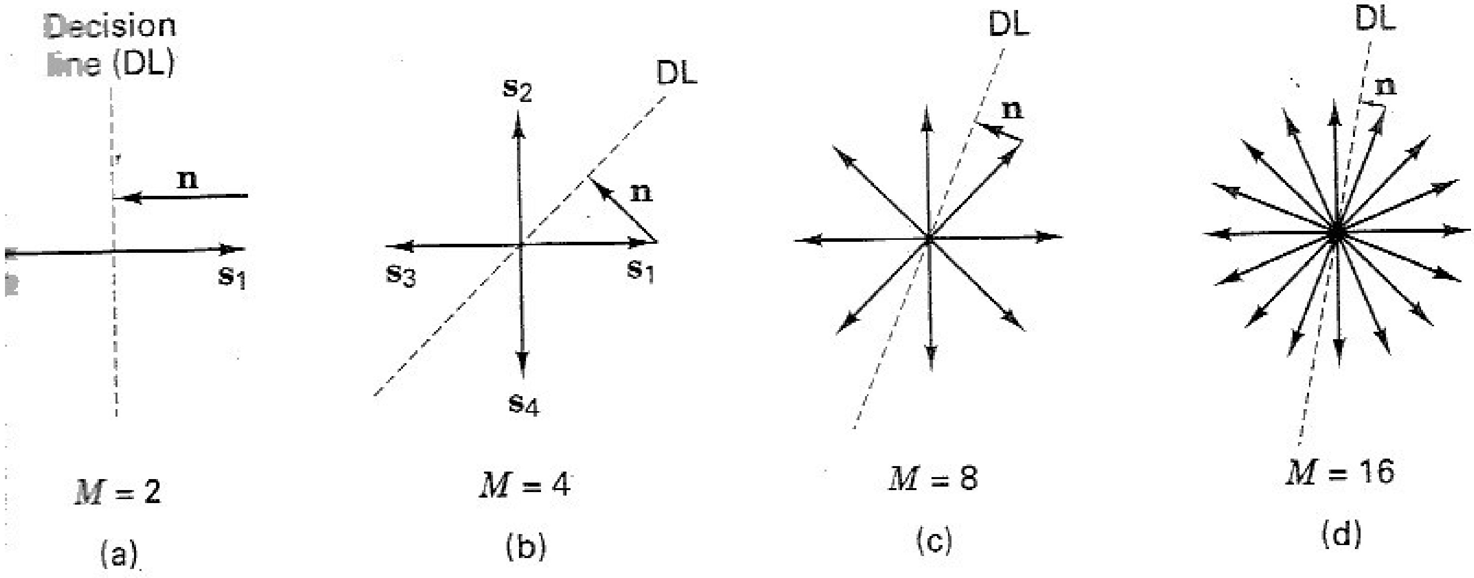

Figure 4.30 MPSK signal sets for $M = 2, 4, 8, 16$.

off in multiple phase signaling. Crowding more signal vectors into the signal space is tantamount to increasing the data rate without increasing the system bandwidth (the vectors are all confined to the same plane). In other words, we have increased the bandwidth utilization at the expense of error performance. Look at Figure 4.30d, where the error performance is worse than any of the other examples in Figure 4.30. How might we "buy back" the degraded error performance? In other words, what can we trade-off so that the distance between neighboring signal vectors in Figure 4.30d is increased to that in Figure 4.30a? We can increase the signal strength (make the signal vectors larger) until the minimum distance from the head of a signal vector to a decision line equals the length of the noise vector in Figure 4.30a. Therefore, in a multiple phase system, as M is increased, we can either achieve improved bandwidth performance at the expense of error performance, or if we increase the E_b/N_0 so that the error probability is not degraded, we can achieve improved bandwidth performance at the expense of increasing E_b/N_0.

Note that Figure 4.30 has been sketched so that all phasors have the same length for any of the M-ary cases. This is tantamount to saying that the comparisons are being considered for a fixed E_s/N_0, where E_s is symbol energy. The figure can also be drawn for a fixed E_b/N_0, in which case the phasor magnitudes would increase with increasing M. The phasors for $M = 4$, 8, and 16 would then have lengths greater than the $M = 2$ case by the factors $\sqrt{2}$, $\sqrt{3}$, and 2 respectively. We would still see crowding and increased vulnerability to noise, with increasing M, but the appearance would not be as pronounced as it is in Figure 4.30.

4.8.4 BPSK and QPSK Have the Same Bit Error Probability

In Equation (3.30) we stated the general relationship between E_b/N_0 and S/N which is rewritten

$$\frac{E_b}{N_0} = \frac{S}{N}\left(\frac{W}{R}\right) \tag{4.101}$$

where S is the average signal power and R is the bit rate. A BPSK signal with the available E_b/N_0 found from Equation (4.101) will perform with a P_B that can be read from the $k = 1$ curve in Figure 4.29. QPSK can be characterized as two orthogonal BPSK channels. The QPSK bit stream is usually partitioned into an even and odd (I and Q) stream; each new stream modulates an orthogonal component of the carrier at half the bit rate of the original stream. The I stream modulates the $\cos \omega_0 t$ term and the Q stream modulates the $\sin \omega_0 t$ term. If the magnitude of the original QPSK vector has the value A, the magnitude of the I and Q component vectors each has a value of $A/\sqrt{2}$, as shown in Figure 4.31. Thus, each of the quadrature BPSK signals has half of the average power of the original QPSK signal. Hence if the original QPSK waveform has a bit rate of R bits/s and an average power of S watts, the quadrature partitioning results in each of the BPSK waveforms having a bit rate of $R/2$ bits/s and an average power of $S/2$ watts.

Therefore, the E_b/N_0 characterizing each of the orthogonal BPSK channels, making up the QPSK signal, is equivalent to the E_b/N_0 in Equation (4.101), since it can be written as

$$\frac{E_b}{N_0} = \frac{S/2}{N_0}\left(\frac{W}{R/2}\right) = \frac{S}{N_0}\left(\frac{W}{R}\right) \tag{4.102}$$

Thus each of the orthogonal BPSK channels, and hence the composite QPSK signal, is characterized by the same E_b/N_0 and hence the same P_B performance as a BPSK signal. The natural orthogonality of the 90° phase shifts between adjacent QPSK symbols results in the *bit error probabilities* being equal for both BPSK and QPSK signaling. It is important to note that the symbol *error probabilities* are *not* equal for BPSK and QPSK signaling. The relationship between bit error probability and symbol error probability is treated in Sections 4.9.3 and 4.9.4. We see that, in effect, QPSK is the equivalent of two BPSK channels in quadrature. This same idea can be extended to any symmetrical M-ary amplitude/phase signaling, such as quadrature amplitude modulation (QAM) described in Section 9.8.3.

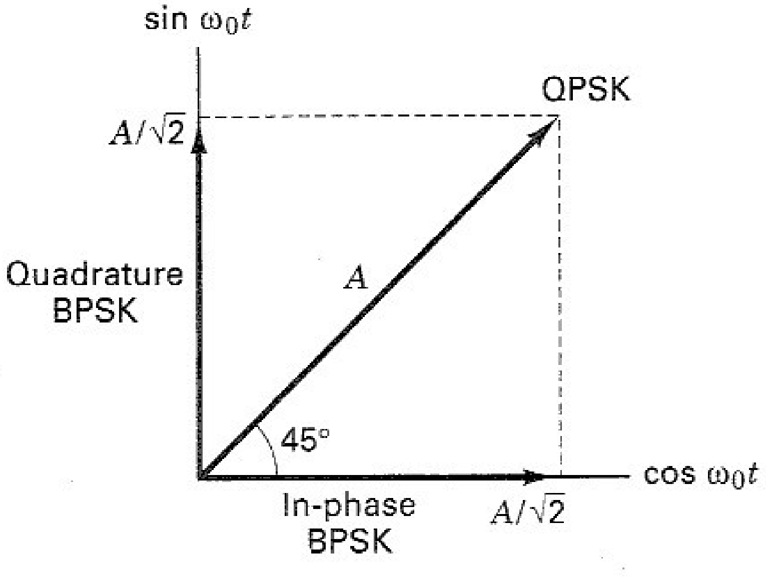

Figure 4.31 In-phase and quadrature BPSK components of QPSK signaling.

