Here $f_s = 1.5 f_{max}$. Then above equation will be,

$$X_\delta(f) = 1.5 f_{max} \cdot X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} 1.5 f_{max} X(f - 1.5 n f_{max})$$

With $f_{max} = 1$ Hz, above equation will be,

$$X_\delta(f) = 1.5 X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} 1.5 X(f - 1.5 n)$$

Fig. 2.1.16 (a) shows the plot of above equation.

Because of under sampling ($f_s = 1.5 f_{max}$), there is aliasing effect and it is visible in Fig. 2.1.16 (a). When the sampled signal is passed through an ideal low pass filter of bandwidth f_{max} [Fig. 2.1.16 (b)], the spectrum of the corresponding output signal is shown in Fig. 1.1.16 (c). It shows clearly that the spectrum of Fig. 2.1.15 and Fig. 2.1.16 (c) are not same because of aliasing.

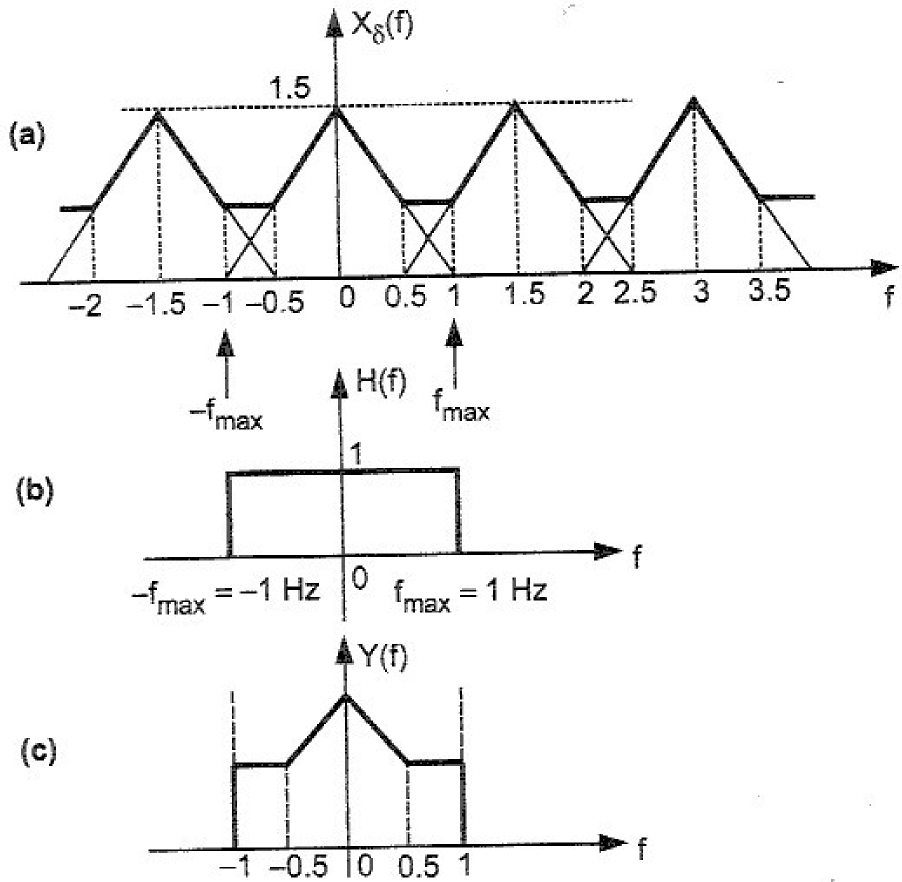

Fig. 2.1.16 (a) Spectrum of the sampled signal at $f_s = 1.5 f_{max}$

(b) Response of ideal low pass filter with $|H(f)| = 1$ for $-f_{max} < f < f_{max}$

(c) Spectrum of signal at the output of the low pass filter

Review Questions

1. *State and prove the sampling theorem in time domain. What is Nyquist rate ?*
2. *Explain the function of low pass filter in sampling.*
3. *State and prove the sampling theorem in frequency domain. Show that the effect of sampling is to produce double sided spectra around each harmonic of sampling frequency.*
4. *A bandlimited signal $f(t)$ is sampled by train of rectangular pulses of width τ and period T.*

   (a) Find an expression for the sampled signal.

   (b) Determine the spectrum of the sampled signal and sketch it.
5. *What is aliasing and how it is reduced ?*

Unsolved Examples

1. *Specify the Nyquist rate and the Nyquist interval for each of the following signals.*

   (a) $x(t) = \text{sinc}\,(200\,t)$

   (b) $x(t) = \text{sinc}^2\,(200\,t)$ (c) $x(t) = \text{sinc}\,(200\,t) + \text{sinc}^2\,(200\,t)$

   [Ans : (a) Rate : 200, Interval : $\frac{1}{200}$, (b) Rate : 400, Interval : $\frac{1}{400}$

   (c) Rate : 400, Interval : $\frac{1}{400}$]
2. *Consider the analog signal* $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$. *What is the Nyquist rate for this signal ?* **[Ans. : Nyquist rate = 300 Hz]**
3. *Find the Nyquist rate and Nyquist interval for the following.*

   i) $m(t) = \frac{1}{2\pi}\cos(400\pi t)\cdot\cos(200\pi t)$ **[Ans. : Rate = 600 Hz, Interval = 1.6667 ms]**

   ii) $m(t) = \frac{\sin \pi t}{\pi t}$ **[Ans. : Rate 1 Hz, Interval = 1 sec]**
4. *The signal* $x(t) = \cos 200\pi t + 0.25\cos 700\pi t$ *is sampled at the rate of 400 samples per second. Sampled waveform is then passed through an ideal low pass filter with 200 Hz bandwidth. Write an expression for filter output. Sketch the frequency spectrum of sampled waveform.*

   [Ans. : $y(t) = \cos(2\pi\times 100t) + 0.25\cos(2\pi\times 50f)$]
5. *A waveform* $[20 + 20\sin(500t + 30°)]$ *is to be sampled periodically and reproduced from these sample values. Find maximum allowable time interval between sample values. How many sample values are needed to be stored in order to reproduce 1 sec of this waveform ?*

   [Ans. : T_s = 6.283 ms, Samples = 160]

2.3 Uniform Quantization (Linear Quantization)

We know that input sample value is quantized to nearest digital level. This quantization can be uniform or nonuniform. In uniform quantization, the quantization step or difference between two quantization levels remains constant over the complete amplitude range. Depending upon the transfer characteristic there are three types of uniform or linear quantizers as discussed next.

2.3.1 Midtread Quantizer

The transfer characteristic of the midtread quantizer is shown in Fig. 2.3.1.

