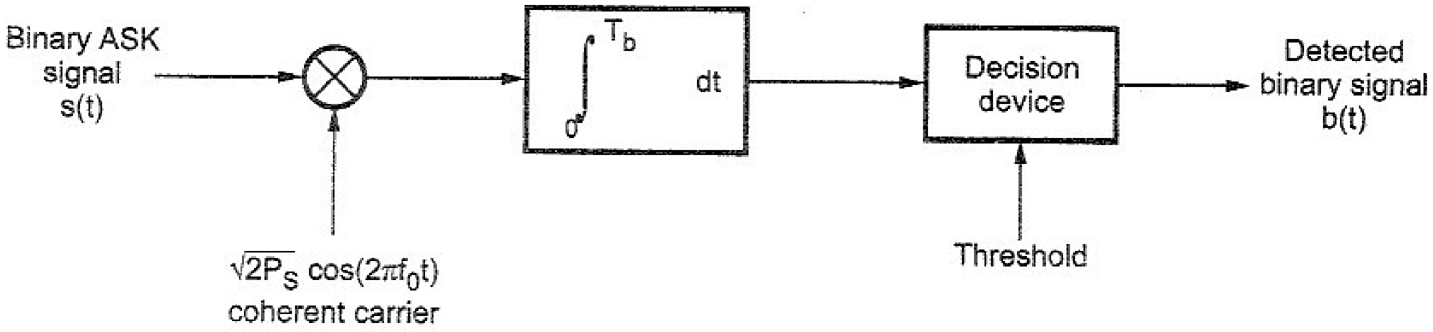


Fig. 5.10.4 Block diagram of coherent ASK detector

5.10.2.3 Noncoherent ASK Reception

Fig. 5.10.5 shows the block diagram of noncoherent ASK receiver. In this figure observe that the received ASK signal is given to bandpass filter. This bandpass filter passes only carrier frequency, f_0. The envelope detector generates high output voltage when carrier is present. When carrier (f_0) is absent, there is only noise at the input of envelope detector. Hence it produces low output. The decision device is basically a regenerator. It generates the binary sequence b(t). Threshold is provided to the decision device to overcome effects due to noise. When y(t) is greater than threshold, b(t) = 1 and when y(t) is less than threshold, b(t) = 0. Non-coherent reception of ASK does not need any carrier synchronization.

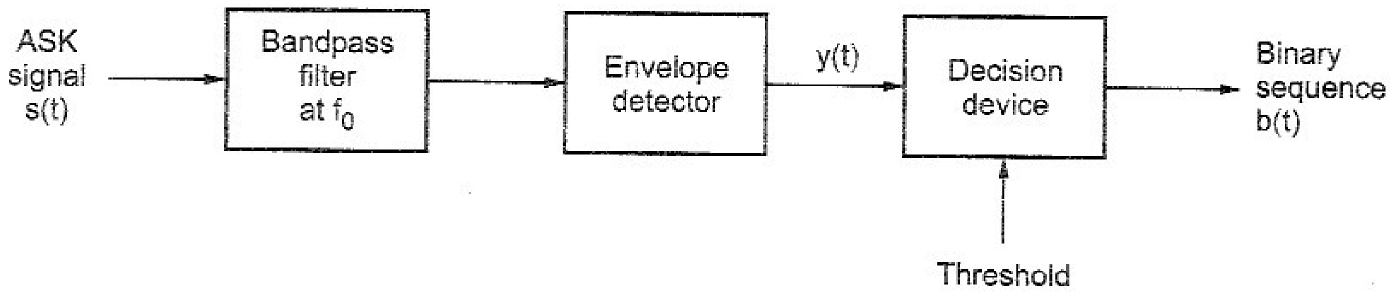


Fig. 5.10.5 Block diagram of noncoherent ASK receiver

5.11 Comparison of Digital Modulation Techniques

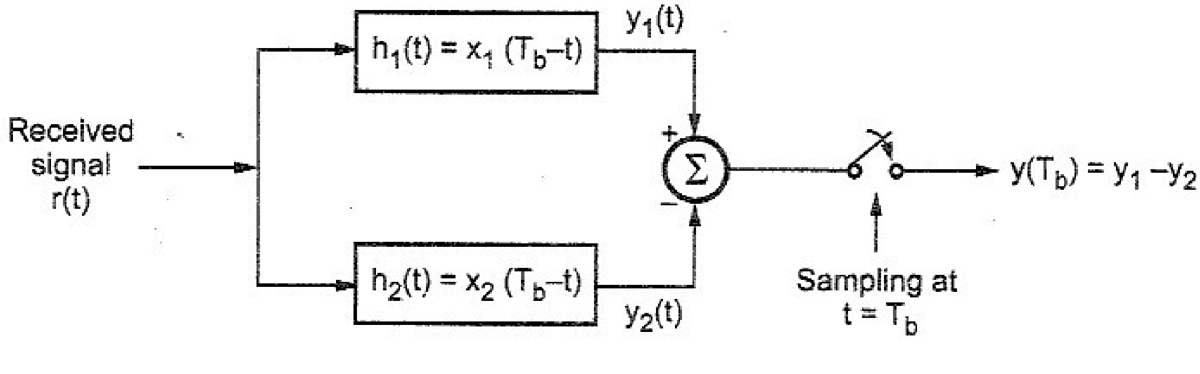
Table 5.11.1 shows the comparison of various digital modulation techniques. They are compared on the basis of various parameters like bits transmitted per symbol, detection method, Euclidean distance, bandwidth, error probability, symbol duration etc. Various other important parameters like bandwidth efficiency, spectrum of transmitted signal etc are not compared. QPSK, ASK have amplitude variations hence noise interference is more in these techniques. Normally PSK and FSK methods have less noise interference. M-ary techniques are more complex compared to binary techniques.

