Therefore the decision boundary will be midway between $x_{01}(T)$ and $x_{02}(T)$. It is given as,

$$\text{Decision boundary} = \frac{x_{01}(T)+x_{02}(T)}{2} \quad \text{... (4.4.1)}$$

> **Review Question**
>
> 1. *Explain what is matched filter.*

4.5 Error Rate due to Noise (Error Probability)

4.5.1 Error Conditions in Matched Filter

- Suppose that $x_2(t)$ was transmitted, but $x_{01}(T)$ is greater than $x_{02}(T)$. If noise $n_0(T)$ is positive and larger in magnitude than the voltage difference $\frac{1}{2}[x_{01}(T)+x_{02}(T)]-x_{02}(T)$, then incorrect decision will be taken. i.e. error will be generated if,

$$n_0(T) \geq \frac{x_{01}(T)+x_{02}(T)}{2}-x_{02}(T)$$

$$\therefore \quad n_0(T) \geq \frac{x_{01}(T)-x_{02}(T)}{2} \quad \text{... (4.5.1)}$$

- Similarly, let $x_1(t)$ is transmitted, but $x_{02}(T)$ is greater than $x_{01}(T)$. Then incorrect decision will be taken if noise $n_0(T)$ is less than $\frac{1}{2}[x_{01}(T)+x_{02}(T)]-x_{01}(T)$. i.e., error will be generated if,

$$n_0(T) \leq -\frac{x_{01}(T)+x_{02}(T)}{2}-x_{01}(T)$$

$$\therefore \quad n_0(T) \leq -\frac{x_{02}(T)-x_{01}(T)}{2} \quad \text{... (4.5.2)}$$

The above two error conditions are summarized in the following table :

Input x(t)	Value of $n_0(t)$ for error in the output	Probability of error
$x_2(t)$	$n_0(T) \geq \frac{x_{01}(T)-x_{02}(T)}{2}$	Probability of error will be obtained by evaluating the probability that $n_0(T) \geq \frac{x_{01}(T)-x_{02}(T)}{2}$
$x_1(t)$	$n_0(T) \leq -\frac{x_{02}(T)-x_{01}(T)}{2}$	Probability of error can be obtained by evaluating the probability that $n_0(T) \leq -\frac{x_{02}(T)-x_{01}(T)}{2}$

Table 4.5.1 : Error conditions in a matched filter

4.8 Receiving Filter-Correlation Receiver

In this section we will study a little different type of receiver which is called *correlator*. Fig. 4.8.1 shows the block diagram of this correlator.

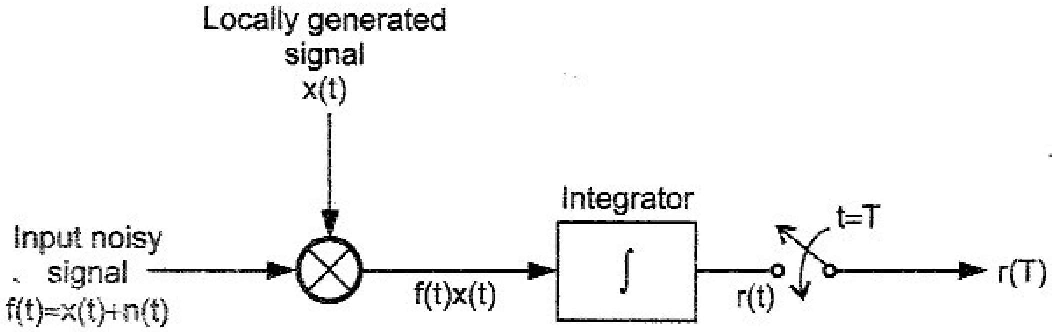

Fig. 4.8.1 Block diagram of the correlator

In the adjacent figure $f(t)$ represents input noisy signal, i.e., $f(t) = x(t) + n(t)$. The signal $f(t)$ is multiplied to the locally generated replica of input signal $x(t)$. This result of multiplication $f(t) \cdot x(t)$ is integrated. The output of the integrator is sampled at $t = T$ (i.e. end of one symbol period). Then based on this sampled value, decision is made. This is how the correlator works. *It is called correlator* since *it correlates the received signal* $f(t)$ *with a stored replica of the known signal* $x(t)$. In the block diagram of above figure, the product $f(t)\,x(t)$ is integrated over one symbol period, i.e. T. Hence output $r(t)$ can be written as,

$$r(t) = \int_0^T f(t)\, x(t)\, dt$$

At $t = T$, the above equation will be,

$$\textbf{Output of correlator : } r(T) = \int_0^T f(t)\, x(t)\, dt \qquad \text{... (4.8.1)}$$

Now let us consider the matched filter as shown in Fig. 4.8.2 below.