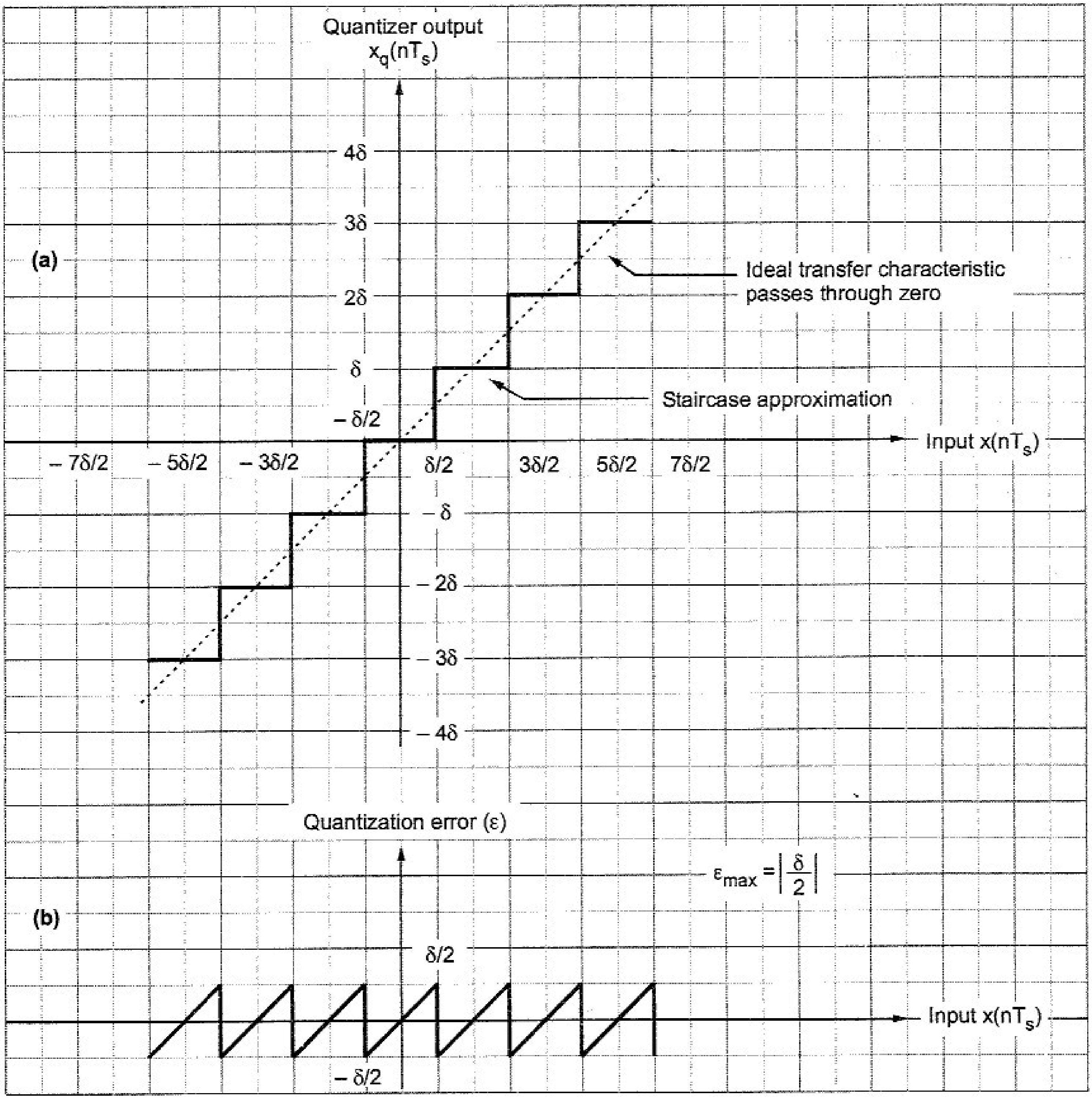

Fig. 2.3.1 (a) Quantization characteristic of midtread quantizer (b) Quantization error

As shown in this Fig. 2.3.1, when an input is between $-\delta/2$ and $+\delta/2$ then the quantizer output is zero. i.e.,

For $\quad -\delta/2 \le x(nT_s) < \delta/2 \,; \quad x_q(nT_s) = 0$

Here 'δ' is the step size of the quantizer.

for $\quad \delta/2 \le x(nT_s) < 3\,\delta/2 \,; \quad x_q(nT_s) = \delta$

Similarly other levels are assigned. It is called midtread because quantizer output is zero when $x(nT_s)$ is zero. Fig. 2.3.1 (b) shows the quantization error of midtread quantizer. Quantization error is given as,

$$\varepsilon = x_q(nT_s) - x(nT_s) \quad \text{... (2.3.1)}$$

In Fig. 2.3.1 (b) observe that when $x(nT_s) = 0$, $x_q(nT_s) = 0$. Hence quantization error is zero at origin. When $x(nT_s) = \delta/2$, quantizer output is zero just before this level. Hence error is $\delta/2$ near this level. From Fig. 2.3.1 (b) it is clear that,

$$-\delta/2 \le \varepsilon \le \delta/2 \quad \text{... (2.3.2)}$$

Thus quantization error lies between $-\delta/2$ and $+\delta/2$. And maximum quantization error is, maximum quantization error, $\varepsilon_{max} = \left|\frac{\delta}{2}\right|$. ... (2.3.3)

2.3.2 Midriser Quantizer

The transfer characteristic of the midriser quantizer is shown in Fig. 2.3.2.

In Fig. 2.3.2 observe that, when an input is between 0 and δ, the output is $\delta/2$. Similarly when an input is between 0 and $-\delta$, the output is $-\delta/2$. i.e.,

For $\quad 0 \le x(nT_s) < \delta \,; \quad x_q(nT_s) = \delta/2$

$\quad -\delta \le x(nT_s) < 0 \,; \quad x_q(nT_s) = -\delta/2$

Similarly when an input is between $3\,\delta$ and $4\,\delta$, the output is $7\,\delta/2$. This is called midriser quantizer because its output is either $+\delta/2$ or $-\delta/2$ when input is zero.

Fig. 2.3.2 (b) shows the quantization error in midriser quantization. When input $x(nT_s) = 0$, the quantizer will assign the level of $\delta/2$. Hence quantization error will be,