Sr. No.	Parameter	BPSK	DPSK	QPSK	M-ary PSK	QASK	BFSK	M-ary FSK	MSK	ASK
1	Modulation of	Phase	Phase	Phase	Phase	Amplitude and phase	frequency	frequency	frequency	amplitude
2	Equation of the transmitted signal $s(t)$	$s(t) = b(t)\sqrt{2P_s} \cos(2\pi f_0 t)$	$s(t) = b(t)\sqrt{2P_S} \cos(2\pi f_0 t)$ $b(t)$ differentially coded	$s(t) = \sqrt{2P_S} \cos [2\pi f_0 t + (2m+1)\frac{\pi}{4}]$ $m = 0, 1, 2, 3$	$s(t) = \sqrt{2P_S} \cos(2\pi f_0 t + \phi m)$ $\phi m = (2m+1)\frac{\pi}{M}$ $m = 0, 1, 2, \ldots, M-1$	$s(t) = k_1\sqrt{0.2P_S} \cos(2\pi f_0 t) + k_2\sqrt{0.2\ P_S} \sin(2\pi f_0 t)$ $k_1, k_2 = \pm 1$ or ± 3 for $M = 16$	$s(t) = \sqrt{2P_S} \cos [2(\pi f_0 + d(t)\Omega) t]$ Ω is frequency shift.	$s(t) = \sqrt{2P_S} \cos(2\pi f_i t)$ $i = 1, 2, \ldots M$	$s(t) = b_o(t)\sqrt{2P_s} \sin 2\pi \left[f_0 + b_e(t) b_o(t) \frac{f_b}{4}\right] t$ $b_e(t), b_o(t)$ = odd/even sequence	$s(t) = 2\sqrt{2P_s} \cos(2\pi f_0 t)$ for symbol '1' = 0 for symbol '0'
3	Bits per symbol	One	One	Two	N	N	One	N	Two	One
4	Number of possible symbols $M = 2^N$	Two	Two	Four	$M = 2^N$	$M = 2^N$	Two	$M = 2^N$	Four	Two
5	Detection method	Coherent	Non-Coherent	Coherent	Coherent	Coherent	Non-coherent	Non-coherent	Coherent	Coherent
6	Minimum Euclidean distance	$2\sqrt{E_b}$	-	$2\sqrt{E_b}$	$2\sqrt{E_s} \sin\frac{\pi}{M}$	$\sqrt{0.4E_s}$ for $M = 16$	$\sqrt{2E_b}$	$\sqrt{2NE_b}$	$2\sqrt{E_b}$	$\sqrt{E_b}$
7	Minimum bandwidth (BW)	$2f_b$	f_b	f_b	$\frac{2f_b}{N}$	$\frac{2f_b}{N}$	$4f_b$	$\frac{2^{N+1}}{N} f_b$	$1.5 f_b$	$2 f_b$
8	Symbol duration (T_s)	T_b	$2T_b$	$2T_b$	NT_b	NT_b	T_b	NT_b	$2T_b$	T_b

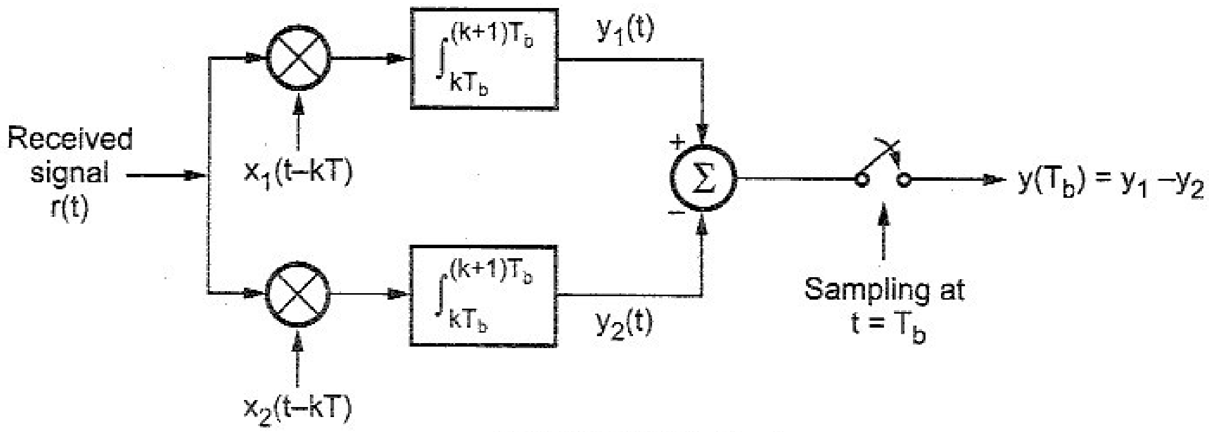
Table 5.11.1 Comparison of digital modulation techniques

5.12 Bandpass Receiving Filter

We have studies two baseband signal receivers in the last chapter. They are matched filter and correlater. They can also be used to receive the bandpass signals as shown in Fig. 5.12.1. These are called bandpass receiving filters. These are the filters for coherent binary modulation methods. Observe that the matched filter has two parallel branches which are matched to each of the binary symbols. The difference of the two is sampled at the end of bit interval (T_b) and given to decision device.



(a) Parallel matched filter



(b) Correlation receiver

Fig. 5.12.1 Bandpass receiving filter

Similarly there are two correlats in parallel to receive the binary symbol. The outputs $y_1(t)$ and $y_2(t)$ are generated according to the modulated binary symbols. The error probabilities of the modulation methods are derived for matched filter reception.

5.13 Error Performance of Binary Systems

In previous sections we studied digital modulation techniques. The noise interferes the signal during transmission. Hence received signal is noisy. The receiver has to detect the signal in presence of noise. We have studied optimum receivers to detect binary signals in presence of noise. When the noise is white gaussian, the receiver is called matched filter. In this section we will study error performance (error probability) of binary systems in presence of noise. Normally white Gaussian noise is considered for error performance.

5.13.1 Error Probability of ASK

In Amplitude Shift Keying (ASK), some number of carrier cycles are transmitted to send '1' and no signal is transmitted for binary '0'. Thus,

$$\text{Binary '1'} \Rightarrow \quad x_1(t) = \sqrt{2P_s}\cos(2\pi f_0 t) \text{ and}$$

$$\text{Binary '0'} \Rightarrow \quad x_2(t) = 0 \quad \text{(i.e. no signal)} \qquad \dots (5.13.1)$$

Here P_s is the normalized power of the signal in 1Ω load. i.e. power $P_s = \frac{A^2}{2}$. Hence $A = \sqrt{2P_s}$. Therefore in above equation for $x_1(t)$ amplitude 'A' is replaced by $\sqrt{2P_s}$.