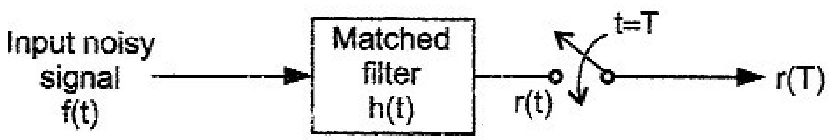

Fig. 4.8.2 Block diagram of a matched fitler receiver

In the above block diagram observe that the matched filter does not need locally generated replica of input signal $x(t)$. The output of the matched filter can be obtained by convolution of input $f(t)$ and its impulse response $h(t)$. i.e.,

$$r(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau)\, d\tau \qquad \text{... (4.8.2)}$$

From equation (4.4.3) we know that impulse response $h(t)$ of the matched filter is given as,

$$h(t) = \frac{2k}{N_0} x(T-t) \qquad \text{... (4.8.3)}$$

$$\therefore \quad h(t-\tau) = \frac{2k}{N_0} x(T-t+\tau)$$

Putting this value of $h(t-\tau)$ in equation (4.8.2) we get,

$$r(t) = \int_{-\infty}^{\infty} f(\tau) \frac{2k}{N_0} x(T-t+\tau)\, d\tau$$

Since the integration is performed over one bit period, we can change integration limits from 0 to T. i.e.,

$$r(t) = \frac{2k}{N_0} \int_0^T f(\tau)\, x(T-t+\tau)\, d\tau$$

At $t = T$, the above equation will be,

$$r(T) = \frac{2k}{N_0} \int_0^T f(\tau)\, x(T-T+\tau)\, d\tau = \frac{2k}{N_0} \int_0^T f(\tau)\, x(\tau)\, d\tau$$

Let us put $\tau = t$ just for convenience of notation,

$$\textbf{Output of matched filter : } r(T) = \frac{2k}{N_0} \int_0^T f(t)\, x(t)\, dt \qquad \text{... (4.8.4)}$$

This equation gives the output of matched filter. Observe that this equation and equation (4.8.1) (which gives output or correlator) are identical. In above equation the constant $\frac{2k}{N_0}$ is present which can be normalized to 1. The similarity between equation (4.8.1) and equation (4.8.4) shows that the matched filter and correlator gives same output. Therefore we can state,

The matched filter and correlator are two distinct, independent techniques which give the same result. These two techniques are used to synthesize the optimum filter.

Review Question

1. *What is correlator ? What is the difference and similarity between correlator and matched filter ?*

4.9 Intersymbol Interference

4.9.1 Baseband Transmission of Binary Data

- We know that the binary data can be transmitted in baseband or passband. In passband transmission the binary data modulates some carrier and the modulated carrier is transmitted over the channel. In baseband transmission, there is no modulation of high frequency carrier.
- One of the baseband system for transmission of digital data is discrete pulse amplitude modulation (PAM). In this discrete PAM, the amplitude of the pulses varies in discrete manner according to the input binary data.
- The discrete PAM can have only two amplitude levels corresponding to binary '1' and '0'. Successive binary digits can be combined into symbols. There can be multiple amplitude levels corresponding to these symbols. They generate discrete PAM signals.
- These signals are transmitted (without any modulation) over the channel in baseband transmission. Fig. 4.9.1 shows the block diagram of such baseband transmission system. The binary data $\{b_k\}$ is applied to the data encoder. The data encoder generates the pulse waveform $x(t)$. This waveform can be represented mathematically as,

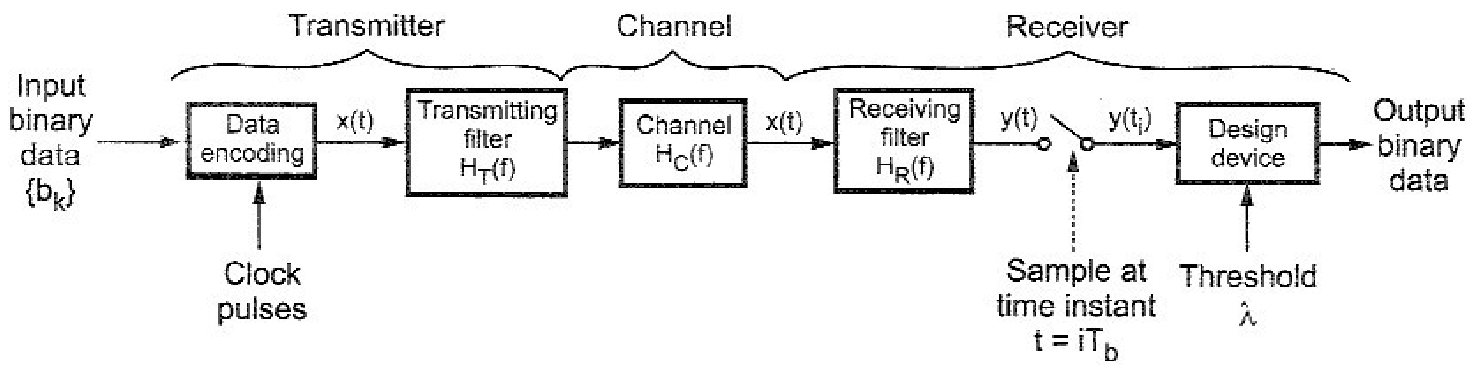

Fig. 4.9.1 Block diagram of baseband binary data transmission system

$$x(t) = \sum_{k=-\infty}^{\infty} A_k \, g\left(t - kT_b\right) \quad \ldots (4.9.1)$$

Here T_b is the duration of each input binary bit.