$$\varepsilon = x_q(nT_s) - x(nT_s)$$

$$= \delta/2 - 0 = \delta/2$$

Thus the quantization error lies between $-\delta/2$ and $+\delta/2$. i.e.,

$$-\delta/2 \le \varepsilon \le \delta/2 \quad \text{... (2.3.4)}$$

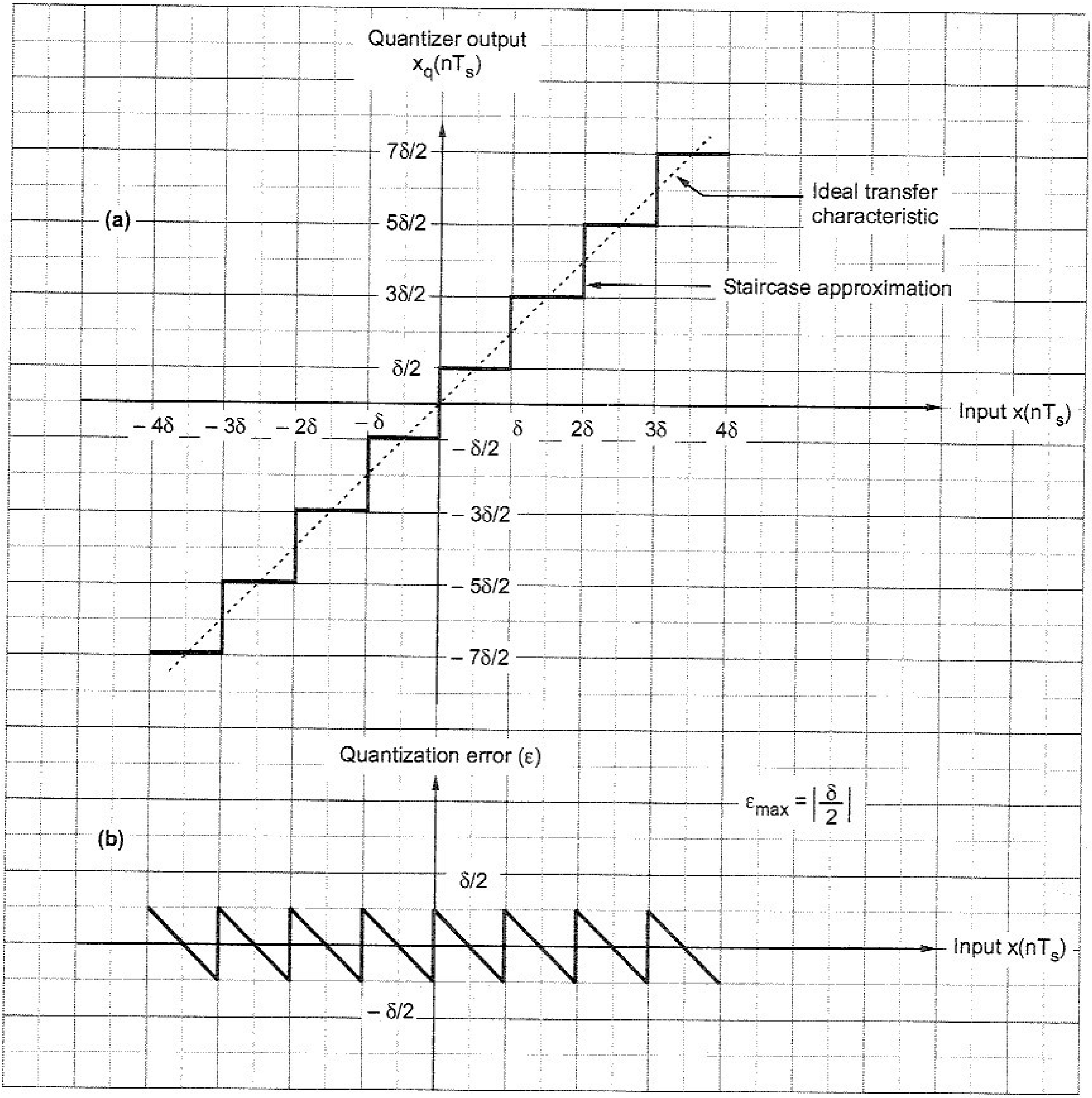

Fig. 2.3.2 (a) Transfer characteristic of midriser quantizer

(b) Quantization error

And the maximum quantization error is,

$$\varepsilon_{max} = \left|\frac{\delta}{2}\right| \qquad \text{... (2.3.5)}$$

In both the midriser and midtread quantizers, the dotted line of unity slope passes through origin. It represents ideal nonquantized input output characteristic. The staircase characteristic is an approximation of this line. The difference between the staircase and unity slope line represents the quantization error.

2.3.3 Biased Quantizer

Fig. 2.3.3 shows the transfer characteristic of biased uniform quantizer.

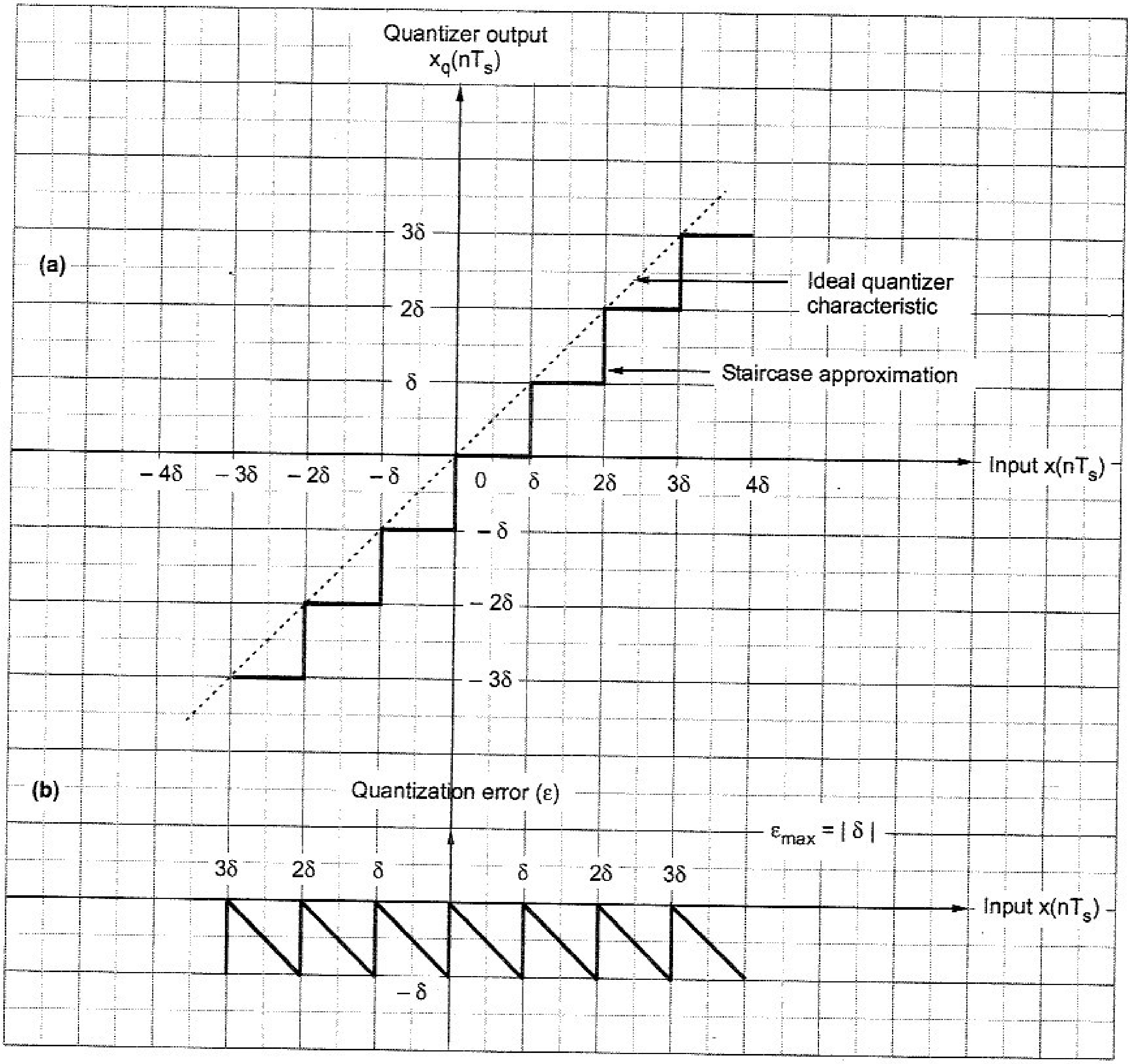

Fig. 2.3.3 (a) Biased quantizer transfer characteristic

(b) Quantization error

The midriser and midtread quantizers are rounding quantizers. But biased quantizer is truncation quantizer. This is clear from above diagram. When input is between 0 and δ, the output is zero. i.e.,

For $\quad 0 \leq x(nT_s) < \delta\,; \quad x_q(nT_s) = 0$

Similarly, for $\quad -\delta \leq x(nT_s) < 0\,; \quad x_q(nT_s) = -\delta$

Fig. 2.3.3 shows quantization error. When input is δ, output is zero. Hence quantization error is,

$$\varepsilon = x_q(nT_s) - x(nT_s)$$
$$= 0 - \delta = -\delta$$

Thus the quantization error lies between 0 and – δ. i.e.,

$$-\delta \leq \varepsilon \leq 0 \quad \text{... (2.3.6)}$$

And the maximum quantization error is,

$$\varepsilon_{max} = |\delta| \quad \text{... (2.3.7)}$$

Thus the quantization error is more in biased quantizer compared to midriser and midtread quantizers. The unity slope dotted line passes through origin. It represents ideal nonquantized transfer characteristic. The difference between staircase and dotted line gives quantization error.

2.4 Encoding Techniques for Analog Sources

Following figure lists the classification of encoding techniques for analog sources :