4.8.5 Vectorial View of MFSK Signaling

In Section 4.8.3, Figure 4.30 provides some insight as to why the error performance of MPSK signaling degrades as k (or M) increases. It would be useful to have a similar vectorial illustration for the error performance of orthogonal MFSK signaling as seen in the curves of Figure (4.28). Since the MFSK signal space is characterized by M mutually perpendicular axes, we can only conveniently illustrate the cases $M = 2$ and $M = 3$. In Figure 4.32a we see the binary orthogonal vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ positioned 90° apart. The decision boundary is drawn so as to partition the signal space into two regions. On the figure is also shown a noise vector $\mathbf{n}$, which represents the minimum noise vector that would cause the detector to make an error.

In Figure 4.32b we see a 3-ary signal space with axes positioned 90° apart. Here decision planes partition the signal space into three regions. Noise vectors $\mathbf{n}$ are shown added to each of the prototype signal vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$; each noise vector illustrates an example of the minimum noise vector that would cause the detector to make a symbol error. The minimum noise vectors in Figure 4.32b are the same length as the noise vector in Figure 4.32a. In Section 4.4.4 we stated that for a given level of received energy, the distance between any two prototype signal vectors $\mathbf{s}_i$ and $\mathbf{s}_j$ in an M-ary orthogonal space is constant. It follows that the minimum distance between a prototype signal vector and any of the decision boundaries remains fixed as M increases. Unlike the case of MPSK signaling, where adding new signals to the signal set makes the signals vulnerable to smaller noise vectors, here, in the case of MFSK signaling, adding new signals to the signal set does *not* make the signals vulnerable to smaller noise vectors.

It would be convenient to illustrate the point by drawing higher dimensional orthogonal spaces, but of course this is not possible. We can only use our "mind's eye" to understand that increasing the signal set M—by adding additional axes, where each new axis is mutually perpendicular to all the others—does not crowd

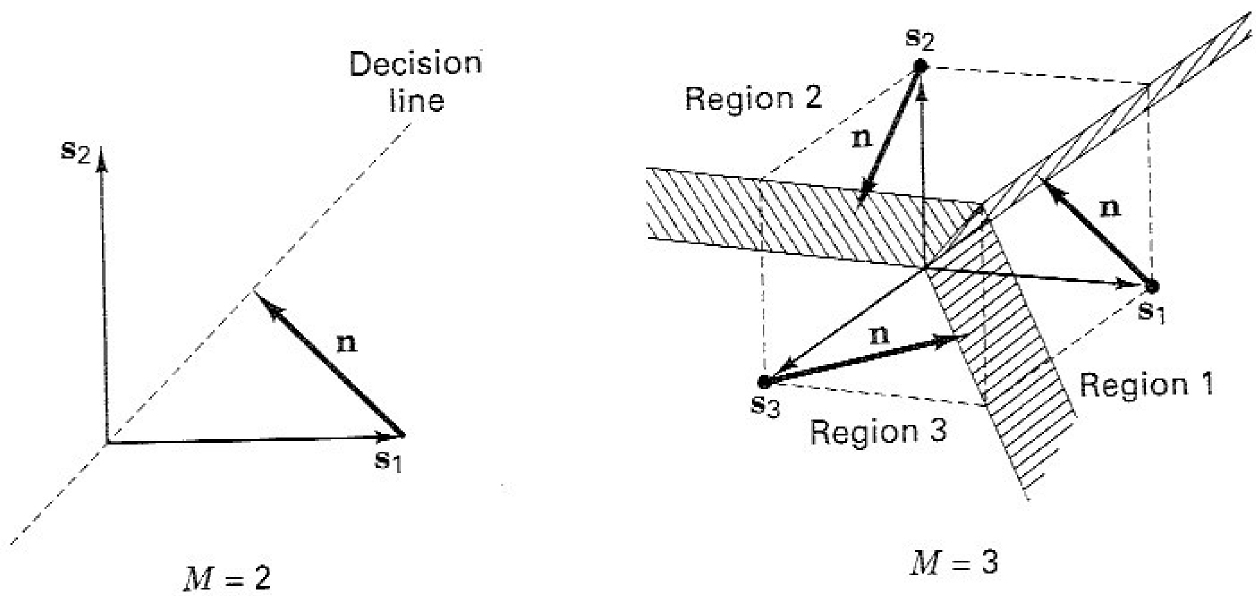

Figure 4.32 MFSK signal sets for $M = 2, 3$.

the signal set more closely together. Thus, a transmitted signal from an orthogonal set is *not* more vulnerable to a noise vector when the set is increased in size. In fact, we see from Figure 4.28 that as k increases, the bit error performance improves.

Understanding the error performance improvement of orthogonal signaling, as illustrated in Figure 4.28, is facilitated by comparing the probability of symbol error (P_E) versus unnormalized SNR, with P_E versus E_b/N_0. Figure 4.33 represents a set of P_E performance curves plotted against unnormalized SNR for coherent FSK signaling. Here we see that P_E degrades as M is increased. Didn't we say that a signal from an orthogonal set is *not* made more vulnerable to a given noise vector, as the orthogonal set is increased in size? It is correct that for orthogonal signaling, with a given SNR it takes a fixed size noise vector to perturb a transmitted signal into an error region; the signals do not become vulnerable to smaller noise vectors as M increases. However, as M increases, more neighboring decision regions are introduced; thus the number of ways in which a symbol error can be made increases. Figure 4.33 reflects the degradation in P_E versus unnormalized SNR as M is increased; there are $(M - 1)$ ways to make an error. Examining performance under the condition of a fixed SNR (as M increases) is not very useful for digital communications. A fixed SNR means a fixed amount of energy per symbol; thus as M increases, there is a fixed amount of energy to be apportioned over a larger number of bits, or there is less energy per bit. The most useful way of comparing one digital system with another is on the basis of *bit-normalized SNR* or E_b/N_0. The error performance improvement with increasing M (see Figure 4.28) manifests itself only when error probability is plotted against E_b/N_0. For this case, as M increases, the required E_b/N_0 (to meet a given error probability) is reduced for a fixed SNR; therefore, we need to map the plot shown in Figure 4.33 into a new plot, similar to that shown in Figure 4.28, where the abscissa represents E_b/N_0 instead of SNR. Figure 4.34 illustrates such a mapping; it demonstrates that curves manifesting degraded P_E with increasing M (such as Figure 4.33) are transformed into curves manifesting improved P_E with increasing M. The basic mapping relationship is expressed in Equation (4.101), repeated here as

$$\frac{E_b}{N_0} = \frac{S}{N}\left(\frac{W}{R}\right)$$

where W is the detection bandwidth. Since

$$R = \frac{\log_2 M}{T} = \frac{k}{T}$$

where T is the symbol duration, we can then write

$$\frac{E_b}{N_0} = \frac{S}{N}\left(\frac{WT}{\log_2 M}\right) = \frac{S}{N}\left(\frac{WT}{k}\right) \tag{4.103}$$

For FSK signaling, the detection bandwidth W (in hertz) is typically equal in value to the symbol rate $1/T$; in other words, $WT \approx 1$. Therefore,