We know that the probability of error of the optimum filter is given as,

$$P_e = \frac{1}{2} erfc\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right\} \qquad \dots (5.13.2)$$

Here $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df$

The above equations can be applied to matched filter when we consider white Gaussian noise. The power spectral density of white Gaussian noise is given as,

$$S_{ni}(f) = \frac{N_0}{2}$$

Putting this value of $S_{ni}(f)$ in above equations we get,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}}\, df$$

$$= \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2\, df \qquad \dots (5.13.3)$$

Parseval's power theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2\, df = \int_{-\infty}^{\infty} x^2(t)\, dt$$

Hence equation 5.13.3 becomes,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} x^2(t)\, dt$$

We know that $x(t)$ is present from 0 to T. Hence limits in above equation can be changed as follows :

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt \quad \text{... (5.13.4)}$$

We know that $x(t) = x_1(t) - x_2(t)$. For ASK $x_2(t)$ is zero, hence $x(t) = x_1(t)$. Hence above equation becomes,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x_1^2(t)\,dt$$

Putting equation of $x_1(t)$ from equation 5.13.1 in above equation we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T \left[\sqrt{2P_s}\cos(2\pi f_0 t)\right]^2 dt$$

$$= \frac{4P_s}{N_0}\int_0^T \cos^2(2\pi f_0 t)\,dt$$

We know that $\cos^2\theta = \frac{1+\cos 2\theta}{2}$. Here applying this formula to $\cos^2(2\pi f_0 t)$ we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{4P_s}{N_0}\int_0^T \frac{1+\cos 4\pi f_0 t}{2}\,dt$$

$$= \frac{4P_s}{N_0}\cdot\frac{1}{2}\left\{\int_0^T dt + \int_0^T \cos 4\pi f_0 t\,dt\right\}$$

$$= \frac{2P_s}{N_0}\left\{[t]_0^T + \left[\frac{\sin 4\pi f_0 t}{4\pi f_0}\right]_0^T\right\}$$

$$= \frac{2P_s}{N_0}\left\{T + \frac{\sin 4\pi f_0 T}{4\pi f_0}\right\} \quad \text{... (5.13.5)}$$

We know that T is the bit period and in this one bit period, the carrier has integer number of cycles. Thus the product $f_0 T$ is an integer. This is illustrated in Fig. 5.13.1

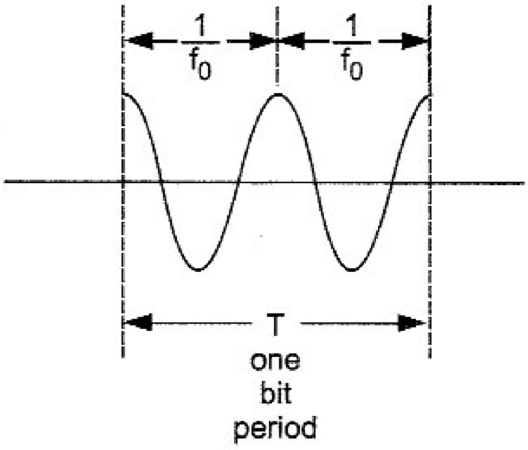


Fig. 5.13.1 In one bit period T, the carrier completes its two cycles. The carrier has frequency f_0. From figure we can write,

$$T = \frac{1}{f_0} + \frac{1}{f_0}$$

i.e. $$T = \frac{2}{f_0}$$

$\therefore$ $$f_0 T = 2 \quad \text{(integer no. of cycles)}$$

As shown in above figure, the carrier completes two cycles in one bit duration. Hence

$$f_0 T = 2$$

Therefore, in general if carrier completes 'k' number of cycles, then,

$$f_0 T = k \qquad \text{(Here } k \text{ is an integer)}$$

Therefore the sine term in equation 5.13.5 becomes, $\sin 4\pi k$ and k is integer.

For all integer values of k, $\sin 4\pi k = 0$. Hence equation 5.13.5 becomes,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2P_s T}{N_0} \qquad \text{... (5.13.6)}$$

$\therefore$ $$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{2P_s T}{N_0}} \qquad \text{... (5.13.7)}$$

Putting this value in equation 5.13.2 we get error probability of ASK using matched filter detection as,

$$P_e = \frac{1}{2} erfc\left\{\frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{2P_s T}{N_0}}\right\} = \frac{1}{2} erfc\sqrt{\frac{P_s T}{4N_0}}$$

Here $P_s T = E$, i.e. energy of one bit hence above equation becomes,

Error probability of ASK : $$P_e = \frac{1}{2} erfc\sqrt{\frac{E}{4N_0}} \qquad \text{... (5.13.8)}$$

This is the expression for error probability of ASK using matched filter detection.

Probability of error for noncoherent detection of ASK

The error probability of noncoherent detection also depends on the ratio $\frac{E_b}{N_0}$. It is given as,

$$P_e = \frac{1}{2} e^{-\frac{E_b}{2N_0}}$$

Note that there is exponential relationship between error probability and the ratio $\frac{E_b}{N_0}$. It shows that noncoherent ASK requires $\frac{E_b}{N_0} >> 1$ for reasonable performance.

Example 5.13.1 : *A signal is either* $s_1(t) = A\cos(2\pi f_0 t)$ *or* $s_2(t) = 0$ *for an interval* $T = \frac{n}{f_0}$ *with 'n' an integer. The signal is corrupted by white noise with PSD* $= \frac{N_0}{2}$. *Find the transfer function of the matched filter for this signal. Write an expression for the probability of error* P_e.

Solution : The signaling scheme given in this example is ASK. Observe that equation 5.13.1 describes the ASK signal as,

$$x_1(t) = \sqrt{2P_s}\cos(2\pi f_0 t) \text{ and}$$

$$x_2(t) = 0$$

If we compare the given equations of $s_1(t)$ and $s_2(t)$ with above equations, we find that,

$$A = \sqrt{2P_s}$$

or power $$P_s = \frac{A^2}{2}$$

The detailed derivation of transfer function and error probability for this ASK scheme is given in section 5.13.1. The error probability of ASK is obtained in equation 5.13.8 as,

$$P_e = \frac{1}{2} erfc\sqrt{\frac{E}{4N_0}}$$

Here $$E = P_s T \quad \text{i.e. energy of the bit}$$

$$= \frac{A^2}{2} T$$

The transfer function of the matched filter is given by equation as,

$$H(f) = \frac{2k}{N_0} X^*(f)\, e^{-j2\pi fT}$$

Here $X(f)$ is the fourier transform of signal $x(t)$. The signal $x(t)$ is given as,

$$x(t) = x_1(t) - x_2(t)$$

or

$$x(t) = s_1(t) - s_2(t) = A\cos 2\pi f_0 t$$

Hence,

$$X(f) = FT\{x(t)\} = FT\{A\cos(2\pi f_0 t)\}$$

The fourier transform of cosine function is given in appendix F. Hence above equation becomes,

$$X(f) = \frac{A}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$$

Since there is no imaginary part in above result,

$$X^*(f) = X(f) = \frac{A}{2}\{\delta(f - f_0) + \delta(f + f_0)\}$$

Hence the transfer function becomes,

$$H(f) = \frac{2k}{N_0} \cdot \frac{A}{2}\{\delta(f - f_0) + \delta(f + f_0)\}\, e^{-j2\pi fT}$$

$$\therefore \quad H(f) = \frac{kA}{N_0}\{\delta(f - f_0) + \delta(f + f_0)\}\, e^{-j2\pi fT}$$

This is the required transfer function.