$g(t)$ is the shaping pulse.

And $$A_k = \begin{cases} +a & if \quad b_k = 1 \\ -a & if \quad b_k = 0 \end{cases} \qquad \dots (4.9.2)$$

- The signal $x(t)$ is then passed through the transmitting filter. The transmitting filter combines all the necessary transmitting circuits and systems. The combined transfer function of the transmitting filter is $H_T(f)$. The signal is then passed through the channel having the transfer function $H_C(f)$. The channel delivers the signal to the receiving filter. It consists of all the necessary receiving circuits and systems. The combined transfer function of the receiving filter is $H_R(f)$. The output of the receiving filter is $y(t)$. This $y(t)$ is noisy replica of the transmitted signal $x(t)$.
- The signal $y(t)$ is sampled synchronously with the transmitter. The sampling instants are $t = i\,T_b$. These sampling instants are synchronous to the clock pulses at the transmitter. The sampled signal $y(t_i)$ is then given to the decision device. The decision device compares the input signal with threshold 'λ'. Then the decision is taken as follows :

  If $y(t_i) > \lambda$ select symbol '1'.

  If $y(t_i) \leq \lambda$ select symbol '0'.

4.9.2 Intersymbol Interference (ISI) Problem

Nov./Dec. - 2005

- Consider the output $y(t)$ of the receiving filter in Fig. 4.9.1. $y(t)$ can be given in terms of A_k as,

$$y(t) = \mu \sum_{k=-\infty}^{\infty} A_k\, p(t - kT_b) \qquad \dots (4.9.3)$$

Here μ is the scaling factor

$p(t)$ is the shape different from that of $g(t)$.

In Fig. 4.9.1 observe that $A_k\, g(t)$ is the signal applied to the input of cascade of transmitting filter, channel and receiving filter. The output of this cascade connection is $\mu\, A_k\, p(t)$.

- Let the fourier transform of $g(t)$ be $G(f)$ and that of $p(t)$ be $P(f)$. Then in frequency domain we can write,

$$\mu\, A_k\, P(f) = H(f)\, A_k\, G(f) \qquad \dots (4.9.4)$$

Here $H(f)$ is the combined transfer function of transmitting filter, channel and receiving filter. It is given for the cascade connection as,

$$H(f) = H_T(f)\, H_C(f)\, H_R(f) \qquad \dots (4.9.5)$$

Hence equation (4.9.4) becomes,

$$\mu P(f) = H_T(f) H_C(f) H_R(f) G(f) \quad \text{... (4.9.6)}$$

The receiving filter output is sampled at $t_i = i\,T_b$.

- From equation (4.9.3), at $t = i\,T_b$ we can write,

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} A_k\, p\left(i\,T_b - k\,T_b\right)$$

$$= \mu \sum_{k=-\infty}^{\infty} A_k\, p\left[(i-k)\,T_b\right] \quad \text{... (4.9.7)}$$

Let us rearrange the above equation as follows :

$$y(t_i) = \mu A_i\, p(0) + \mu \sum_{\substack{k=-\infty \\ k\neq i}}^{\infty} A_k\, p\left[(i-k)\,T_b\right] \quad \text{... (4.9.8)}$$

- The first term represents the value of $y(t_i)$ when $i = k$. $p(t)$ is normalized such that $p(0) = 1$. Hence above equation becomes,

$$y(t_i) = \mu A_i + \mu \sum_{\substack{k=-\infty \\ k\neq i}}^{\infty} A_k\, p\left[(i-k)\,T_b\right] \quad \text{... (4.9.9)}$$

And $i = 0, \pm 1, \pm 2, \pm 3, \ldots$

Comments

i) The first term in above equation is μA_i. It is the contribution of the i^{th} transmitted bit.

ii) The second term represents the residual effect of all other bits transmitted before and after the sampling instant t_i.

Definition of ISI

The presence of outputs due to other bits (symbols) interfere with the output of required bit (symbol). This effect is called Intersymbol Interference (ISI). Here note that we have not considered the effect of channel noise. Actually, channel noise and ISI both interfere the transmitted signal.

- If the intersymbol interference is absent, then the second term will not be present in equation (4.9.7) i.e.,

$$y(t_i) = \mu A_i \quad \text{... (4.9.10)}$$

At $t = i\,T_b$, the correct bit is A_i. Observe that it is decoded correctly in absence of ISI. It is not possible to eliminate the second term of equation (4.9.7) (and hence ISI) totally. The ISI can be reduced by proper design of pulse spectrum $G(f)$, transmit filter $H_T(f)$, receive filter $H_R(f)$ and the channel $H_C(f)$. We will discuss some of these issues in subsequent sections.