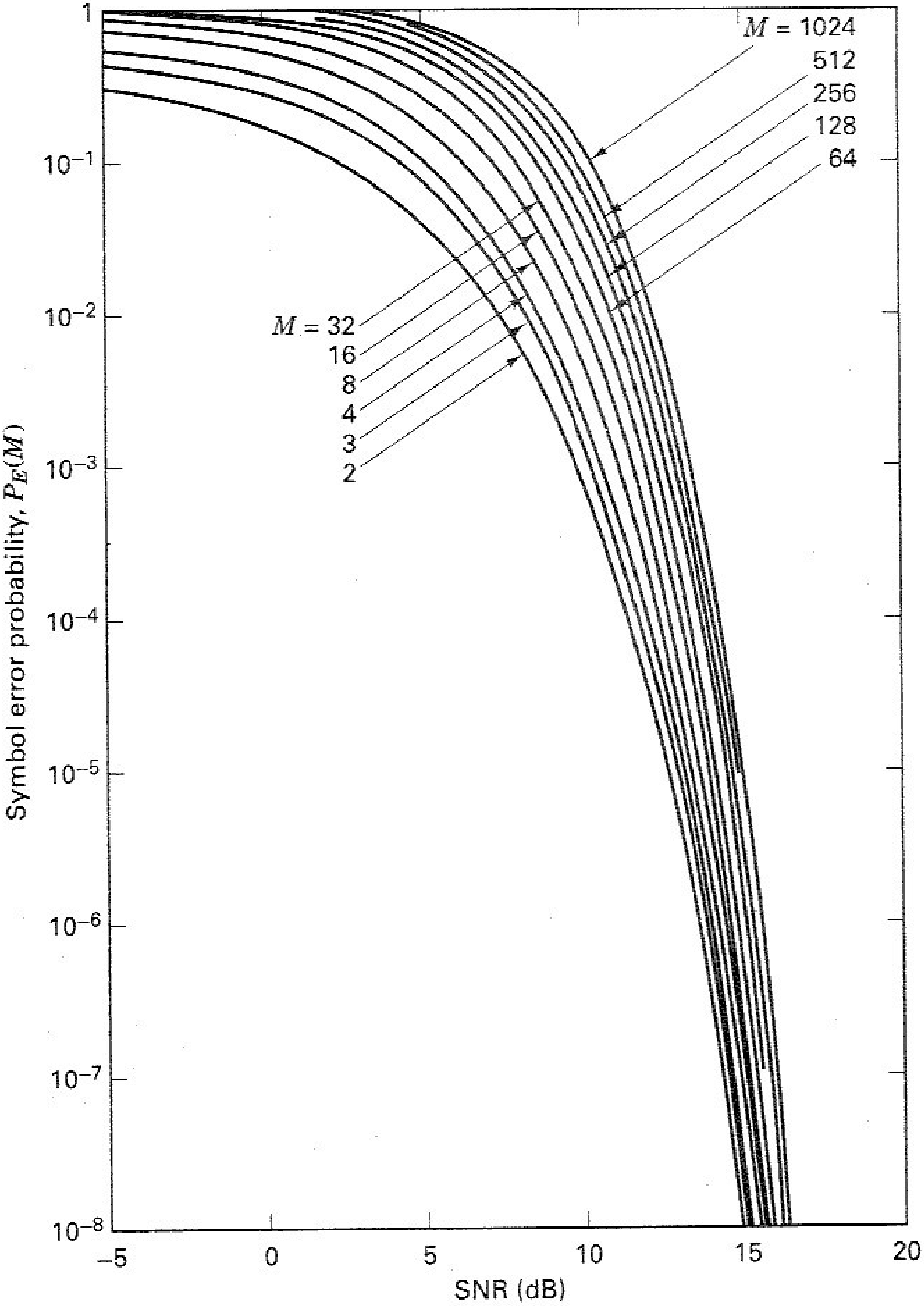

Figure 4.33 Symbol error probability versus SNR for coherent FSK signaling. (From Bureau of Standards, *Technical Note 167,* March 1963.) (Reprinted from *Central Radio Propagation Laboratory Technical Note 167,* March 25, 1963, Fig. 1, p. 5, courtesy of National Bureau of Standards.)

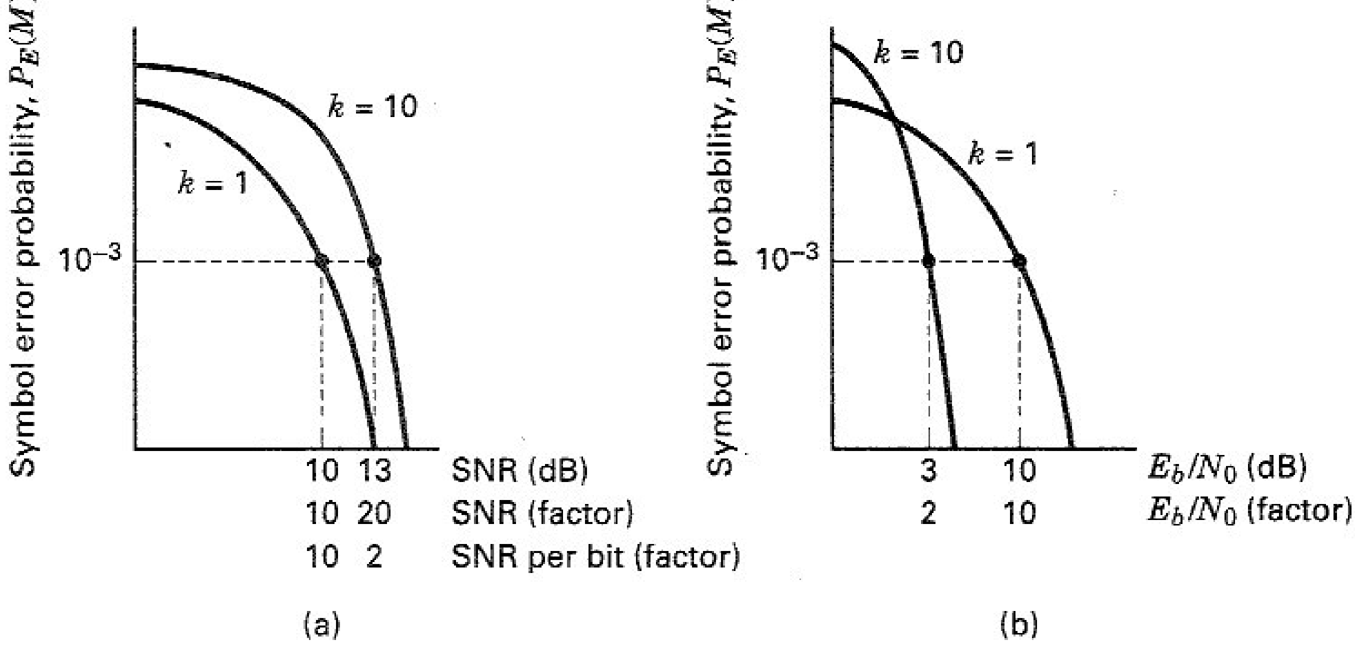

Figure 4.34 Mapping P_E versus SNR into P_E versus E_b/N_0 for orthogonal signaling. (a) Unnormalized. (b) Normalized.

$$\frac{E_b}{N_0} \approx \frac{S}{N}\left(\frac{1}{k}\right) \tag{4.104}$$

Figure 4.34 illustrates the mapping from P_E versus SNR to P_E versus E_b/N_0 for coherently detected M-ary orthogonal signaling, with "ballpark" numbers on the axes. In Figure 4.34a, on the $k = 1$ curve is shown an operating point corresponding to $P_E = 10^{-3}$ and SNR = 10 dB. On the $k = 10$ curve is shown an operating point at the same $P_E = 10^{-3}$ but with SNR = 13 dB (approximate values taken from Figure 4.33). Here we clearly see the degradation in error performance as k increases. To appreciate where the performance improvement comes from, let us convert the abscissa from the nonlinear scale of SNR in decibels to a linear one—SNR expressed as a factor. This is shown in Figure 4.34a as the factors 10 and 20 for the $k = 1$ and $k = 10$ cases, respectively. Next, we further convert the abscissa scale to SNR per bit (expressed as a factor). This is shown in Figure 4.34a as the factors 10 and 2 for the $k = 1$ and $k = 10$ cases, respectively. It is convenient to think of the 1024-ary symbol or waveform ($k = 10$ case) as being interchangeable with its 10-bit meaning. Thus, if the symbol requires 20 units of SNR then the 10 bits belonging to that symbol require that same 20 units; or, in other words, each bit requires 2 units.