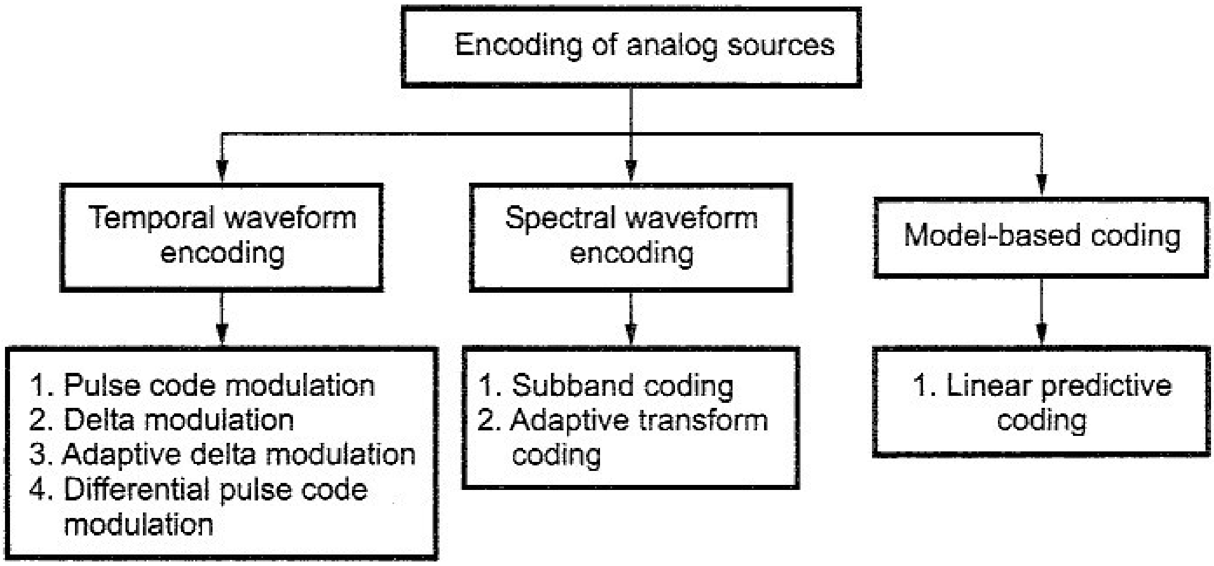

Fig. 2.4.1 Enconding techniques for analog sources

The above encoding methods are discussed further in this chapter.

2.5 Temporal Waveform Encoding - Pulse Code Modulation

2.5.1 PCM Generator

The pulse code modulator technique samples the input signal $x(t)$ at frequency $f_s \geq 2W$. This sampled 'Variable amplitude' pulse is then digitized by the analog to digital converter. The parallel bits obtained are converted to a serial bit stream. Fig. 2.5.1 shows the PCM generator.

We know that quantization levels q and number of bits v are related as,

$$q = 2^v$$

$$\therefore \quad 1024 = 2^v$$

$$\therefore \quad v = 10 \text{ bits}$$

The number of bits/sec generated by PCM system is called bit rate or signalling rate. i.e.,

$$\text{Signalling rate, } r = v f_s$$

$$= 10 \times 10.8 \times 10^6 \text{ bits/sec}$$

$$= 108 \times 10^6 \text{ bits/sec}$$

The output bit rate does not change if linear PCM is converted into companded PCM. Companded PCM is used to improve the signal to noise ratio.

2.7 Nonuniform Quantization

In nonuniform quantization, the step size is not fixed. It varies according to certain law or as per input signal amplitude. Fig. 2.7.1 shows the transfer characteristic and error in nonuniform quantization. (See Fig. 2.7.1 on next page.)

In this Fig. 2.7.1 observe that step size is small at low input signal levels. Hence quantization error is also small at these inputs. Therefore signal to quantization noise power ratio is improved at low signal levels. Step size is higher at high input levels. Hence signal to noise power ratio remains almost same throughout the dynamic range of quantizer.

2.7.1 Necessity of Nonuniform Quantization

In uniform quantization, the quantizer has a linear characteristics as we have seen in Fig. 2.3.2 (a). The step size also remains same throughout the range of quantizer. Therefore over the complete range of inputs, the maximum quantization error also remains same. From equation (2.3.5) the quantization error is given as,

$$\text{Maximum quantization error} = \varepsilon_{max} = \left|\frac{\delta}{2}\right| \qquad \dots (2.7.1)$$

From equation (2.6.3) step size 'δ' is given as,

$$\delta = \frac{2\, x_{max}}{q}$$

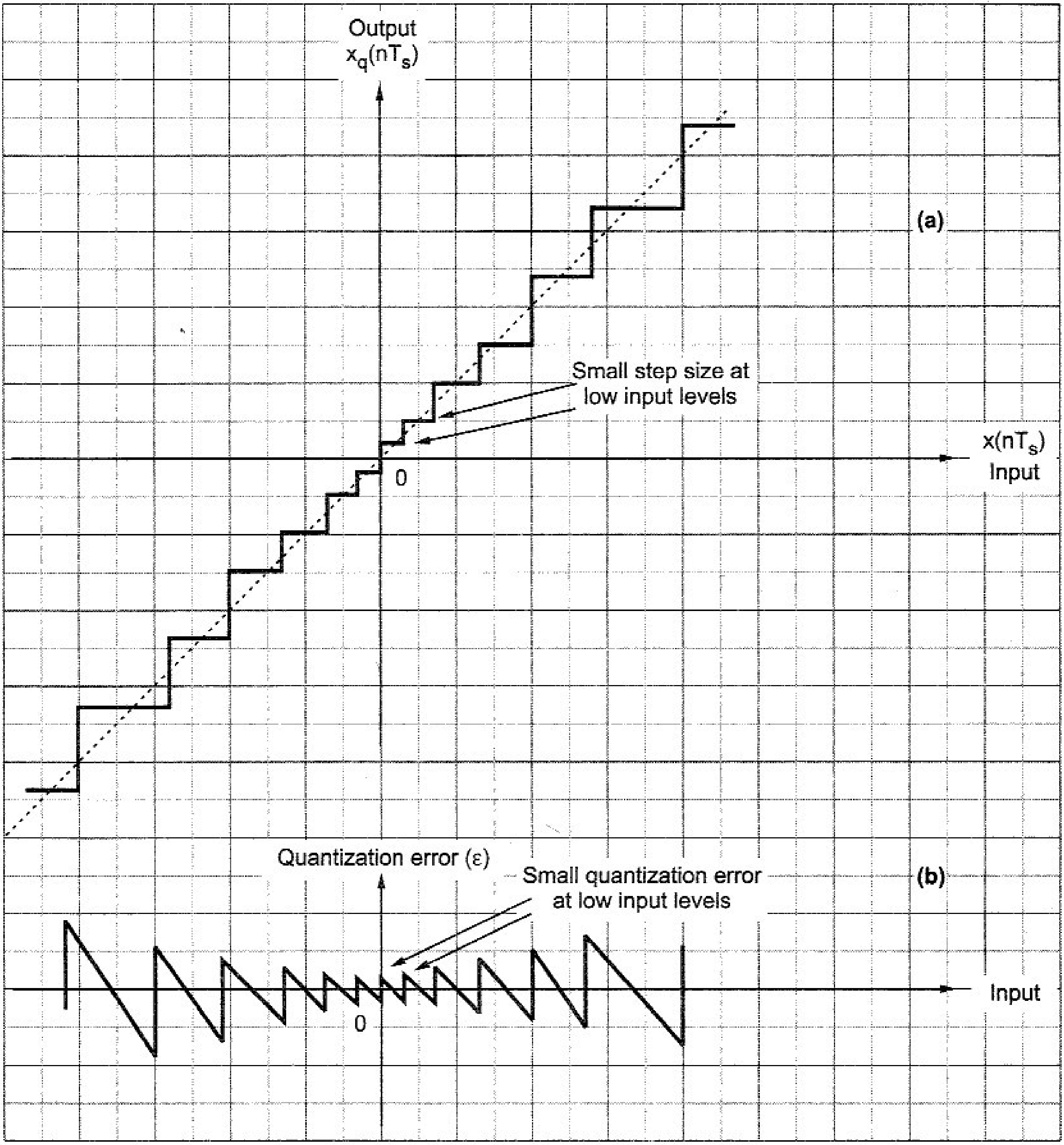

Fig. 2.7.1 (a) Nonuniform quantization transfer characteristic (b) Quantization error

If $x(t)$ is normalized, its maximum value i.e. $x_{\max} = 1$.

$$\therefore \qquad \delta = \frac{2}{q} \qquad \dots (2.7.2)$$

Let us consider an example of PCM system in which $v = 4$ bits.