4.10 Signal and System Design for ISI Elimination

April/May-2005, Nov./Dec.-2005

4.10.1 Nyquist Pulse Shaping Criterion

Time Domain Criterion

From equation 4.9.9 we know that the second term (summation) must be zero to eliminate effect of ISI. This is possible if the received pulse p(t) is controlled such that,

$$p\left[(i-k)T_b\right] = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \qquad \text{... (4.10.1)}$$

If p(t) satisfies the above condition, then we get a signal which is free from ISI. i.e.,

$$y(t_i) = \mu A_i \qquad \text{from equation (4.9.10)}$$

Hence equation (4.10.1) gives the condition for perfect reception in absence of noise. Equation (4.10.1) is the condition in time domain. This condition gives more useful criteria in frequency domain.

Criterion in Frequency Domain

- Let $p(nT_b)$ represent the impulses at which p(t) is sampled for decision. These samples are taken at the rate of T_b. Fourier spectrum of these impulses is given as

$$P_\delta(f) = f_b \sum_{n=-\infty}^{\infty} P(f - nf_b) \qquad \text{... (4.10.2)}$$

This means the spectrums of p(t) are periodic with period f_b. Here note that the sampling frequency (instants) is f_b. Here $P_\delta(f)$ represents the spectrum of $p(nT_b)$ and P(f) is the spectrum of p(t).

- We can think of $p(nT_b)$ as the infinite length of impulses with period T_b, which are weighted with amplitudes of p(t). i.e.,

$$p_\delta(t) = \sum_{n=-\infty}^{\infty} p(nT_b)\,\delta(t - nT_b) \qquad \text{... (4.10.3)}$$

Example 4.10.1 : *The output of a digital computer is at a rate of 64 kbps. If the roll off factors (i) $\alpha = 1$, (ii) $\alpha = 0.5$, (iii) $\alpha = 0.25$, (iv) $\alpha = 0$, find the bandwidth required to transmit the data in each case.*

Solution : Data rate is

$$f_b = 64 \text{ kbps}$$

$$\therefore \text{Bandwidth} \quad B_0 = \frac{f_b}{2} = 32 \text{ kbps}$$

Equation 4.9.18 gives the bandwidth required using raised cosine spectrum i.e.,

$$B = B_0(1+\alpha)$$

i) For $\alpha = 1$, bandwidth becomes,

$$B = 32 \times 10^3 (1+1) = 64 \text{ kHz}$$

ii) For $\alpha = 0.5$, $B = 32 \times 10^3 (1+0.5) = 48 \text{ kHz}$

iii) For $\alpha = 0.25$ $B = 32 \times 10^3 (1+0.25) = 40 \text{ kHz}$

iv) For $\alpha = 0$ $B = B_0 = 32 \text{ kHz}$

Thus as rolloff factor decreases bandwidth also decreases.

4.11 Eye Pattern Analysis

April/May- 2005

- The Eye Pattern is used to study the effect of ISI in baseband digital transmission.
- When the sequence is transmitted over a baseband binary data transmission system of Fig. 4.11.1 the signal obtained at the output i.e. $y(t)$ is a continuous time signal as shown in Fig. 4.11.1. Ideally this signal should go high and low depending on the symbol that was transmitted. But because of the nature of transmission channel, the signal becomes continuous with increasing and decreasing amplitudes. Fig. 4.11.1 (a) shows the binary sequence that is transmitted and Fig. 4.11.1 (b) shows the signal $y(t)$ obtained at the output. Fig. 4.11.1 (b) also shows various sampling instants t_1, t_2, t_3 .... etc. Thus based on the signal obtained over the period T_b between two sampling instants, decision is taken by the decision device. If we cut the signal $y(t)$ shown in Fig. 4.11.1 (b) in each interval (T_b) and place it over one another, then we obtain the diagram as shown in Fig. 4.11.1 (c). This diagram is called Eye pattern of the signal $y(t)$.

- The name 'eye' is given because it looks like an eye. This pattern can also be obtained on CRO if we apply $y(t)$ to one of the input channels and apply an external trigger signal of $1 / T_b$ Hz. This makes one sweep of beam equal to 'T_b' seconds. Therefore the pattern shown in Fig. 4.11.1 (c) will be obtained. When there are large number of bits of the sequence, then eye patterns will be as shown in Fig. 4.11.1 (d).