5.13.2 Probability of Error for Coherently Detected BPSK **Nov./Dec.-2005**

In Binary PSK (BPSK), the phase of the carrier is shifted by 180° for two symbols. These two symbols (bits) are represented by two signals as follows :

$$\text{Binary '1'} \Rightarrow x_1(t) = \sqrt{2P}\cos(2\pi f_0 t) \quad \text{... (5.13.9)}$$

$$\text{and Binary '0'} \Rightarrow x_2(t) = -\sqrt{2P}\cos(2\pi f_0 t) \quad \text{... (5.13.10)}$$

Here P is normalized power of the carrier; and $P = \frac{A^2}{2}$, where A is amplitude of the carrier. From the above two equations we can write,

$$x_2(t) = -x_1(t) \quad \text{... (5.13.11)}$$

In the last subsection (see 5.13.1) we have seen that probability of error of the matched filter is given as (By equation 5.13.2),

$$P_e = \frac{1}{2} \operatorname{erfc}\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right\} \quad \text{... (5.13.12)}$$

And for a matched filter detection in presence of white Gaussian noise,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt \qquad \text{By equation 5.13.4} \quad \text{... (5.13.13)}$$

Here we know that $x(t) = x_1(t) - x_2(t)$. For PSK, $x_2(t) = -x_1(t)$ as we have seen (equation 5.13.11).

$$\therefore \quad x(t) = x_1(t) - [-x_1(t)]$$

$$= 2x_1(t)$$

Hence equation 5.13.13 becomes,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T 4\,x_1^2(t)\,dt$$

$$= \frac{8}{N_0}\int_0^T x_1^2(t)\,dt \quad \text{... (5.13.14)}$$

As in the previous subsection it can be shown very easily that,

$$\int_0^T x_1^2(t)\,dt = \int_0^T 2P\cos^2(2\pi f_0 t)\,dt$$

$$= 2P \cdot \frac{1}{2}\int_0^T [1 + \cos 4\pi f_0 t]\,dt = P\left[\int_0^T dt + \int_0^T \cos 4\pi f_0 t\,dt\right]$$

In the above equation second integration will be zero since it is integration of cosine wave over one bit period. This we have proved in last subsection (i.e. section 5.13.1). Hence above equation becomes,

$$\int_0^T x_1^2(t)\,dt = P\int_0^T dt \qquad \text{Since second term is zero.}$$

$$= P[t]_0^T = PT$$

$$= E \quad \text{... (5.13.15)}$$

Thus energy E = Power (P) × bit duration (T). Putting the above result in equation 5.13.14 we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{8}{N_0}\cdot E$$

$$\therefore \quad \left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{8E}{N_0}} \quad \text{... (5.13.16)}$$

Putting this result in equation 5.13.12 we get,

$$P_e = \frac{1}{2}\, erfc\left\{\frac{1}{2\sqrt{2}}\sqrt{\frac{8E}{N_0}}\right\}$$

On simplification of above equation we get,

$$\therefore \quad \boxed{\textbf{Error probability in PSK : } P_e = \frac{1}{2}\, erfc\sqrt{\frac{E}{N_0}}} \quad \text{... (5.13.17)}$$

This is the expression which gives error probability of PSK using matched filter detection.

Example 5.13.2 : *In a PSK system, the received waveforms $S_1(t) = A\cos\omega t$ and $S_2(t) = -A\cos\omega t$ are coherently detected with a matched filter. The value of A is 20 mV, and the bit rate is 1 Mbps. Assume that the noise power spectral density $\frac{N_0}{2} = 10^{-11}$ W/Hz. Find the probability of error Pe.*

May/June-2007, 6 Marks

Solution : Here $A = 20 \times 10^{-3}$ V

$$f_b = 1 \text{ Mbps, hence } T_b = \frac{1}{1\times 10^6} = 1 \times 10^{-6} \text{ sec}$$

$$\frac{N_0}{2} = 10^{-11} \text{ W/Hz} \Rightarrow N_0 = 2\times 10^{-11}$$

$$\therefore \quad E_b = P_s T_b$$

$$= \frac{1}{2}A^2 T_b, \quad \text{since } P_s = \frac{1}{2}A^2$$

$$= \frac{1}{2}\times(20\times10^{-3})^2\times 1\times 10^{-6} = 2 \times 10^{-10} \text{ J}$$

Error probability of BPSK is given as,

$$P_e = \frac{1}{2}\, erfc\sqrt{\frac{E}{N_0}}$$

$$= \frac{1}{2} erfc \sqrt{\frac{2\times10^{-10}}{2\times10^{-11}}} = \frac{1}{2} erfc\ (3.1622777)$$

Since erfc (x) is higher than erfc (1.5) we can use the following approximation,

$$\text{erfc}\ (x) = \frac{e^{-x^2}}{\sqrt{\pi}\, x}$$

$$\therefore \quad \text{erfc}\ (3.1622777) = \frac{e^{-3.1622777^2}}{\sqrt{\pi}\,(3.1622777)}$$

$$= 8 \times 10^{-6}$$

$$\therefore \quad P_e = \frac{1}{2} erfc\ (3.1622777) = \frac{1}{2} \times 8\times10^{-6} = 4\times10^{-6}$$

5.13.3 Probability of Error for Coherently Detected Binary Orthogonal FSK

April/May-2005

In the binary FSK transmission, two different carrier frequencies are used to transmit two binary levels. As we have seen these two signals are as follows :

$$\text{Binary '1'} \Rightarrow x_1(t) = \sqrt{2P}\cos(2\pi f_0 + \Omega)t$$

$$\text{and Binary '0'} \Rightarrow x_2(t) = \sqrt{2P}\cos(2\pi f_0 - \Omega)t \qquad \text{... (5.13.18)}$$