Then number of levels q will be,

Here power P is defined as, $P = \frac{V_{signal}^2}{R} = \frac{x^2(t)}{R}$

$$V_{signal}^2 = \text{Mean square value of signal voltage}$$

$$= x^2(t)$$

$\therefore$ Normalized power will be, $P = \frac{x^2(t)}{1}$ [with $R=1$]

$$P = x^2(t) \qquad \text{... (2.7.6)}$$

Crest factor is given as,

$$\text{Crest factor} = \frac{\text{Peak value}}{\text{RMS value}} = \frac{x_{max}}{[x^2(t)]^{1/2}} \qquad \text{... (2.7.7)}$$

$$= \frac{x_{max}}{\sqrt{P}} \qquad \text{since } P = x^2(t) \qquad \text{... (2.7.8)}$$

When we normalize the signal $x(t)$, then

$$x_{max} = 1 \qquad \text{... (2.7.9)}$$

Putting above value of x_{max} in equation (2.7.8),

$$\text{Crest factor} = \frac{1}{\sqrt{P}} \qquad \text{... (2.7.10)}$$

For a large crest factor of voice (speech) and music signals P should be very very less than one in above equation.

i.e., $P << 1$ for large crest factor in equation (2.7.10)

Therefore actual signal to noise ratio will be significantly less than the value that is given by equation (2.7.5), since in this equation $P=1$. Consider equation (2.7.4),

$$\left(\frac{S}{N}\right) = 3 \times 2^{2v} \times P$$

$$(3 \times 2^{2v} \times P)\Big|_{P<<1} << (3 \times 2^{2v} \times P)\Big|_{P=1} \qquad \text{... (2.7.11)}$$

This equation shows that the signal to noise ratio for large crest factor signal ($P<<1$) will be very very less than that of the calculated theoretical value. The theoretical value is obtained for normalized power ($P=1$) by equation (2.7.4).

Therefore such large crest factor signals (speech and music) should use nonuniform quantization to overcome the problem just discussed. Signal to noise ratio reduces at low power levels ($P<<1$) just now we have seen by equation (2.7.11). That is at low signal levels, signal to noise ratio reduces mean noise increases. The quantization noise is directly related to step size. Therefore at low signal levels ($P \ll 1$) noise can be kept low by keeping step size low. This means that at low signal levels signal to noise ratio can be increased by decreasing step size 'δ'. This means step size 'δ' should be varied according to the signal level to keep signal to noise ratio at the required value. This is nothing but nonuniform quantization. Now let's see how *nonuniform quantization* is achieved through companding.

2.7.3 Companding in PCM

April/May - 2004

Normally we don't know how the signal level will vary in advance. Therefore the nonuniform quantization (variable step size 'δ') becomes difficult to implement. Therefore the signal is amplified at low signal levels and attenuated at high signal levels. After this process, uniform quantization is used. This is equivalent to more step size at low signal levels and small step size at high signal levels. At the receiver a reverse process is done. That is signal is attenuated at low signal levels and amplified at high signal levels to get original signal. Thus the compression of signal at transmitter and expansion at receiver is called combinely as *companding*. Fig. 2.7.2 shows compression and expansion curves.

As can be seen from Fig. 2.7.2, at the receiver, the signal is expanded exactly opposite to compression curve at transmitter to get original signal. A dotted line in the Fig. 2.7.2 shows uniform quantization. The compression and expansion is obtained by passing the signal through the amplifier having nonlinear transfer characteristic as shown in Fig. 2.7.2. That is nonlinear transfer characteristic means compression and expansion curves.

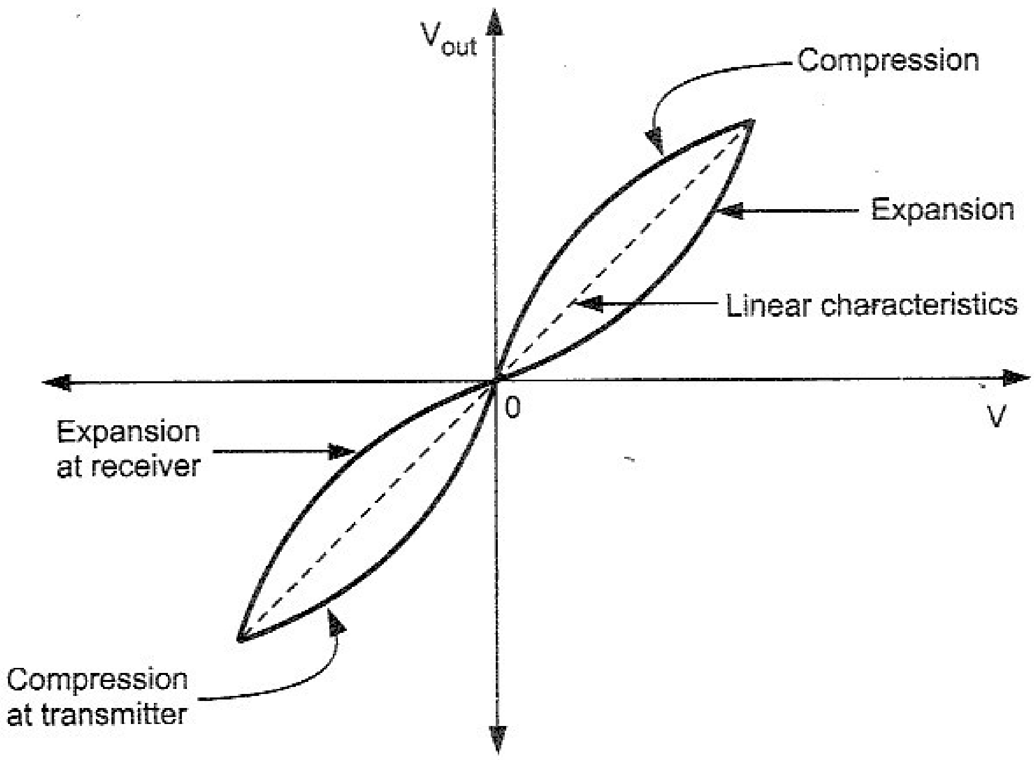

Fig. 2.7.2 Companding curves for PCM

2.7.4 μ - Law Companding for Speech Signals

Normally for speech and music signals a μ - law compression is used. This compression is defined by the following equation,

$$Z(x) = (\text{Sgn } x) \frac{ln(1+\mu|x|)}{ln(1+\mu)} \quad |x| \leq 1 \qquad \dots (2.7.12)$$

Fig. 2.7.3 shows the variation of signal to noise ratio with respect to signal level without companding and with companding.

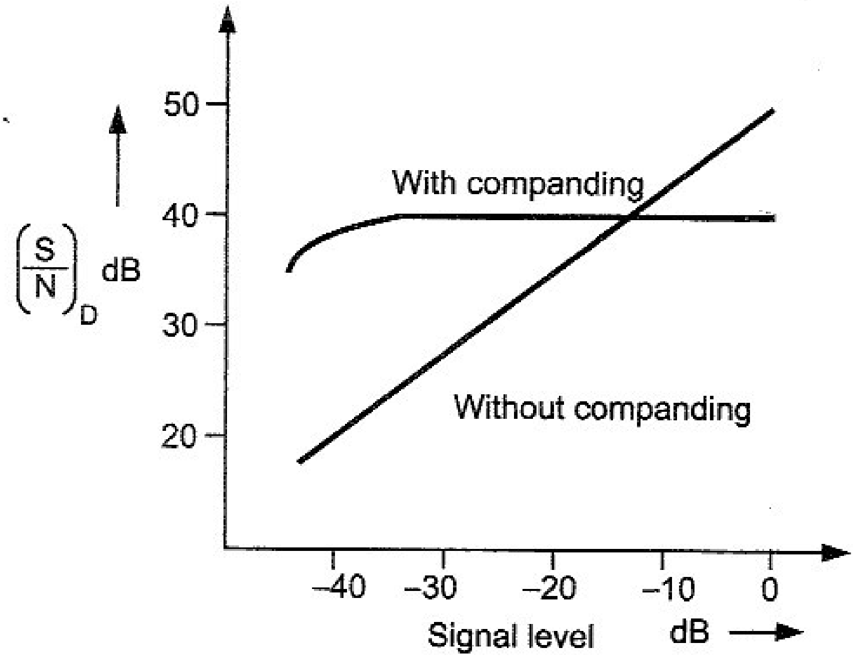

Fig. 2.7.3 PCM performance with μ - law companding

It can be observed from above Fig. 2.7.3 that signal to noise ratio of PCM remains almost constant with companding.