Rather than performing such computations, we can simply map these same $k = 1$ and $k = 10$ cases onto the Figure 4.34b plane, representing P_E versus E_b/N_0. The $k = 1$ case looks exactly the same as it does in Figure 4.34a. But for the $k = 10$ case, there is a dramatic change. We can immediately see that signaling with the $k =$ 10-bit symbol requires only 2 units (3 dB) of E_b/N_0 compared with 10 units (10 dB) for the binary symbol. The mapping that gives rise to the required E_b/N_0 for the $k =$ 10 case is obtained from Equation (4.104) as follows: $E_b/N_0 = 20\ (1/10) = 2$ (or 3-dB), which shows the error performance improvement as k is increased. In digital communication systems, error performance is almost always considered in terms of

E_b/N_0, since such a measurement makes for a meaningful comparison between one system's performance and another. Therefore, the curves shown in Figures 4.33 and 4.34a are hardly ever seen.

Although Figure 4.33 is not often seen, we can still use it for gaining insight into why orthogonal signaling provides improved error performance as M or k increases. Let us consider the analogy of purchasing a commodity—say, grade A cottage cheese. The choice of the grade corresponds to some point on the P_E axis of Figure 4.33—say, 10^{-3}. From this point, construct a horizontal line through all of the curves (from $M = 2$ through $M = 1024$). At the grocery store we buy the very smallest container of cottage cheese, containing 2 ounces and costing \$1. On Figure 4.33 we can say that this purchase corresponds to our horizontal construct intercepting the $M = 2$ curve. We look down at the corresponding SNR and call the intercept on this axis our cost of \$1. The next time we purchase cottage cheese, we remember that the first purchase seemed expensive at 50 cents an ounce. So, we decide to buy a larger carton, containing 8 ounces and costing \$2. On Figure 4.33, we can say that this purchase corresponds to the point at which our horizontal construct intercepts the $M = 8$ curve. We look down at the corresponding SNR, and call this intercept our cost of \$2. Notice that we bought a larger container so the price went up, but because we bought a greater quantity, the price per ounce went down (the unit cost is now only 25 cents per ounce). We can continue this analogy by purchasing larger and larger containers so that the price of the container (SNR) keeps going up, but the price per ounce keeps going down. This is the age-old story called the *economy of scale*. Buying larger quantities at a time is commensurate with purchasing at the wholesale level; it makes for a lower unit price. Similarly, when we use orthogonal signaling with symbols that contain more bits, we need more power (more SNR), but the requirement per bit (E_b/N_0) is reduced.

4.9 SYMBOL ERROR PERFORMANCE FOR *M*-ARY SYSTEMS ($M > 2$)

4.9.1 Probability of Symbol Error for MPSK

For large energy-to-noise ratios, the symbol error performance $P_E(M)$, for equally likely, coherently detected M-ary PSK signaling, can be expressed [7] as

$$P_E(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right) \tag{4.105}$$

where $P_E(M)$ is the probability of symbol error, $E_s = E_b(\log_2 M)$ is the energy per symbol, and $M = 2^k$ is the size of the symbol set. The $P_E(M)$ performance curves for coherently detected MPSK signaling are plotted versus E_b/N_0 in Figure 4.35.

The symbol error performance for differentially coherent detection of M-ary DPSK (for large E_s/N_0) is similarly expressed [7] as

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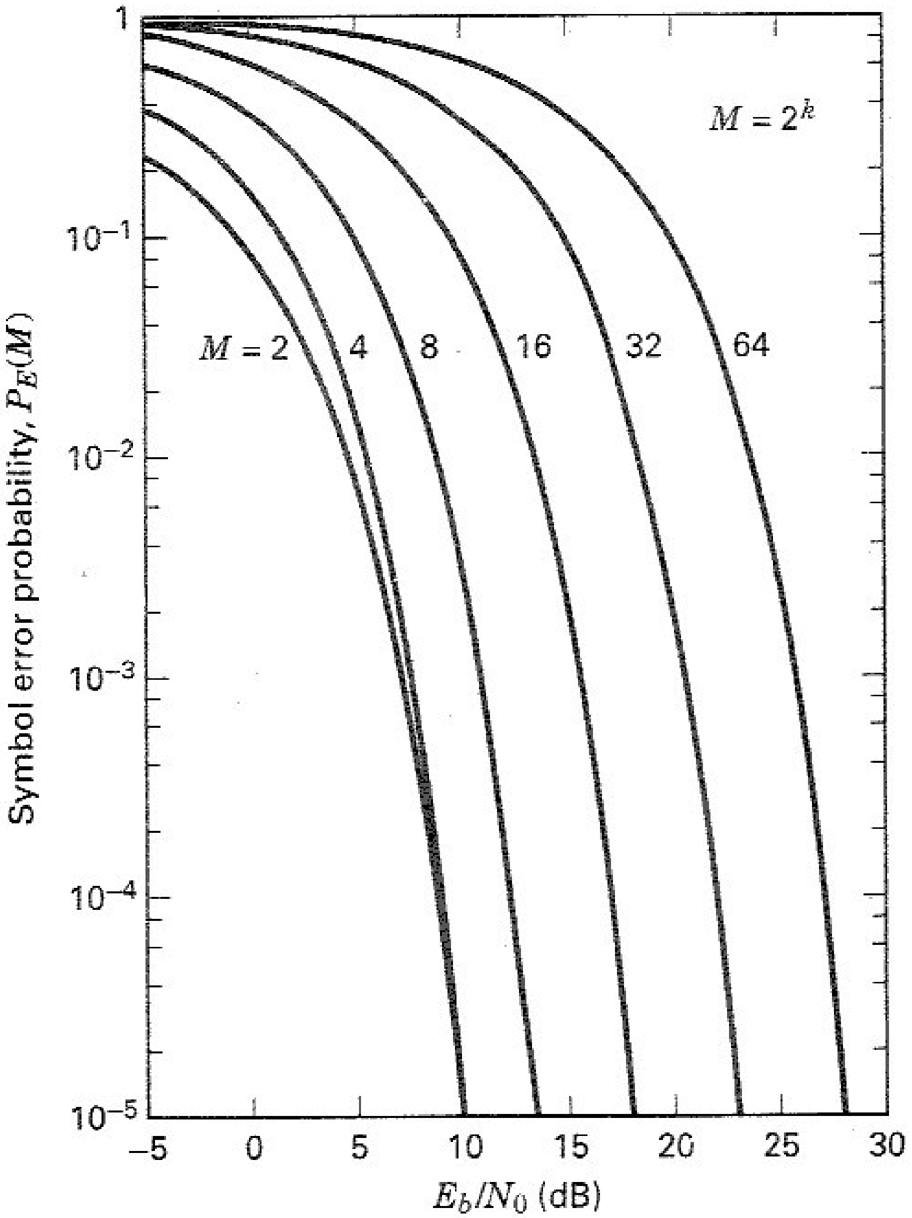

Figure 4.35 Symbol error probability for coherently detected multiple phase signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering,* Prentice-Hall, Inc. Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

$$P_E(M) \approx 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{\sqrt{2}M}\right) \tag{4.106}$$

4.9.2 Probability of Symbol Error for MFSK

The symbol error performance $P_E(M)$, for equally likely, *coherently* detected M-ary orthogonal signaling can be upper bounded [5] as

$$P_E(M) \leq (M-1)\, Q\left(\sqrt{\frac{E_s}{N_0}}\right) \tag{4.107}$$

where $E_s = E_b(\log_2 M)$ is the energy per symbol and M is the size of the symbol set. The $P_E(M)$ performance curves for coherently detected M-ary orthogonal signaling are plotted versus E_b/N_0 in Figure 4.36.