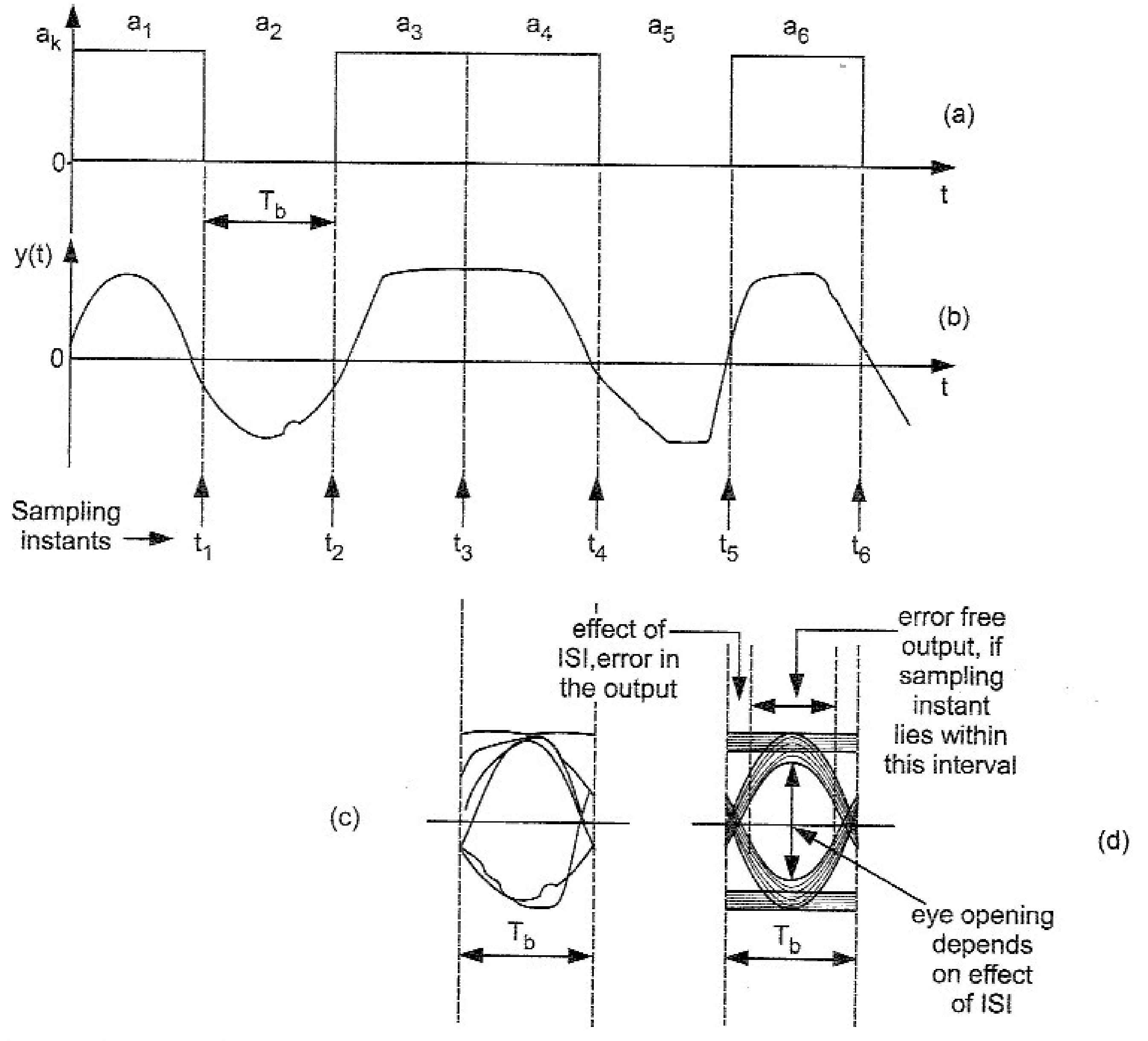

Fig. 4.11.1 (a) Binary sequence transmitted (b) Received signal by baseband transmission system (c) Eye pattern of signal in (b) (d) Eye pattern for large number of bits in waveform $y(t)$

4.11.1 Performance of the Data Transmission System using Eye Pattern

- Various important conclusions can be derived from eye pattern. Fig. 4.11.2 shows various points related to eye pattern.

  i) The width of the eye opening defines the interval over which the received wave can be sampled without error from intersymbol interference. It is preferable to sample the instant at which eye is open widest. The instant is shown as best sampling time in Fig. 4.11.2.

ii) The sensitivity of the system to timing error is determined by the rate of closure of the eye as the sampling time is varied.

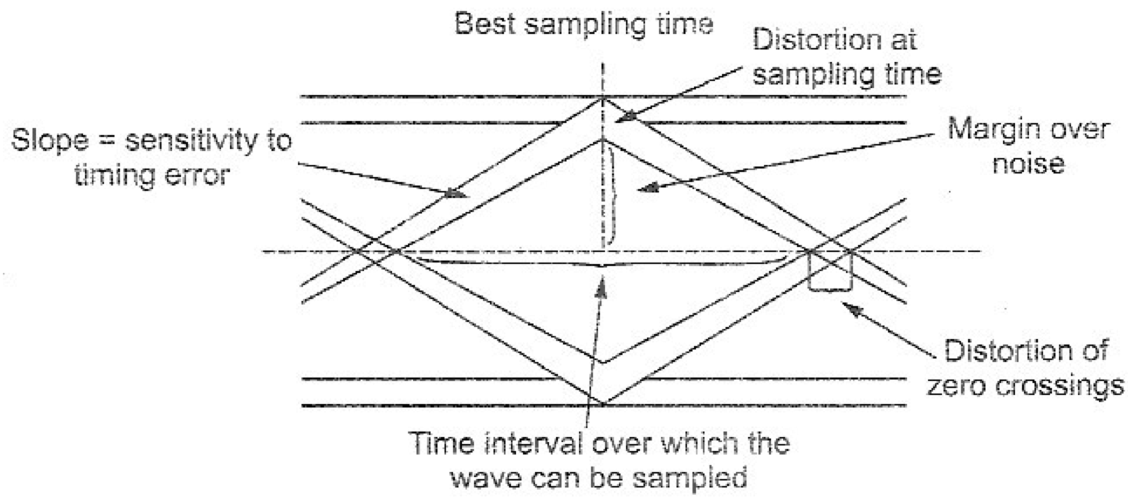

Fig. 4.11.2 Interpretation of the eye pattern

iii) The height of the eye opening, at the specified sampling time, is called margin over the noise.

