~ it, is called ~~weight~~ rate distortion theory.

It has applications in two types of situation:-

(i) Source coding where the permited coding alphabet doesn't exactly represent the information source, & hence we are forced to do lossy data comprehension.

(ii) Information transmission at a rate greater than Channel Capacity.

Rate Distortion Function



→ Rate distortion function $R(D)$ is defined as the smallest coding rate, possible for which the average distortion is guaranteed not to exceed D (distortion).

for a fixed D,

$$R(D) = \min_{\cancel{P(x_i/y_j) \in P_D} \; P(y_i/x_i) \in P_D} I(X;Y)$$

where

$P_D \rightarrow$ set to which $P(y_i/x_i)$ belongs to ~~pre~~ prescribed D.

Subject to the Constraint :-

$$\sum_{i=1}^{N} P(y_j / x_i) = 1 , \quad \text{for } i = 1, 2 \ldots N.$$

$$\boxed{R_D \propto \frac{1}{D}}$$

Unit of Rate of D is : <u>bits</u>.

⇒ To summarize the rate distortion theory,
" ~~Even~~ given the source Symbol $\{x_i\}$ their probabilities $\{P_i\}$, distortion measure $d(x_i, y_j)$, the Calculation of $R(D)$ involves Conditional probability assignment $P(y_j / x_i)$ Subject to the Constraints imposed on $P(y_j / x_i)$ "

DATA COMPRESSION.

→ Data Compression is a lossy operation in which the source entropy is reduced (information is lost). ~~with re~~ irrespective of type of source.

→ In Case of Continuous source, the entropy is infinite & hence the ~~single~~ signal compression code has to be encode the source o/p at a finite rate, but it is impossible to encode an analog signal without producing distortion. Thus quantizer is used as compresser.

→ There are ~~t~~ two types of quantizer:-

(i) Scalar (ii) Vector.

→ Scalar quantizers deals with analog signal one at a time. Each sample is converted into quantised value.

→ Vector Quantizer use blocks of ~~the~~ consecutive samples to form vector, each of which is treated as single entity.

→ The Vector is encoded by comparing it with a codebook, which consists of set of stored ~~p~~reference vector called Code vector.

Let $N \rightarrow$ no. of Code Vectors in the Code book

$k \rightarrow$ Dimension of each vector.

$r \rightarrow$ Code Tx^n rate in bits/Samp

$$\boxed{r = \frac{\log_2 N}{k}}$$

Signal to Noise ratio (SNR) for

~~Single~~ Vector quantizer is

$$10 \log_{10} (SNR) = 6\left(\frac{\log_2 N}{k}\right) + C_k \text{ (dB)}$$

Where

$C_k \rightarrow$ Constant

→ Note:- Advantage of vector quantizer over Scalar is that C_k (Constant) has higher value

J.S.
28/08/11