The symbol error performance for equally likely, *noncoherently* detected M-ary orthogonal signaling is [9]

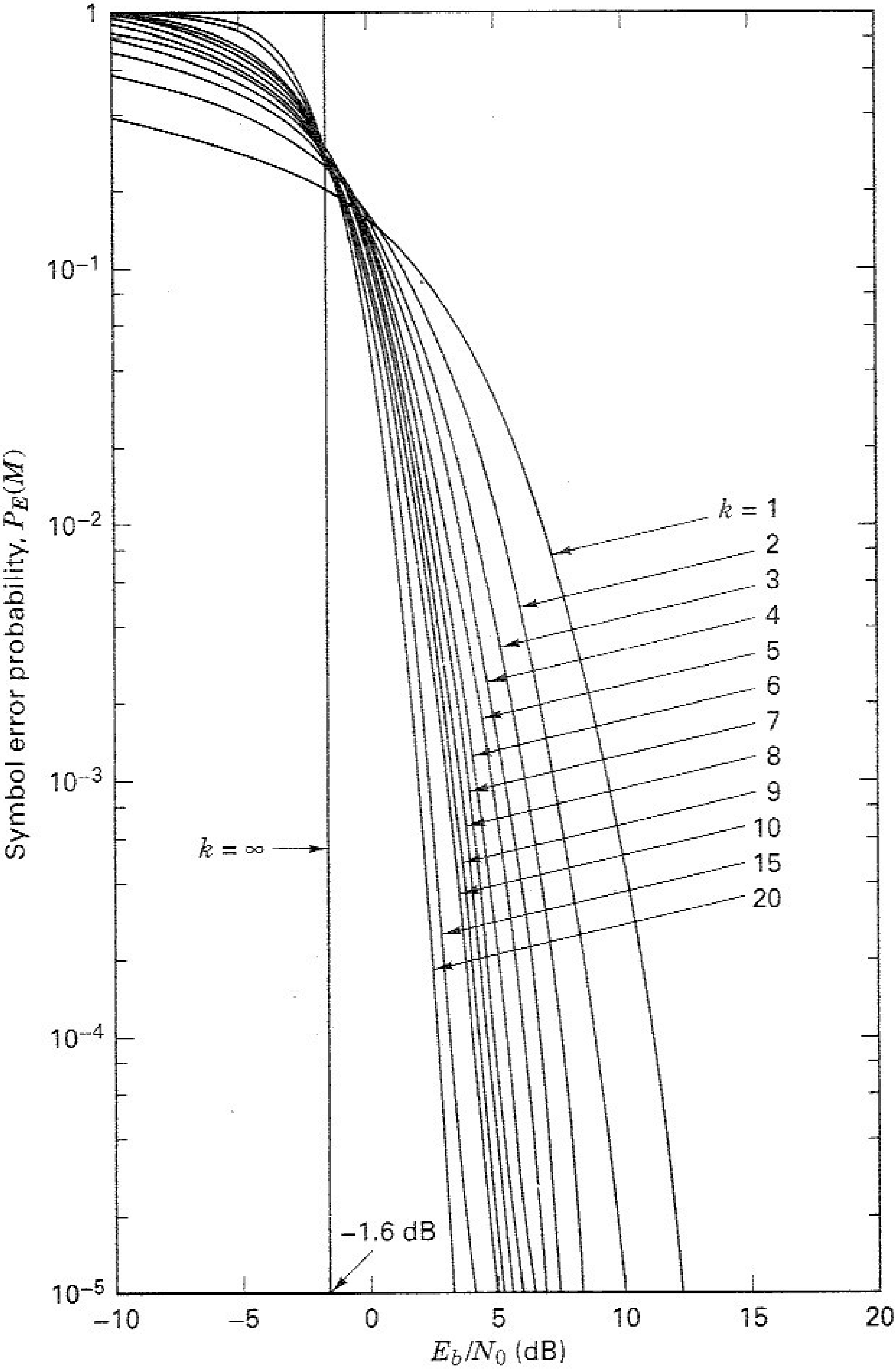

Figure 4.36 Symbol error probability for coherently detected *M*-ary orthogonal signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering,* Prentice-Hall, Inc., Englewod Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

$$P_E(M) = \frac{1}{M}\exp\left(-\frac{E_s}{N_0}\right)\sum_{j=2}^{M}(-1)^j\binom{M}{j}\exp\left(\frac{E_s}{jN_0}\right) \tag{4.108}$$

where

$$\binom{M}{j} = \frac{M!}{j!\,(M-j)!} \tag{4.109}$$

is the standard binomial coefficient yielding the number of ways in which j symbols out of M may be in error. Note that for binary case, Equation (4.108) reduces to

$$P_B = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right) \tag{4.110}$$

which is the same result as that described by Equation (4.96). The $P_E(M)$ performance curves for noncoherently detected M-ary orthogonal signaling are plotted versus E_b/N_0 in Figure 4.37. If we compare this noncoherent orthogonal $P_E(M)$ performance with the corresponding $P_E(M)$ results for the coherent detection of orthogonal signals in Figure 4.36, it can be seen that for $k > 7$, there is a negligible difference. An upper bound for coherent as well as noncoherent reception of orthogonal signals is [9]

$$P_E(M) < \frac{M-1}{2}\exp\left(-\frac{E_s}{2N_0}\right) \tag{4.111}$$

where E_s is the energy per symbol and M is the size of the symbol set.

4.9.3 Bit Error Probability versus Symbol Error Probability for Orthogonal Signals

It can be shown [9] that the relationship between probability of bit error (P_B) and probability of symbol error (P_E) for an M-ary orthogonal signal set is

$$\frac{P_B}{P_E} = \frac{2^{k-1}}{2^k - 1} = \frac{M/2}{M-1} \tag{4.112}$$

In the limit as k increases, we get

$$\lim_{k\to\infty}\frac{P_B}{P_E} = \frac{1}{2}$$

A simple example will make Equation (4.112) intuitively acceptable. Figure 4.38 describes an octal message set. The message symbols (assumed equally likely) are to be transmitted on orthogonal waveforms such as FSK. With orthogonal signaling, a decision error will transform the correct signal into any one of the $(M-1)$ incorrect signals with equal probability. The example in Figure 4.38 indicates that the symbol comprising bits 0 1 1 was transmitted. An error might occur in any one of the other $2^k - 1 = 7$ symbols, with equal probability. Notice that just because a symbol error is made does not mean that all the bits within the symbol will be in error. In Figure 4.38, if the receiver decides that the transmitted symbol is the bot-