From above equation we can write,

$$x_1(t) - x_2(t) = \sqrt{2P}\{\cos(2\pi f_0 + \Omega)t - \cos(2\pi f_0 - \Omega)t\}$$

$$[x_1(t) - x_2(t)]^2 = 2P\left[\cos(2\pi f_0 + \Omega)t - \cos(2\pi f_0 - \Omega)t\right]^2$$

Here let us use $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$, then above equation becomes,

$$\left[x_1(t) - x_2(t)\right]^2 = 2P[-2\sin 2\pi f_0\, t \sin\Omega t]^2$$

$$= 2P[4\sin^2\omega_0 t \sin^2\Omega t] \qquad \text{By putting } 2\pi f_0 = \omega_0$$

$$= 2P\left\{(2\sin^2\omega_0 t)(2\sin^2\Omega t)\right\} \qquad \text{By rearranging the equation}$$

Here let us use $2\sin^2(x) = 1 - \cos(2x)$, then above equation becomes,

$$[x_1(t) - x_2(t)]^2 = 2P\{(1-\cos 2\omega_0 t)(1-\cos 2\Omega t)\}$$

$$= 2P\{1-\cos 2\Omega t - \cos 2\omega_0 t + \cos 2\omega_0 t \cos 2\Omega t\}$$

In the above equation let us use $\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y)+\cos(x+y)]$ then above equation will be,

$$[x_1(t) - x_2(t)]^2 = 2P\left\{1-\cos 2\Omega t - \cos 2\omega_0 t + \frac{1}{2}[\cos 2(\omega_0 - \Omega)t + \cos 2(\omega_0+\Omega)t]\right\}$$

Let us take integration of both the sides from 0 to T then we get,

$$\int_0^T [x_1(t) - x_2(t)]^2 dt = \int_0^T 2P\{1-\cos 2\Omega t - \cos 2\omega_0 t + \frac{1}{2}[\cos 2(\omega_0 - \Omega)t + \cos 2(\omega_0+\Omega)t]\}dt$$

$$= 2P\left\{\int_0^T dt - \int_0^T \cos 2\Omega t\, dt - \int_0^T \cos 2\omega_0 t\, dt + \frac{1}{2}\int_0^T \cos 2(\omega_0-\Omega)t\, dt + \frac{1}{2}\int_0^T \cos 2(\omega_0+\Omega)t\, dt\right\}$$

$$= 2P\left\{T - \frac{\sin 2\Omega T}{2\Omega} - \frac{\sin 2\omega_0 T}{2\omega_0} + \frac{1}{2}\frac{\sin 2(\omega_0-\Omega)T}{2(\omega_0-\Omega)} + \frac{1}{2}\frac{\sin 2(\omega_0+\Omega)T}{2(\omega_0+\Omega)}\right\} \quad \text{... (5.13.19)}$$

$$= 2PT\left\{1 - \frac{\sin 2\Omega T}{2\Omega T} - \frac{\sin 2\omega_0 T}{2\omega_0 T} + \frac{1}{2}\frac{\sin 2(\omega_0-\Omega)T}{2(\omega_0-\Omega)T} + \frac{1}{2}\frac{\sin 2(\omega_0+\Omega)T}{2(\omega_0-\Omega)T}\right\} \quad \text{... (5.13.20)}$$

We know that the frequency shift 'Ω' is very small in comparison with the carrier frequency ω_0. Then the last three terms in above equation will be of the form $\frac{\sin 2\omega_0 T}{2\omega_0 T}$. Here ω_0 is the angular frequency of the carrier signal and T is the period of one bit. Actually many cycles of carrier are completed in one bit period, i.e. $\omega_0 T >> 1$. Hence the ratio $\frac{\sin 2\omega_0 T}{2\omega_0 T}$ approaches to zero as $\omega_0 T$ increases. Therefore we can neglect last three terms in equation 5.13.20. Then the equation becomes,

$$\int_0^T [x_1(t) - x_2(t)]^2 dt = 2PT\left\{1 - \frac{\sin 2\Omega T}{2\Omega T}\right\} \quad \text{... (5.13.21)}$$

We know that error probability of the matched filter is given as,

$$P_e = \frac{1}{2} erfc \left\{ \frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma} \right\} \quad \text{By equation 5.13.2} \qquad \dots (5.13.22)$$

And for a matched filter detection, in presence of white Gaussian noise,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]^2_{max} = \frac{2}{N_0} \int_0^T x^2(t)\, dt \quad \text{By equation 5.13.4} \qquad \dots (5.13.23)$$

Here we know that $x(t) = x_1(t) - x_2(t)$. Therefore above equation becomes,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]^2_{max} = \frac{2}{N_0} \int_0^T [x_1(t) - x_2(t)]^2\, dt$$

Putting the value of integral of RHS from equation 5.13.21 we get,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]^2_{max} = \frac{2}{N_0} \cdot 2PT \left\{ 1 - \frac{\sin 2\Omega T}{2\Omega T} \right\}$$

$$= \frac{4PT}{N_0} \left\{ 1 - \frac{\sin 2\Omega T}{2\Omega T} \right\} \qquad \dots (5.13.24)$$

The above ratio obtains largest value when $2\Omega T = \frac{3\pi}{2}$

Putting this value in above equation we get,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]^2_{max} = \frac{4PT}{N_0} \left\{ 1 - \frac{\sin\left(\frac{3\pi}{2}\right)}{\frac{3\pi}{2}} \right\}$$

$$= \frac{4.84\, PT}{N_0} \qquad \dots (5.13.25)$$

$$\therefore \quad \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]_{max} = \sqrt{\frac{4.84\, PT}{N_0}} \qquad \dots (5.13.26)$$

Putting this value of $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]$ in equation 5.13.22 we get probability of error as,

$$P_e = \frac{1}{2} erfc \left\{ \frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{4.84\, PT}{N_0}} \right\} = \frac{1}{2} erfc \sqrt{\frac{0.6\, PT}{N_0}}$$

We know that the product $PT = E$ (energy of one bit). Hence above equation becomes,

$$\text{Error probability of FSK : } P_e = \frac{1}{2}\, erfc \sqrt{\frac{0.6\,E}{N_0}} \qquad \ldots (5.13.27)$$

This is the expression for probability of error of FSK which uses matched filter detection.