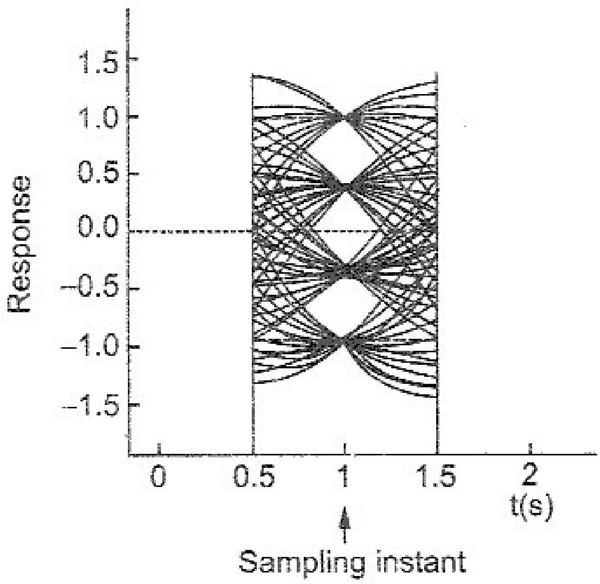

Fig. 4.11.3 Eye diagram for 4-level system

- As the effect of inter symbol interference increases, the eye opening reduces. If the eye is closed completely, then it is not possible to avoid errors in the output.
- All the above description is for two level (binary) system. If there are M-levels (M-ary system), then eye pattern contains (M-1) eye openings stacked vertically one upon the other. Fig. 4.11.3 shows the eye diagram for 4 level (M = 4) system. Therefore there are 3 eye openings.

4.12 Equalizing Filters

In the baseband transmission system, channel noise and intersymbol interference act together. The optimum linear receiver can be used for such transmission system.

4.12.1 Zero Forcing Equalizer

The optimum linear receiver is realized with the help of zero forcing equalizer. This equalizer forces the ISI to zero. Refer the block diagram of baseband data transmission system given is Fig. 4.9.1. The signal $y(t)$ at the output of receiving filter is given as,

$$y(t) = \int_{-\infty}^{\infty} c(\tau)\; x(t-\tau)\; d\tau$$

Here $c(t)$ is impulse response of receive filter and $x(t)$ is an input signal to receive filter. It is given as,

$$x(t) = \sum_{k=-\infty}^{\infty} A_k q(t - kT_b) + w(t)$$

Here A_k is the symbol transmitted at $t = kT_b$ and $w(t)$ is the channel noise. If $y(t)$ is sampled at iT_b,

$$y(iT_b) = z_i + n_i$$

Here z_i is the signal component and n_i is the noise component. Let A_i be the transmitted symbol. Then error between transmitted and received symbol will be,

$$e_i = y(iT_b) - A_i$$

Therefore mean square error is,

$$\text{MSE} = \frac{1}{2} E[e_i^2]$$

This mean square error is minimum for following condition,

$$\int_{-\infty}^{\infty} \left[R_q(t-\tau) + \frac{N_0}{2} \delta(t-\tau) \right] c(\tau)\; d\tau = q(-t)$$

Here $R_q(t, \tau)$ is temporal autocorrelation function of the sequence $q(kT_b)$. Above equation can be solved to get $c(t)$. It gives impulse response of the zero forcing equalizer.

4.13 Equalization Techniques

Necessity of Equalization

When the signal is passed through the channel, distortion is introduced in terms of i) amplitude and ii) delay. This distortion creates the problems of ISI. The detection of the signal also becomes difficult. This distortion can be compensated with the help of *equalizers*. Equalizers are basically filters, which correct the channel distortion. Fig. 4.13.1 shows channel and equalizer for correction of distortion.

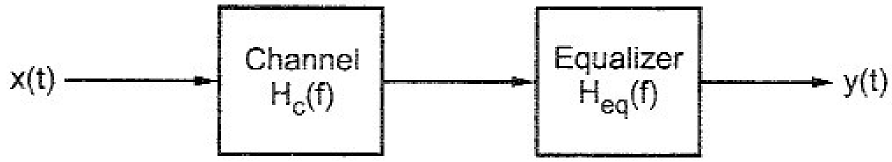

Fig. 4.13.1 Equalizer for correction of distortion introduced in the channel

- The transfer function of distortionless system is given as,

$$H(f) = Ke^{-j2\pi f t_0}$$

- The cascade connection of channel + equalizer shown in above figure will be distortionless if,

$$H_c(f) \cdot H_{eq}(f) = Ke^{-j2\pi f t_0}$$

- Hence transfer function of the equalizer will be,

$$H_{eq}(f) = \frac{Ke^{-j2\pi f t_0}}{H_c(f)} \qquad \text{... (4.13.1)}$$

The equation is difficult to realize directly, but approximations are available. It can be implemented with the help of tapped delay line filters.