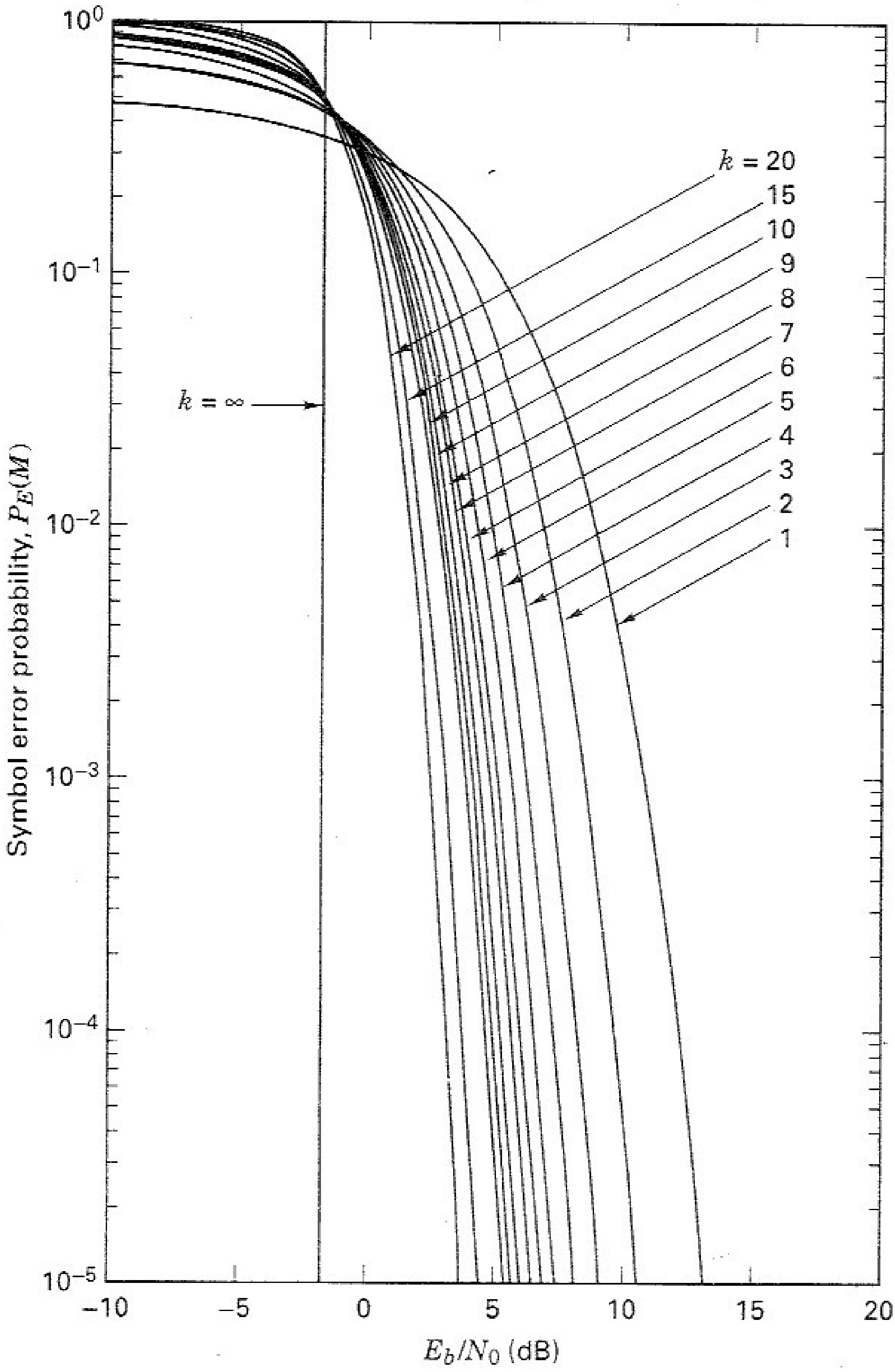

Figure 4.37 Symbol error probability for noncoherently detected *M*-ary orthogonal signaling. (Reprinted from W. C. Lindsey and M. K. Simon, *Telecommunication Systems Engineering,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1973, courtesy of W. C. Lindsey and Marvin K. Simon.)

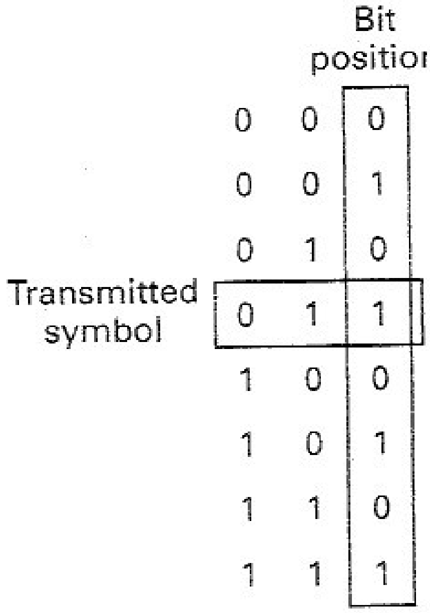

Figure 4.38 Example of P_B versus P_E.

tom one listed, comprising bits 1 1 1, two of the three transmitted symbol bits will be correct; only one bit will be in error. It should be apparent that for nonbinary signaling, P_B will always be less than P_E (keep in mind that P_B and P_E reflect the frequency of making errors on the *average*.)

Consider any of the bit-position columns in Figure 4.38. For each bit position, the digit occupancy consists of 50% ones and 50% zeros. In the context of the first bit position (rightmost column) and the transmitted symbol, how many ways are there to cause an error to the binary one? There are $2^{k-1} = 4$ ways (four places where zeros appear in the column) that a bit error can be made; it is the same for each of the columns. The final relationship P_B/P_E, for orthogonal signaling, in Equation (4.112), is obtained by forming the following ratio: the number of ways that a bit error can be made (2^{k-1}) divided by the number of ways that a symbol error can be made ($2^k - 1$). For the Figure 4.38 example, $P_B/P_E = 4/7$.

4.9.4 Bit Error Probability Versus Symbol Error Probability for Multiple Phase Signaling

For the case of MPSK signaling, P_B is less than or equal to P_E, just as in the case of MFSK signaling. However, there is an important difference. For orthogonal signaling, selecting any one of the $(M-1)$ erroneous symbols is equally likely. In the case of MPSK signaling, each signal vector is not equidistant from all of the others. Figure 4.39a illustrates an 8-ary decision space with the pie-shaped decision regions denoted by the 8-ary symbols in binary notation. If symbol (0 1 1) is transmitted, it is clear that should an error occur, the transmitted signal will most likely be mistaken for one of its closest neighbors, (0 1 0) or (1 0 0). The likelihood that (0 1 1) would get mistaken for (1 1 1) is relatively remote. If the assignment of bits to symbols follows the binary sequence shown in the symbol decision regions of Figure 4.39a, some symbol errors will usually result in two or more bit errors, even with a large signal-to-noise ratio.

For nonorthogonal schemes, such as MPSK signaling, one often uses a binary-to-M-ary code such that binary sequences corresponding to adjacent sym-

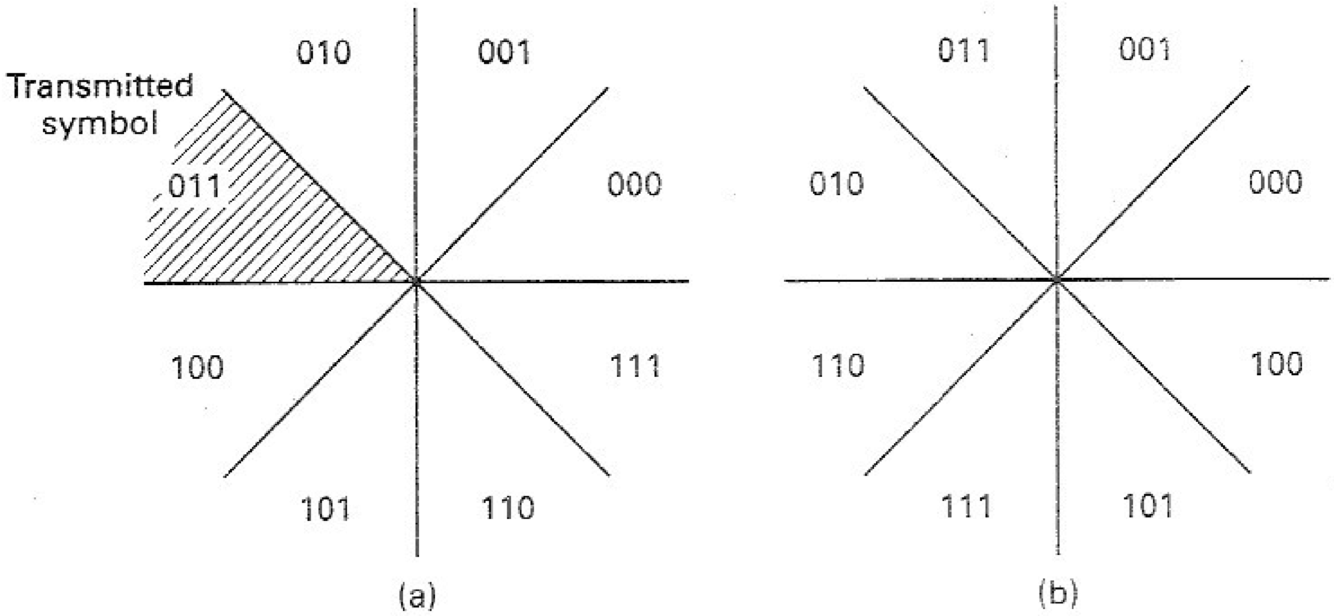

Figure 4.39 Binary-coded versus Gray-coded decision regions in an MPSK signal space. (a) Binary coded. (b) Gray coded.

bols (phase shifts) differ in only one bit position; thus when an M-ary symbol error occurs, it is more likely that only one of the k input bits will be in error. A code that provides this desirable feature is the Gray code [7]; Figure 4.39b illustrates the bit-to-symbol assignment using a Gray code for 8-ary PSK. Here it can be seen that neighboring symbols differ from one another in only one bit position. Therefore, the occurrence of a multibit error, for a given symbol error, is much reduced compared to the uncoded binary assignment seen in Figure 4.39a. Implementing such a Gray code, represents one of the few cases in digital communications where a benefit can be achieved without incurring any cost. The Gray code is simply an assignment that requires no special or additional circuitry. Utilizing the Gray code assignment, it can be shown [5] that

$$P_B \approx \frac{P_E}{\log_2 M} \qquad (\text{for } P_E \ll 1) \tag{4.113}$$

Recall from Section 4.8.4 that BPSK and QPSK signaling have the same bit error probability. Here, in Equation (4.113), we verify that they do not have the same symbol error probability. For BPSK, $P_E = P_B$. However, for QPSK, $P_E \approx 2P_B$.