2.7.5 A - Law for Companding

The A - law provides piecewise compressor characteristic. It has linear segment for low level inputs and logarithmic segment for high level inputs. It is defined as,

$$Z(x) = \begin{cases} \dfrac{A|x|}{1+ln A} & \text{for } 0 \leq |x| \leq \dfrac{1}{A} \\ \dfrac{1+ln(A|x|)}{1+ln A} & \text{for } \dfrac{1}{A} \leq |x| \leq 1 \end{cases} \qquad \dots (2.7.13)$$

When $A = 1$, we get uniform quantization. The practical value for A is 87.56. Both A - law and μ - law companding is used for PCM telephone systems.

2.7.6 Signal to Noise Ratio of Companded PCM

The signal to noise ratio of companded PCM is given as,

$$\frac{S}{N} = \frac{3q^2}{[ln(1+\mu)]^2} \qquad \dots (2.7.14)$$

Here $q = 2^v$ is number of quantization levels.

Example 2.7.1 : *For a random variable are the mean square value and variance always equal ? Calculate these quantities for the quantization noise or error in PCM system.*

Solution : For a random variable X, the variance σ_x^2 is given as,

$$\sigma_x^2 = \overline{X^2} - m_x^2$$

Here $\overline{X^2}$ is the mean square value

and m_x is the mean value.

Above equation shows that variance (σ_x^2) and mean square value $\left(\overline{X^2}\right)$ will be same if mean (m_x) is zero.

From quantization characteristics of Fig. 2.3.2 (b) it is clear that quantization error (ε) has zero mean or average value. And it follows uniform distribution from $-\frac{\delta}{2}$ to $+\frac{\delta}{2}$. Hence probability density function of quantization error will be,

$$f_\varepsilon(\varepsilon) = \begin{cases} \frac{1}{\delta} & \text{for} -\frac{\delta}{2} \le \varepsilon \le \frac{\delta}{2} \\ 0 & \text{elsewhere} \end{cases} \qquad \text{By equation (2.6.9)}$$

Mean square value can be calculated as,

$$\overline{X^2} = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

Putting values in above equation,

$$\overline{X^2} = \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \varepsilon^2 \frac{1}{\delta} d\varepsilon$$

$$= \frac{1}{\delta}\left[\frac{\varepsilon^3}{3}\right]_{-\frac{\delta}{2}}^{\frac{\delta}{2}}$$

$$= \frac{\delta^2}{12}$$

This is the mean square value of quantization error.

Example 2.7.2 : *A Compact Disc (CD) records audio signals digitally by using PCM. Assume the audio signal bandwidth to be 15 kHz.*

(i) What is Nyquist rate ?

(ii) If the Nyquist samples are quantized into L = 65,536 levels and then binary coded, determine the number of binary digits required to encode a sample.

(iii) Determine the number of binary digits per second (bits/sec) required to encode the audio signal.

(iv) For practical reasons, the signals are sampled at a rate well above Nyquist rate at 44100 samples per second. If L = 65,536, determine number of bits per second required to encode the signal and transmission bandwidth of encoded signal.

Solution : (i) To obtain Nyquist rate

The bandwidth of the signal is, $W = 15$ kHz.

$$\therefore \quad \text{Nyquist rate} = 2W$$

$$= 2 \times 15 \text{ kHz} = 30 \text{ kHz}$$

(ii) To determine number of bits

Number of levels, $q = L = 65{,}536$

Hence binary digits required to encode each sample will be,

$$q = 2^v$$

or

$$v = \log_2 q$$

$$= \log_2 65536 = 16 \text{ bits}$$

(iii) To determine signalling rate

$v = 16$ bits/sample are used. The samples are taken at the rate of $f_s = 30{,}000$ samples/sec. Hence signalling rate will be,

$$r = v f_s$$

$$= 16 \times 30{,}000 = 480 \text{ kbits/sec}$$

(iv) To obtain B_T if f_s = 44.1 kHz

$$\text{Levels used are } q = 65{,}536$$

$$\therefore \quad v = \log_2 q = 16 \text{ bits}$$

$$f_s = 44100 \text{ samples/sec}$$

For PCM, the transmission bandwidth required to encode the signal will be,

$$B_T = \frac{1}{2} v f_s$$

$$= \frac{1}{2} \times 16 \times 44100$$

$$= 352.8 \text{ kHz}$$

Example 2.7.3 : *The output signal to noise ratio of a 10 bit PCM was found to be 30 dB. The desired SNR is 42 dB. It was decided to increase the SNR to the desired value by increasing the number of quantization levels. Find the fractional increase in transmission bandwidth required for this increase in SNR.*

Solution : (i) To obtain number of bits for 42 dB

Signal to noise ratio of PCM is given as,

$$\left(\frac{S}{N}\right) \text{dB} = (4.8 + 6v) \text{ dB}$$

Above equation shows that signal to noise ratio increases by 6 dB with every bit. It is given that

$$\frac{S}{N} = 30 \text{ dB for 10 bits}$$

The desired signal to noise ratio is 42 dB. Hence rise in $\left(\frac{S}{N}\right)$ ratio is 42 – 30 = 12 dB

We know that $\left(\frac{S}{N}\right)$ ratio increases by 6 dB for 1 bit. Hence 2 bits are required to increase signal to noise ratio by 12 dB.

Hence, $\quad v = 10 + 2 = 12$ bits are required.

(ii) To obtain fractional increase in bandwidth

Bandwidth in PCM is given as,

$$B_T = \frac{1}{2} v f_s$$

$$\therefore \quad B_T \text{ (10 bits)} = \frac{1}{2}\times 10\times f_s = 5f_s$$

and $$B_T \text{ (12 bits)} = \frac{1}{2}\times 12\times f_s = 6f_s$$

$$\therefore \text{ Fractional increase in } B_T = \frac{6f_s - 5f_s}{5f_s}\times 100\ \% = 20\ \%$$

Example 2.7.4 : *A telephone signal with cutoff frequency of 4 kHz is digitized into 8 bit PCM, sampled at Nyquist rate. Calculate baseband transmission bandwidth and quantization $\frac{S}{N}$ ratio.*

Solution : Given data is,

$$W = 4 \text{ kHz}$$

$$v = 8 \text{ bits}$$

For PCM the transmission bandwidth is given as,

$$B_T = vW = 4\times 8 = 32 \text{ kHz}$$

Telephone signal is nonsinusoidal signal. Its signal to quantization noise ratio is given by equation (2.6.25) as,

$$\frac{S}{N} = 4.8 + 6v$$

$$= 4.8 + 6\times 8 = 52.8 \text{ dB}$$

Review Questions

1. *With the help of neat diagrams, explain the transmitter and receiver of pulse code modulation.*
2. *What is uniform (linear) quantization ?*
3. *Explain quantization error and derive an expression for maximum signal to noise ratio in PCM system that uses linear quantization.*
4. *Derive the relations for signalling rate and transmission bandwidth in PCM system.*
5. *What is the necessity of nonuniform quantization and explain companding ?*

Unsolved Examples

1. *A 40 MB hard disk is used to store PCM data. The signal is sampled at 8 kHz and the encoded PCM is to have an average signal to noise ratio of atleast 30 dB. For how many minutes the PCM data can be stored on the hard disk ?* **[Ans. : 133 min]**

2. *In the binary PCM system, find out the minimum number of bits required so that quantizing noise is less than ±k percent of the analog level.* **[Ans. : $v \geq \log_2(50/k)$]**

3. *The Gaussian distributed random variable with zero mean and unit variance is applied to the input of uniform quantizer.*

   (a) What is the probability that the amplitude of this input lies outside the range ±4 ?

   (b) Using the result of past (a), find out the signal to quantization noise ratio.