4.13.1 Tapped Delay Line Filter

Fig. 4.13.2 shows tapped delay line filter.

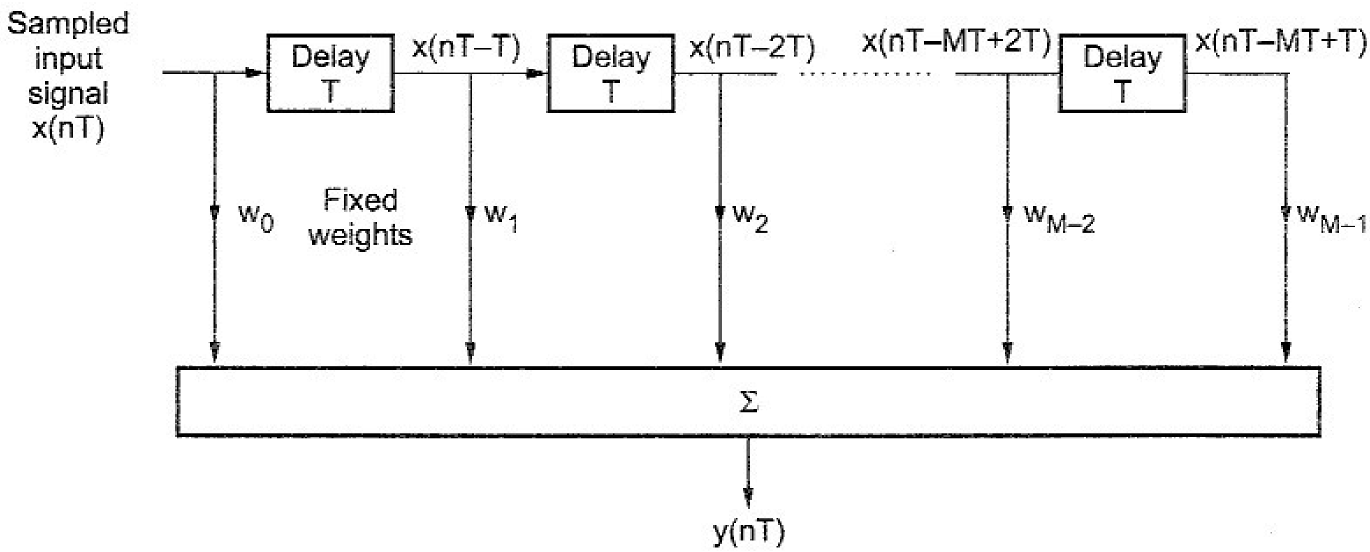

Fig. 4.13.2 Tapped delay line filter

The output of above filter is given as,

$$y(nT) = \sum_{i=0}^{M-1} w_i x(nT - iT) \qquad \text{... (4.13.2)}$$

Here w_i is the weight of i^{th} tap.

M is the total number of taps

and T is the symbol duration of the signal.

The weights are basically filter coefficients. This filter approximates the equalizer transfer function of equation (4.13.1). The approximation will be more accurate if we use more taps in the filter. The weights are calculated once as per the characteristics of channel. Hence this is fixed filter.

4.13.2 Adaptive Equalization

April/May- 2005

Necessity :

Most of the channels are made up of individual links. For example, in the switched telephone network, the distortion induced depends upon

i) transmission characteristics of individual links and

ii) number of links in connnection

Hence, the fixed pair of transmit and receive filters will not serve the equalization problem completely. The transmission characteristics of the channel keep on changing. Hence *adaptive equalization* is used.

Basic Principle :

In adaptive equalization, the filters adapt themselves to the dispersive effects of the channel. That is the coefficients of the filters are changed continuously according to the received data. The filter coefficients are changed in such a way that the distortion in the data is reduced.

Types :

When an equalization is done at the transmitting side it is called prechannel equalization. This type of equalization requires feedback to know the amount of distortion in the received data. When an equalization is done at the receiving side, it is called postchannel equalization. In this case, no feedback is required. The equalizer is placed after the receiving filter in the receiver.