5.13.3.1 Probability of Error for Non-Coherently Detected Binary Orthogonal FSK

Earlier we have studied noncoherent detection of orthogonal FSK. The system uses two frequencies $\omega_0 + \Omega$ and $\omega_0 - \Omega$ to represent two symbols. The detector consists of two bandpass filters centered at $\omega_0 + \Omega$ and $\omega_0 - \Omega$. The receiver then simply decides in favour of the symbol corresponding to which bandpass filter has higher output. The receiver does not use other information of the signal for detection. Hence probability of detecting the signal is reduced. It can be shown that probability of error of noncoherently detected orthogonal FSK is given as,

$$P_e = \frac{1}{2} e^{-\frac{E}{2N_0}}$$

In this expression note that probability of error depends upon signal energy.

Example 5.13.3 : *In a FSK system, following data are observed.*

$$\textit{Transmitted binary data rate} = 2.5 \times 10^6 \text{ bits/second}$$

$$\textit{PSD of zero mean AWGN} = 10^{-20} \textit{ watts/Hz}$$

$$\textit{Amplitude of received} = 1 \textit{ micro volt}$$

signal in the absence of noise.

Determine the average probability of symbol error assuming coherent detection.

Solution : From equation 5.13.18 we know that the amplitude of the transmitted signal is $\sqrt{2P}$. Here P is the transmitted power i.e.

$$A = \sqrt{2P}$$

$$\therefore \qquad P = \frac{A^2}{2}$$

In this example the value of A is given as 1 μV. In absence of noise, the amplitude of transmitted and received signals will be same. Hence taking A = 1μV, power will be,

$$P = \frac{(1\times10^{-6})^2}{2}$$

$$= 5\times10^{-13} \text{ watts}$$

The bit duration is given as,

$$T = \frac{1}{data\ rate}$$

Data rate is given as 2.5×10^6 bits/second. Hence,

$$T = \frac{1}{2.5\times10^6} = 4\times10^{-7} \text{ sec.}$$

The PSD of white noise is given in this example as,

$$\frac{N_0}{2} = 10^{-20} \text{ watts/Hz}$$

$$\therefore \quad N_0 = 2\times10^{-20}$$

The error probability of FSK system is given as,

$$P_e = \frac{1}{2} erfc\sqrt{\frac{0.6\,PT}{N_0}}$$

Putting the values in above equation,

$$P_e = \frac{1}{2} erfc\sqrt{\frac{0.6\times5\times10^{-13}\times4\times10^{-7}}{2\times10^{-20}}}$$

$$= \frac{1}{2} erfc\ (2.449)$$

$$= \frac{1}{2}\{1 - erf\,(2.449)\} \qquad \text{since } erfc(x) = 1 - erf\,(x)$$

In the error function tables in appendix G, observe that erf(2.5) = 0.99959. Hence taking this as the approximate value of erfc(2.449) also, the above equation becomes,

$$P_e = \frac{1}{2}\ \{1 - 0.99959\}$$

$$\therefore \quad P_e = 2.05\times10^{-4}$$

This is the required error probability for given FSK system.

5.13.4 Probability Error for Binary Orthogonal DPSK

Fig. 5.13.2 (a) shows the phasor diagram of DPSK signal when no noise is present. That is in the absence of noise and transmission delay, the phase shift of the DPSK signal is either '0' or 'π'. Therefore a decision boundary is drawn at $\frac{\pi}{2}$ as shown in Fig. 5.13.2(a). Therefore we consider that the transmitted symbol is '1', if the phase difference between two consecutive bits differs by less than $\frac{\pi}{2}$. If the phase difference between two consecutive bits differs by more than $\frac{\pi}{2}$, then decision is taken in favour of zero.

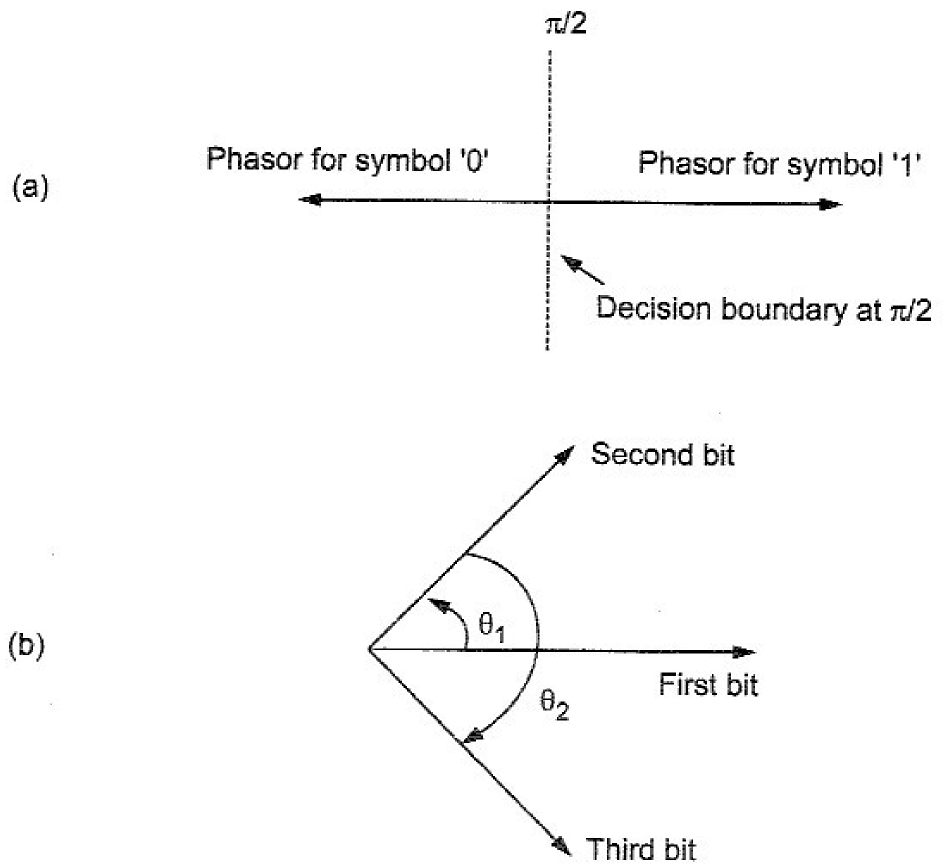


Fig. 5.13.2 (a) DPSK phasors in absence of noise (b) Shows three consecutive bits

Fig. 5.13.2 (b) shows three consecutive bits. The first bit signal contains no noise, hence its phasor is along the horizontal line. Therefore the symbol transmitted in first bit is assumed to be '1'. Because of noise there is a phase difference of 'θ_1' between first and second bit. Since $\theta_1 < \frac{\pi}{2}$, second bit is also taken as symbol '1'. The phase difference between second and third bit is θ_2. From figure it is clear that $\theta_2 > \frac{\pi}{2}$, hence third bit is taken as symbol zero. Since the phase differences are not exact between two successive bits, some error is introduced in the decision. Therefore DPSK system is called "suboptimum" in nature. If the synchronization is used to make phase differences exact, then it becomes PSK system.