   [Ans. : (a) 1 in 10^4 (b) $(S/N)dB = 6v - 7.2dB$]

2.8 Channel Noise and Error Probability

2.8.1 Types of Noises in PCM

The PCM system is affected by two types of noises, (1) channel noise and (2) quantization noise. The channel noise is introduced across the transmission path due to interference. If this noise is large then symbol '0' may be detected as '1' and viceversa. Thus errors are created due to channel noise. Quantization noise is introduced at the time of quantization process. It is an irreversible process in which there is no chance of getting the original signal level back. Quantization noise does not affect the signal detection at the receiver. But it changes the signal shape itself. Quantization noise can be reduced by increasing number of bits per sample.

In this section we will study the effect of channel noise and its effect on the signal detection. The performance of PCM system is evaluated interms of error probability.

2.8.2 Error Probability

The matched filter detection of PCM signal is discussed in 4th chapter. Error is introduced when symbol '0' is detected as '1' and viceversa. This error occurs when the channel noise is higher than decision threshold. Let the channel noise be white gaussian with zero mean and power spectral density of $\frac{N_0}{2}$. Then probability of error P_e is given as,

$$P_e = \frac{1}{2} erfc \sqrt{\frac{E}{4N_0}} \qquad \dots (2.8.1)$$

Here *erfc* is complementary error function. It is defined in an appendix. 'E' is an energy of one bit. It is given as,

$$E = A^2 T$$

Here A is an amplitude of the PCM signal and T is one bit period.

We can write equation 2.8.1 as,

4.2.2 Channel Noise

This noise is also called electrical noise. It corrupts the signal during the transmission across the channel. It is produced due to switching transients, intermodulation, random motion of electrons, lightning etc. Channel noise hampers easy detection of the signal.

4.2.3 Intersymbol Interference

The sharp rising and falling edges of the signal are smoothened due to band limited (lowpass) nature of the channel. Hence the signal disperses or spreads while passing through the channel. This overlaps the timings of individual pulses. The overlapping of pulses (symbols) is called inter symbol interference.

4.3 Detection-Maximum Likelihood Detector

- The transmitted signal $s_i(t)$ is often corrupted by noise when it is received. Such a received signal is represented as,

$$x(t) = s_i(t) + n(t) \quad \text{for } 0 \leq t \leq T \quad \ldots(4.3.1)$$

$$\text{and } i = 1, 2, \ldots. M$$

Here $n(t)$ is the sample function of additive white Gaussian noise process $N(t)$.

- The main job of the receiver is to observe the signal $x(t)$ and make best estimate of $s_i(t)$. The transmitted signal vector s_i, the observation vector x and the noise vector n can be represented in the N-dimentional Euclidean space is called a signal constellation as shown in Fig. 4.3.1.

It shows the signal constallation for N = 3 (i.e. 3-dimensions). Observe that the received signal point shifts from message point 'm_i' by a noise vector 'n'.

- The detector observes the vector 'x' and performs the mapping to an estimate $\hat{m}$ of the transmitted message point m_i. This mapping is done in such a way that average probability of symbol error is minimum in the decision.

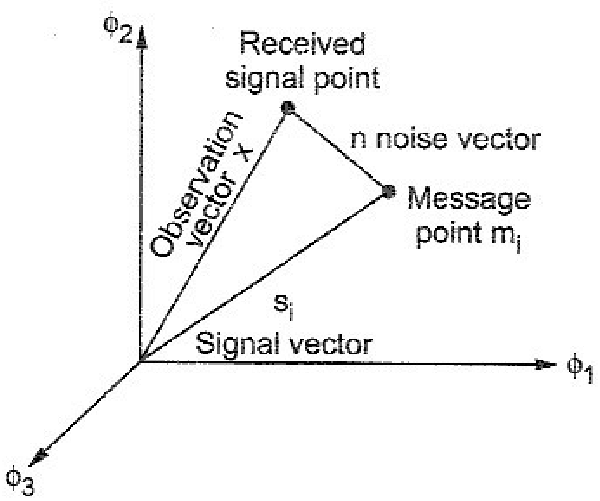

Fig. 4.3.1 Signal constellation

4.3.1 Maximum Likelihood Detector

- Let the observation vector be x. The decision is made as $\hat{m} = m_i$. The average probability of symbol error in this decision is

$$P_e(m_i, x) = P(m_i \text{ not sent} / x) = 1 - P(m_i \text{ sent} / x) \quad \ldots(4.3.2)$$

Here $P_e(m_i, x)$ indicates average probability of symbol error when 'x' is the observation vector and message m_i is selected. To minimize the error probability $P_e(m_i, x)$ given by above equation the optimum decision rule can be stated as,

$$set\ \hat{m} = m_i,\ if\ P(m_i\ sent\ /\ x) \geq P(m_k\ sent\ /\ x)\ for\ all\ k \neq i \quad ...(4.3.3)$$

- This decision rule can be represented graphically. Let 'R'-denote the N-dimensional space of all possible vectors 'x'. And this region be partitioned into 'M' decision regions, $R_1, R_2, ... R_M$. The decision rule can then be written as,

$$Vector\ x\ lies\ in\ region\ R_i\ if\ ln\ [f_X(x\ /\ m_k)]\ is\ maximum\ for\ k = i \quad ...(4.3.4)$$

Here $f_X(x / m_k)$ is the likelihood function which results when symbol m_k is transmitted. This rule is called *maximum likelihood* and the corresponding detector which uses this rule is called *maximum likelihood detector.*

- The decision rule of equation 4.3.4 can be further written alternately as,

$$Vector\ x\ lies\ in\ region\ R_i\ if\ \|\ x - s_k\|\ is\ minimum\ for\ k = i \quad ...(4.3.5)$$

Here $\| x - s_k \|$ is the distance between the received signal point and the message point. The maximum likelihood decision rule chooses the message point closest to the received signal point.

- **An example of maximum likelihood decision for N = 2 and M = 4**

Fig. 4.3.2 shows the decision regions and boundaries for N = 2 and M = 4 symbols. Observe that the region is $\phi_1 - \phi_2$ i.e. 2-dimensional. There are four regions R_1, R_2, R_3 and R_4 corresponding to four messages m_1, m_2, m_3 and m_4. The decision boundaries are shown by the dotted lines. If the observation vector falls in region R_1 then message m_1 is selected.

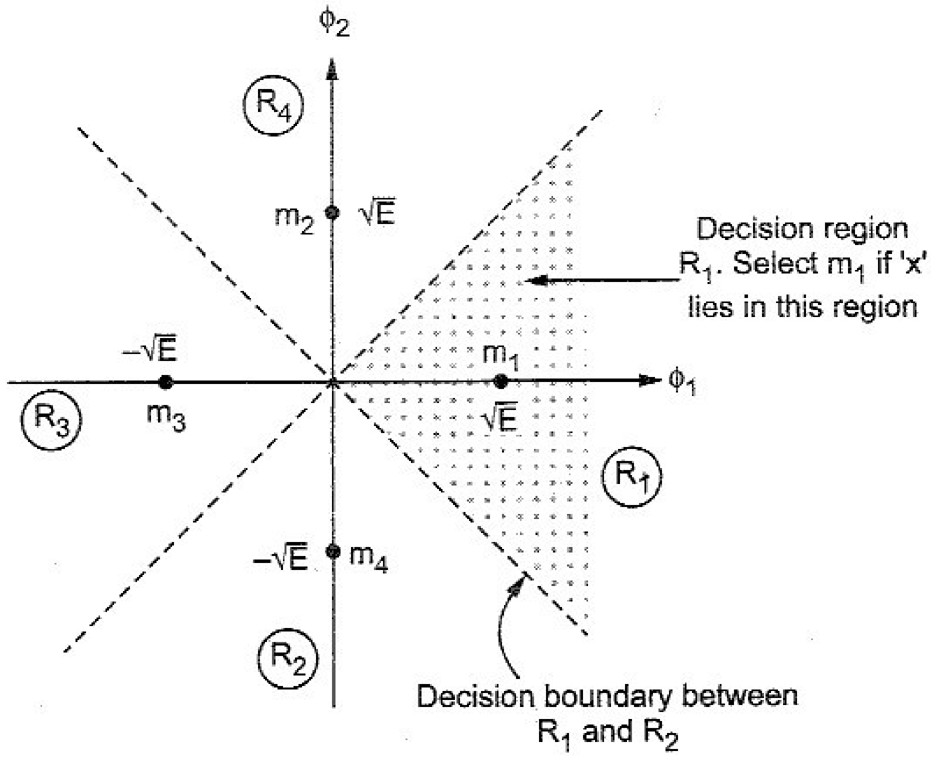

Fig. 4.3.2 Decision regions for N = 2

4.4 Receiving Filter-Matched Filter

Matched filter is used as a receiver for baseband and bandpass signals. Hence it is also called receiving filter.