Block diagram

The adaptive equalizer shown in figure below is a tapped-delay-line filter. It consists of set of delay elements and variable multipliers. The sequence $x(nT)$ is applied to the input of the adaptive filter. The output $y(nT)$ of the adaptive filter will be,

$$y(nT) = \sum_{i=0}^{M} w_i \, x(nT - iT) \qquad \dots (4.13.3)$$

The weights w_i on the taps are basically adaptive filter coefficients. A known sequence $\{d(nT)\}$ is transmitted first. This sequence is known to the receiver. The response sequence $y(nT)$ is observed. As shown in Fig. 4.13.3, the error sequence between the two sequences is calculated. i.e.,

$$e(nT) = d(nT) - y(nT), \qquad n = 0, 1, \ldots N-1 \qquad \ldots (4.13.4)$$

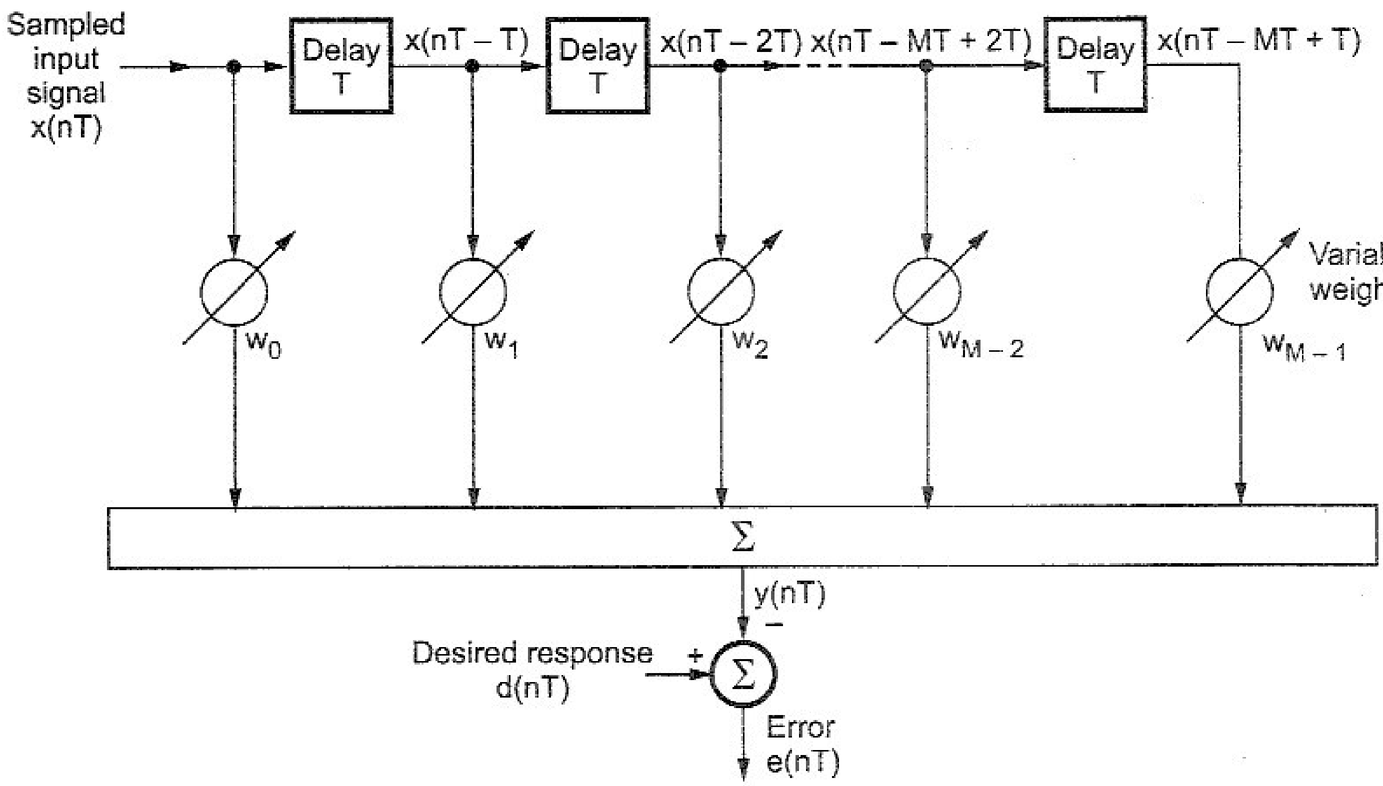

Fig. 4.13.3 Structure of adaptive equalizer

Here note that if there is no distortion in the channel, then $d(nT)$ and $y(nT)$ will be exactly same producing zero error sequence. Then the weights of the filter i.e. w_i are changed recursively such that error $e(nT)$ is minimized. There are standard algorithms to change weights of the filter recursively.

Least Mean Square (LMS) Algorithm :

This is one of the algorithm to change the tap weights of the adaptive filter recursively. The tap weights are adapted by this algorithm as follows :

$$\hat{w}_i\left(nT+T\right) = \hat{w}_i(nT) + \mu\, e(nT)\, x(nT - iT) \qquad \ldots (4.13.5)$$

Here $i = 0, 1, \ldots\ldots M-1$

$\hat{w}_i(nT)$ is the present estimate for tap 'i' at time nT.

$\hat{w}_i(nT+T)$ is the updated estimate for tap 'i' at time nT

μ is the adaption constant.

$x(nT - iT)$ is the filter input and

$e(nT)$ is the error signal.

The parameter μ controls the amount of correction applied to the old estimate to produce updated estimate. With the help equation (4.13.5), the tap weights are obtained in recursive manner. In this algorithm, initial tap weights are assumed zero.