An exact closed-form expression for the bit-error probability P_B of 8-ary PSK, together with tight upper and lower bounds on P_B for M-ary PSK with larger M, may be found in Lee [10].

4.9.5 Effects of Intersymbol Interference

In the previous sections and in Chapter 3 we have treated the detection of signals in the presence of AWGN under the assumption that there is no intersymbol interference (ISI). Thus the analysis has been straightforward, since the zero-mean AWGN process is characterized by its variance alone. In practice we find that ISI is often a second source of interference which must be accounted for. As explained in Section 3.3, ISI can be generated by the use of bandlimiting filters at the transmitter output, in the channel, or at the receiver input. The result of this additional interference is to degrade the error probabilities for coherent as well as for noncoherent reception. Calculating error performance in the presence of ISI in addition to AWGN is much more complicated since it involves the impulse response of the channel. The subject will not be treated here; however, for those readers interested in the details of the analysis, References [11–16] should prove interesting.

4.10 CONCLUSION

We have catalogued some basic bandpass digital modulation formats, particularly phase shift keying (PSK) and frequency shift keying (FSK). We have considered a geometric view of signal vectors and noise vectors, particularly antipodal and orthogonal signal sets. This geometric view allows us to consider the detection problem in the light of an orthogonal signal space and signal regions. This view of the space, and the effect of noise vectors causing transmitted signals to be received in the incorrect region, facilitates the understanding of the detection problem and the performance of various modulation and demodulation techniques. In Chapter 9 we reconsider the subjects of modulation and demodulation, and we investigate some bandwidth-efficient modulation techniques.

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PROBLEMS

4.1. Find the expected number of bit errors made in one day by the following continuously operating coherent BPSK receiver. The data rate is 5000 bits/s. The input digital waveforms are $s_1(t) = A \cos \omega_0 t$ and $s_2(t) = -A \cos \omega_0 t$ where $A = 1$ mV and the single-sided noise power spectral density is $N_0 = 10^{-11}$ W/Hz. Assume that signal power and energy per bit are normalized relative to a 1-Ω resistive load.

4.2. A continuously operating coherent BPSK system makes errors at the average rate of 100 errors per day. The data rate is 1000 bits/s. The single-sided noise power spectral density is $N_0 = 10^{-10}$ W/Hz.

(a) If the system is ergodic, what is the average bit error probability?

(b) If the value of received average signal power is adjusted to be 10^{-6} W, will this received power be adequate to maintain the error probability found in part (a)?

4.3. If a system's main performance criterion is bit error probability, which of the following two modulation schemes would be selected for an AWGN channel? Show computations.

Binary noncoherent orthogonal FSK with $E_b/N_0 = 13$ dB

Binary coherent PSK with $E_b/N_0 = 8$ dB

4.4. The bit stream

1 0 1 0 1 0 1 1 1 1 0 1 0 1 0 1 0 0 0 0 1 1 1 1

is to be transmitted using DPSK modulation. Show four different differentially encoded sequences that can represent the data sequence above, and explain the algorithm that generated each.

4.5. (a) Calculate the minimum required bandwidth for a noncoherently detected orthogonal binary FSK system. The higher-frequency signaling tone is 1 MHz and the symbol duration is 1 ms.

(b) What is the minimum required bandwidth for a noncoherent MFSK system having the same symbol duration?

4.6. Consider a BPSK system with equally likely waveforms $s_1(t) = \cos \omega_0 t$ and $s_2(t) = -\cos \omega_0 t$. Assume that the received $E_b/N_0 = 9.6$ dB, giving rise to a bit-error probability of 10^{-5}, when the synchronization is perfect. Consider that carrier recovery with the PLL suffers some fixed error ϕ associated with the phase estimate, so that the reference signals are expressed as $\cos(\omega_0 t + \phi)$ and $-\cos(\omega_0 t + \phi)$. Note that the error-degradation effect of a fixed known bias can be computed by using the closed-end relationships presented in this chapter. However, if the phase error were to consist of a random jitter, computing its effect would then require a stochastic treatment. (See Chapter 10.)

(a) How badly does the bit-error probability degrade when $\phi = 25°$?

(b) How large a phase error would cause the bit-error probability to degrade to 10^{-3}?

4.7. Find the probability of bit error, P_B, for the coherent matched filter detection of the equally likely binary FSK signals

$$s_1(t) = 0.5 \cos 2000\pi$$

and

$$s_2(t) = 0.5 \cos 2020\pi t,$$

where the two-sided AWGN power spectral density is $N_0/2 = 0.0001$. Assume that the symbol duration is $T = 0.01$ s.

4.8. Find the optimum (minimum probability of error) threshold γ_0, for detecting the equally likely signals $s_1(t) = \sqrt{2E/T} \cos \omega_0 t$ and $s_2(t) = \sqrt{\frac{1}{2} E/T} \cos(\omega_0 t + \pi)$ in AWGN, using a correlator receiver as shown in Figure 4.7b. Assume a reference signal of $\psi_1(t) = \sqrt{2/T} \cos \omega_0 t$.

4.9. A system using matched filter detection of equally likely BPSK signals, $s_1(t) = \sqrt{2E/T} \cos \omega_0 t$ and $s_2(t) = \sqrt{2E/T} \cos(\omega_0 t + \pi)$, operates in AWGN with a received E_b/N_0 of 6.8 dB. Assume that $\mathbf{E}\{z(T)\} = \pm\sqrt{E}$.

(a) Find the minimum probability of bit error, P_B, for this signal set and E_b/N_0.

(b) If the decision threshold is $\gamma = 0.1\sqrt{E}$, find P_B.

(c) The threshold of $\gamma = 0.1\sqrt{E}$ is optimum for a particular set of a priori probabilities, $P(s_1)$ and $P(s_2)$. Find the values of these probabilities (refer to Section B.2).

4.10. **(a)** Describe the impulse response of a matched filter for detecting the discrete signal shown in Figure P4.1. With this signal at the input to the filter, show the output as a function of time. Neglect the effects of noise. What is the maximum output value?

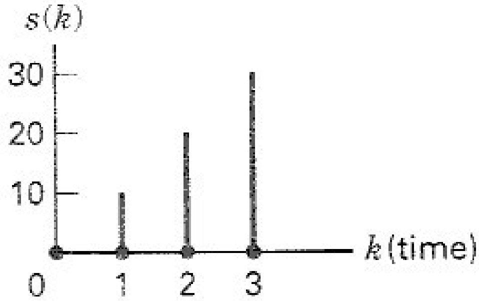

Figure P4.1

(b) In a matched filter (MF), a signal is convolved with a time-reversed function of the signal (impulse response of the MF). Convolution reverses the function again; thus, the MF yields the correlation of a signal and its "look alike" copy (even though the MF operation is designated as convolution). In implementing a

MF, suppose that you accidently connect the circuitry so as to yield the correlation of a signal and its time-reversed copy; the output would then yield a signal convolved with itself. Show the output as a function of time. What is the maximum output value? Note that the maximum output value for part (a) will occur at a different time index compared with that of part (b).

(c) By examining the filter's output vales from the flawed circuit of part (b), compared with the correct values in part (a), can you find a clue that can help you predict when such an output sequence appears to be a valid matched filter output, and when it does not?

(d) If noise were added to the signal, compare the output SNR for the correlator versus the convolver. Also, if the input consists of noise only, compare the output of the correlator versus the convolver.