Because of a the sub optimum nature of DPSK, the error probability is higher than that of BPSK. The average probability of error of non-coherent DPSK receiver is given as

$$P(e) = \frac{1}{2} e^{-E/2N_0} \quad \text{... (5.13.28)}$$

Here E is the energy per symbol and $\frac{N_0}{2}$ is spectral density of white Gaussian noise. We know that symbol duration $T = 2T_b$. Hence energy of the symbol is,

$$E = 2E_b \quad \text{... (5.13.29)}$$

∴ Equation 5.13.28 becomes,

$$P(e) = \frac{1}{2} e^{-E_b/N_0} \quad \text{... (5.12.30)}$$

This is called average probability of error or bit error rate (BER) of DPSK system.

➠ **Example 5.13.4 :** *Binary data is transmitted at a rate of 10^6 bits/second over a channel having a bandwidth 3 MHz. Assume that the noise PSD at the receiver is $\frac{N_0}{2} = 10^{-10}$ watts/Hz. Find the average carrier power required at the receiver input for coherent PSK and DPSK-signaling schemes to maintain a probability of error $P_e = 10^{-4}$*

Solution : i) PSK signaling

From equation 5.13.17 we know that, the average probability of error of PSK system is given as,

$$P_e = \frac{1}{2} erfc \sqrt{\frac{E}{N_0}}$$

Since probability of error is $P_e = 10^{-4}$, above equation becomes,

$$10^{-4} = \frac{1}{2} erfc \sqrt{\frac{E}{N_0}}$$

$$\therefore \quad 2 \times 10^{-4} = erfc \sqrt{\frac{E}{N_0}}$$

We know that

$$erf(x) = 1 - erfc(x)$$

$$\therefore \quad erf\sqrt{\frac{E}{N_0}} = 1 - erfc\sqrt{\frac{E}{N_0}}$$

$$\therefore \quad erf\sqrt{\frac{E}{N_0}} = 1 - 2\times10^{-4}$$

$$= 0.9998$$

The values of error function are listed in appendix G. Observe that $erf(2.5) = 0.99959$ and $erf(3) = 0.99998$. Hence $erf(2.6)$ will be nearly equal to 0.9998. Hence above equation becomes,

$$\sqrt{\frac{E}{N_0}} = 2.6$$

$$\frac{E}{N_0} = 6.76$$

It is given that $\frac{N_0}{2} = 10^{-10}$.

Hence $N_o = 2\times10^{-10}$. Then above equation becomes,

$$\frac{E}{2\times10^{-10}} = 6.76$$

$$\therefore \quad E = 1.352\times10^{-9} \text{ Joules}$$

The bit rate and bit duration are related as,

$$T = \frac{1}{\text{Bit rate}}$$

$$= \frac{1}{10^6}$$

The average power and energy are related as,

$$E = PT$$

Putting the values in above equation,

$$1.352\times10^{-9} = P\times\frac{1}{10^6}$$

$$\therefore \quad P = 1.352 \text{ mW}$$

Thus 1.352 mW of carrier power will be required.

ii) DPSK signaling

The error probability is given for DPSK system by equation 5.13.30 as,

$$P_e = \frac{1}{2} e^{-E_b/N_0}$$

Since $P_e = 10^{-4}$ above equation becomes,

$$10^{-4} = \frac{1}{2} e^{-E_b/N_0}$$

$$\therefore \qquad \frac{E_b}{N_o} = 8.5$$

We know that $\frac{N_0}{2} = 10^{-10}$. Hence $N_0 = 2 \times 10^{-10}$.

Then above equation becomes,

$$\frac{E_b}{2 \times 10^{-10}} = 8.5$$

$$\therefore \qquad E_b = 1.7 \times 10^{-9} \text{ Joules.}$$

As we have seen just now,

$$T = \frac{1}{\text{Bit rate}} = \frac{1}{10^6}$$

and $E = PT$

Here E is the bit energy, which is also denoted as E_b. Hence,

$$E_b = PT$$

Putting the values in above equation,

$$1.7 \times 10^{-9} = P \times \frac{1}{10^6}$$

$$\therefore \qquad P = 1.7 \text{ mW}$$

Thus 1.7 mW of carrier power will be required. This means more power is required by DPSK.

Example 5.13.5 : *A bandpass data transmission scheme uses a PSK signaling scheme with $x_2(t) = A\cos(2\pi f_0 t)$ and $x_1(t) = -A\cos(2\pi f_0 t)$, $0 \le t \le T_b$,*

$T_b = 0.2$ m sec and $f_0 = 5 f_b$. The carrier amplitude at the receiver input is 1 millivolt and

power spectral density of the additive white gaussian noise at the input is 10^{-11} watt/Hz. Assume that an ideal corrrelation receiver is used. Calculate probability of error of the receiver.

Solution : The amplitude of the carrier is,

$$A = 1\times 10^{-3}\ V$$

psd of white noise is, $\frac{N_0}{2} = 10^{-11}$ watt / Hz

Bit duration $\quad T_b = 0.2\times 10^{-3}$ sec

The normalized power of the carrier in 1Ω load is,

$$P_s = \frac{1}{2}A^2$$

Bit energy is, $\quad E_b = P_s\, T_b$ (i.e. Power × Bit duration)

$$= \frac{1}{2}A^2\, T_b$$

$$= \frac{1}{2}\times(1\times 10^{-3})^2\times 0.2\times 10^{-3} = 1\times 10^{-10}$$

Error probability of PSK is given from equation 5.13.17 as,

$$P_e = \frac{1}{2}\,erfc\sqrt{\frac{E}{N_0}}$$

$$= \frac{1}{2}\,erfc\sqrt{\frac{1\times 10^{-10}}{2\times 10^{-11}}} \qquad \text{Here } E = E_b = \text{bit energy}$$

$$= \frac{1}{2}\,erfc\,\sqrt{5}$$

Example 5.13.6 : *Find out the error probability of BFSK (Binary FSK) system for following parameters.*

Psd of white noise is, $\frac{N_0}{2} = 10^{-10}$ *watt / Hz*

Amplitude of carrier is, A = 1 millivolt at receiver input

Frequency of baseband NRZ signal is, $f_b = 1\,kHz$

Solution : The frequency of baseband signal is 1 kHz. Hence bit duration $T_b = \frac{1}{f_b} = \frac{1}{1\,\text{kHz}} = 1\,m\,sec.$

Normalized power of carrier is equal to $\left(\frac{A}{\sqrt{2}}\right)^2$ in 1Ω load resistance. Hence,

$$\text{Normalized power of carrier } P = \frac{(A/\sqrt{2})^2}{1} = \frac{A^2}{2}$$

$$\text{Bit energy} \quad E_b = PT_b = \frac{A^2}{2}T_b = \frac{(1\times10^{-3})^2}{2}\times1\times10^{-3}$$

$$= 5\times10^{-10}$$

From equation 5.13.27 error probability of FSK signal is given as,

$$P_e = \frac{1}{2}\,erfc\sqrt{\frac{0.6\,E}{N_0}} = \frac{1}{2}\,erfc\sqrt{\frac{0.6\times5\times10^{-10}}{2\times10^{-10}}}$$

$$= \frac{1}{2}\,erfc\sqrt{1.5}$$

5.13.5 Probability of Error for QPSK

May/June-2006

We have seen the signal space representation of QPSK in Fig. 5.13.3. It is shown below for convenience. In the figure observe that transmitted reference carriers are

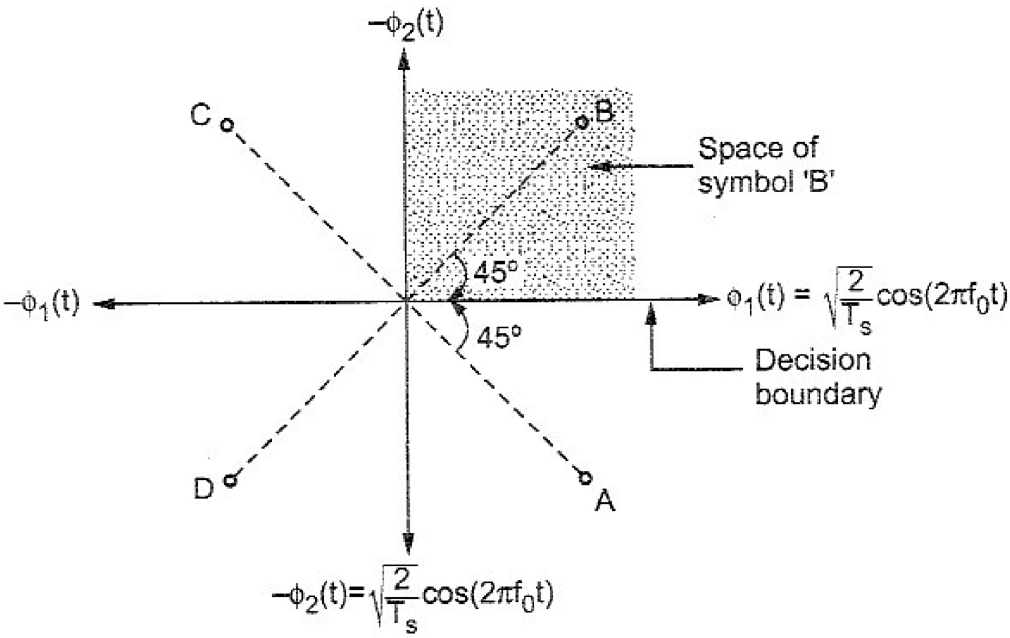


Fig. 5.13.3 Signal space diagram of QPSK

$\phi_1(t)$ and $\phi_2(t)$. All the signal vectors A, B, C and D are at 45° to these reference carriers. Consider that signal vector 'A' is transmitted. If phase shift of the reference carrier is more than 45°, it will be detected as 'B' or 'D'. It will depend upon phase shift of $\phi_2(t)$ also. Fig. 5.13.5 shows the receiver for QPSK signal. Observe that there are two correlators for two reference carriers. These two correlators are actually BPSK receivers. Error probability of BPSK, due to imperfect phase is given as,

$$P_e = \frac{1}{2} erfc \sqrt{\frac{E_b \cos^2 \theta}{N_0}} \quad \text{... (5.13.31)}$$

Hence error probability of correlator 1 is given as,

$$P_{e1} = \frac{1}{2} erfc \sqrt{\frac{E_b \cos^2 \theta}{N_0}}$$

Since both the correlators of BPSK, the error probability of correlator 2 will be same as correlator 1. i.e.,

$$P_{e2} = P_{e1} = \frac{1}{2} erfc \sqrt{\frac{E_b \cos^2 \theta}{N_0}}$$

From Fig. 5.13.5 observe that correlators detect wrong symbol if phase shift of the carrier is more that 45°. Hence putting $\theta = 45°$ in above equation,

$$P_{e1} = P_{e2} = \frac{1}{2} erfc \sqrt{\frac{E_b \cos^2 (45)}{N_0}}$$

$$= \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}} \quad \text{... (5.13.32)}$$

Hence probability of getting correct symbol can be expressed as,

$$P_c = \left(1 - P_{e_1}\right)\left(1 - P_{e_2}\right)$$

Observe that P_c is the product of probabilities of corrector 1 and 2 for getting correct symbol. From equation 5.13.32 we know that $P_{e1} = P_{e2}$. Hence above equation becomes

$$P_c = (1 - P_{e1})(1 - P_{e1})$$

$$= 1 - 2P_{e1} + P_{e1}^2$$

Normally P_{e1} is very very small (<<1). Hence P_{e1}^2 will be negligible. i.e.,

$$P_c = 1 - 2P_{e1}$$

Probability of error is given in terms of 'P_c' as,

$$P_e = 1 - P_c$$

$$= 1 - (1 - 2P_{e1})$$

$$= 2P_{e1}$$