4.11. A binary source with equally likely symbols controls the switch position in a transmitter operating over an AWGN channel, as shown in Figure P4.2. The noise has two-sided spectral density $N_0/2$. Assume antipodal signals of time duration T seconds and energy E joules. The system clock produces a clock pulse every T seconds, and the binary source rate is $1/T$ bits/s. Under *normal* operation, the switch is up when the source produces a binary zero, and it is down when the source produces a binary one. However, the switch is *faulty*. With probability p, it will be thrown in the wrong direction during a given T-second interval. The presence of a switch error during any interval is independent of the presence of a switch error at any other time. Assume that $\mathbf{E}\{z(T)\} = \pm\sqrt{E}$.

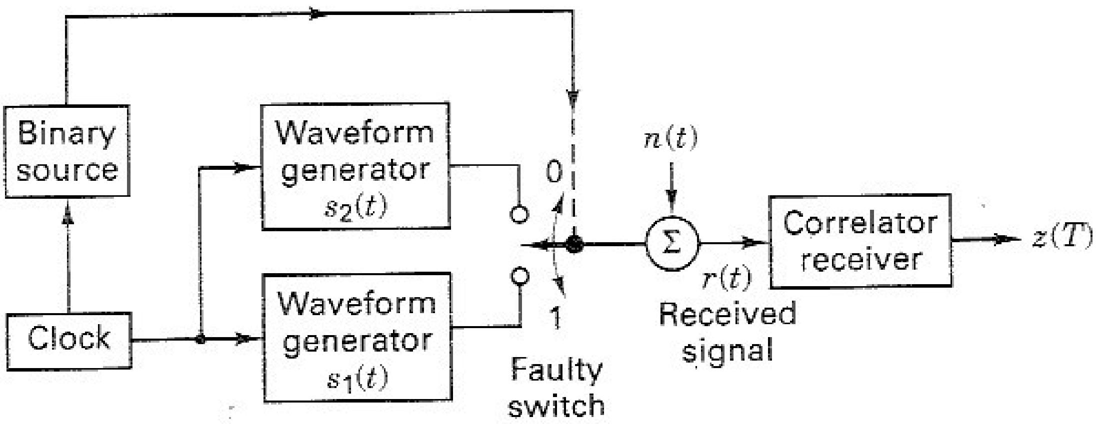

Figure P4.2

(a) Sketch the conditional probability functions, $p(z|s_1)$ and $p(z|s_2)$.

(b) The correlator receiver observes $r(t)$ in the interval $(0, T)$. Sketch the block diagram of an optimum receiver for minimizing the bit error probability when it is known that the switch is faulty with probability, p.

(c) Which one of the following two systems would you prefer to have?

$$p = 0.1 \text{ and } \frac{E_b}{N_0} = \infty$$

or

$$p = 0 \text{ and } \frac{E_b}{N_0} = 7 \text{ dB}$$

4.12. **(a)** Consider a 16-ary PSK system with symbol error probability $P_E = 10^{-5}$. A Gray code is used for the symbol to bit assignment. What is the approximate bit error probability?

(b) Repeat part (a) for a 16-ary orthogonal FSK system.

4.13. Consider a coherent orthogonal MFSK system with $M = 8$ having the equally likely waveforms $s_i(t) = A \cos 2\pi f_i t$, $i = 1, \ldots, M$, $0 \le t \le T$, where $T = 0.2$ ms. The received carrier amplitude, A, is 1 mV, and the two-sided AWGN spectral density, $N_0/2$, is 10^{-11} W/Hz. Calculate the probability of bit error, P_B.

4.14. A bit error probability of $P_B = 10^{-3}$ is required for a system with a data rate of 100 kbits/s to be transmitted over an AWGN channel using coherently detected MPSK modulation. The system bandwidth is 50 kHz. Assume that the system frequency transfer function is a raised cosine with a roll-off characteristic of $r = 1$ and that a Gray code is used for the symbol to bit assignment.

(a) What E_s/N_0 is required for the specified P_B?

(b) What E_b/N_0 is required?

4.15. A differentially coherent MPSK system operates over an AWGN channel with an E_b/N_0 of 10 dB. What is the symbol error probability for $M = 8$ and equally likely symbols?

4.16. If a system's main performance criterion is bit-error probability, which of the following two modulation schemes would be selected for transmission over an AWGN channel?

$$\text{coherent 8-ary orthogonal FSK with } \frac{E_b}{N_0} = 8 \text{ dB}$$

or

$$\text{coherent 8-ary PSK with } \frac{E_b}{N_0} = 13 \text{ dB}$$

(Assume that a Gray code is used for the MPSK symbol-to-bit assignment, and show computations.)

4.17. Consider that a BPSK demodulator/detector has a synchronization error consisting of a time bias pT, where p is a fraction $(0 \le p \le 1)$ of the symbol time T. In other words, the detection of a symbol starts early (late) and concludes early (late) by an amount pT. Assume equally likely signaling and perfect frequency and phase synchronization. Note that the error-degradation effect of a fixed known bias can be computed by using the closed-end relationships presented in this chapter. However, if the timing error were to consist of a random jitter, computing its effect would then require a stochastic treatment. (See Chapter 10.)

(a) Find the general expression for bit-error probability P_b as a function of p.

(b) If the received $E_b/N_0 = 9.6$ dB and $p = 0.2$, compute the value of degraded P_b due to the timing bias.

(c) If one did not compensate for the timing bias in this example, how much additional E_b/N_0 in dB must be provided in order to restore the P_b that exists when $p = 0$?

4.18. Using all of the stated specifications, repeat problem 4.17 for the case of coherently detected, binary frequency-shift-keying (BFSK) modulation.

4.19. Consider that a BPSK demodulator/detector has a synchronization error consisting of a time bias pT, where p is a fraction $(0 \le p \le 1)$ of the symbol time T. Consider that there is also a constant phase-estimation error ϕ. Assume equally-likely signaling and perfect frequency synchronization.

(a) Find the general expression for bit-error probability P_b as a function of p and ϕ.

(b) If received $E_b/N_0 = 9.6$ dB, $p = 0.2$, and $\phi = 25°$, compute the value of degraded P_b due to the combined effects of timing and phase bias.

(c) If one did not compensate for the biases in this example, how much additional E_b/N_0 in dB must be provided in order to restore the P_b that exists when $p = 0$ and $\phi = 0°$?

4.20. Correlating to a known Barker sequence is an often used synchronization technique, since the Barker sequence yields a prominent correlation peak when properly synchronized, and a small correlation output when not synchronized. Using the short Barker sequence 1 0 1 1 1, where the leftmost bit is the earliest bit, devise a discrete matched filter similar to the one in Figure 4.10 that is matched to this sequence. Verify its usefulness for synchronization by plotting the output versus input as a function of time, when the input is the 1 0 1 1 1 sequence.

QUESTIONS

4.1. At what location in the system is E_b/N_0 defined? (See Section 4.3.2.)

4.2. Amplitude- or phase-shift keying is visualized as a constellation of points or phasors on a plane. Why can't we use a similarly simple visualization for orthogonal signaling such as FSK? (See Section 4.4.4.)

4.3. In the case of MFSK signaling, what is the minimum tone spacing that insures signal *orthogonality?* (See Section 4.5.4.)

4.4. What benefits are there in using *complex notation* for representing sinusoids? (See Sections 4.2.1 and 4.6.)

4.5. Digital modulation schemes fall into one of the two classes with opposite behavior characteristics: *orthogonal* signaling, and *phase/amplitude* signaling. Describe the behavior of each class. (See Sections 4.8.2.)

4.6. Why do binary phase shift keying (BPSK) and quaternary phase shift keying (QPSK) manifest the same bit-error-probability relationship? (See Section 4.8.4.)

4.7. In the case of multiple-phase shift keying (MPSK), why does *bandwidth efficiency* improve with higher dimensional signaling? (See Sections 4.8.2 and 4.8.3.)

4.8. In the case of orthogonal signaling such as MFSK, why does *error-performance* improve with higher dimensional signaling? (See Section 4.8.5.)

4.9. The use of a Gray code for assigning bits to symbols, represents one of the few cases in digital communications where a benefit can be achieved *free-of-charge*. Explain why there is no cost. (See Section 4.9.4.)

EXERCISES

Using the Companion CD, run the exercises associated with Chapter 4.