4.4.1 Definition

- The matched filter is used for detection of signals in baseband and passband transmission.
- Fig. 4.4.1 shows the transmitted digital signal and received noisy signal. In this figure observe that the transmitted signal sequence is 1 0 0 1 1. The pulsė is checked at the point 'T' of every bit period. Because of noise pulse present in the third bit at the instant 'T' of checking, it is detected in error.

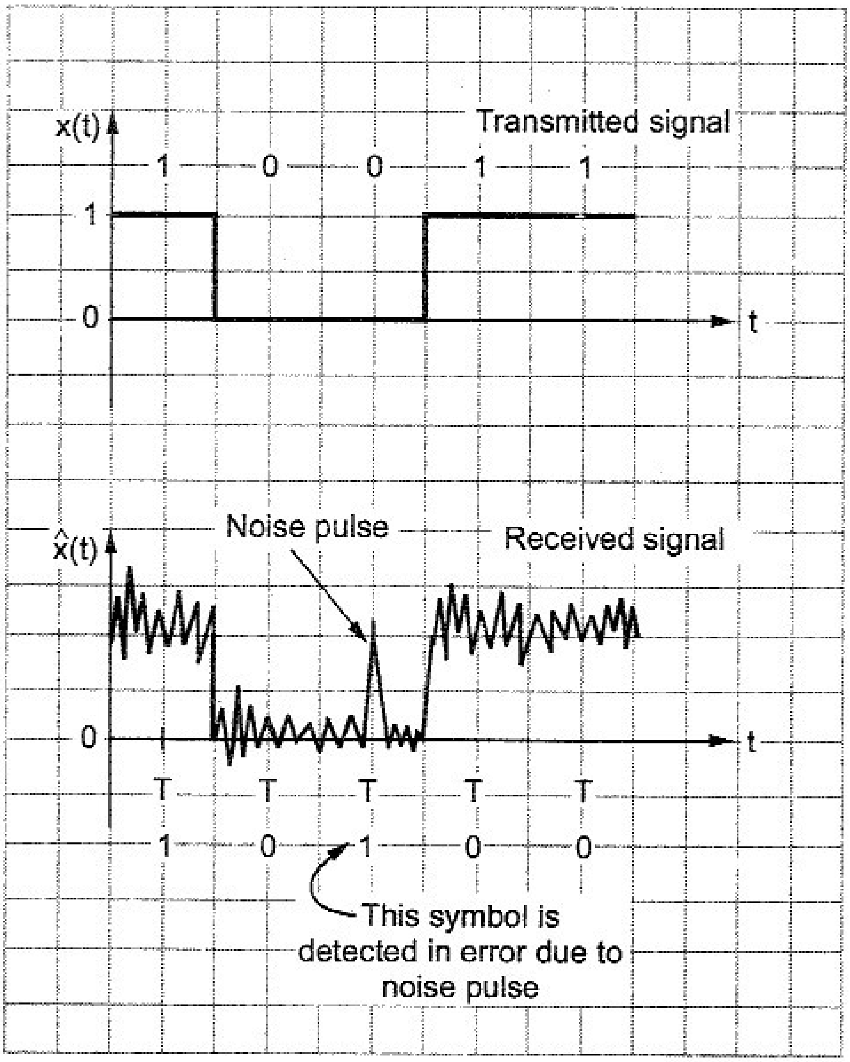

Fig. 4.4.1 Error due to noise

- **Requirements of detection receiver**
  i) Signal to noise ratio of the receiver must be improved.
  ii) The signal must be checked at the instant in bit period, when signal to noise ratio is maximum.
  iii) The error probability should be minimum.

- **Matched Filter**

  i) It satisfies all the above requirements.

  ii) It is called matched filter since its impulse response is *matched* to the shape of input signal.

4.4.2 Decision Threshold in Matched Filters

- **Assumption**

Let us assume that the received signal is a binary waveform. Let us consider that the polar NRZ signal is used to represent binary 1s and 0s. i.e.,

for binary '1'; $x_1(t) = + A$ for one bit period 'T'

and for binary '0'; $x_2(t) = - A$ for one bit period 'T'.

Thus the input signal $x(t)$ will be either $x_1(t)$ or $x_2(t)$ depending upon the polarity of the NRZ signal. Fig. 4.4.2 shows the block diagram for such binary coded signal.

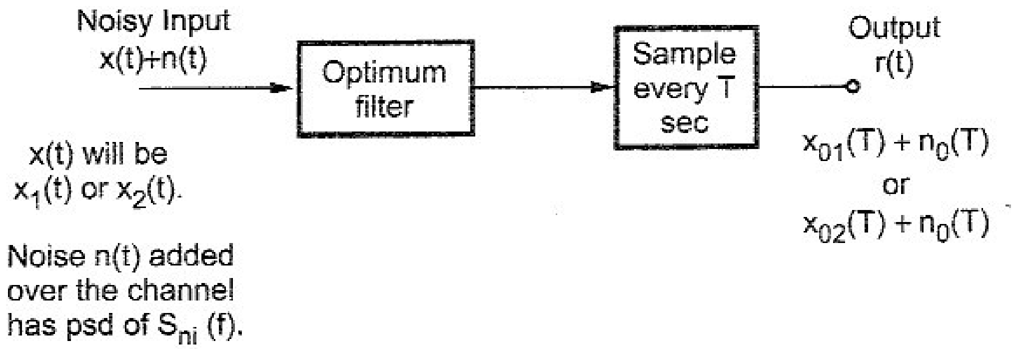

Fig. 4.4.2 A receiver for binary coded signal

- As shown in the above figure, noise $n(t)$ is added to the signal $x(t)$ over the channel during transmission. Hence input to the optimum filter is $x(t)+n(t)$ i.e.,

$$\text{Input to the receiver} = x(t)+n(t)$$

and $$\text{Output from the receiver} = x_{01}(T)+n_0(T) \text{ or } x_{02}(T)+n_0(T)$$

When noise is absent :

The output of the receiver will be,

$$r(T) = x_{01}(T) \quad \text{if} \quad x(t) = x_1(t)$$

and $$r(T) = x_{02}(T) \quad \text{if} \quad x(t) = x_2(t)$$

Thus in the absence of noise, decisions are taken clearly.

When noise is present :

i) select $x_1(t)$ if $r(T)$ is closer to $x_{01}(T)$ than $x_{02}(T)$ and,

ii) select $x_2(t)$ if $r(T)$ is closer to $x_{02}(T)$ than $x_{02}(T)$.

Therefore the decision boundary will be midway between $x_{01}(T)$ and $x_{02}(T)$. It is given as,

$$\text{Decision boundary} = \frac{x_{01}(T) + x_{02}(T)}{2} \quad \text{... (4.4.1)}$$

Review Question

1. *Explain what is matched filter.*

4.5 Error Rate due to Noise (Error Probability)

4.5.1 Error Conditions in Matched Filter

- Suppose that $x_2(t)$ was transmitted, but $x_{01}(T)$ is greater than $x_{02}(T)$. If noise $n_0(T)$ is positive and larger in magnitude than the voltage difference $\frac{1}{2}[x_{01}(T) + x_{02}(T)] - x_{02}(T)$, then incorrect decision will be taken. i.e. error will be generated if,

$$n_0(T) \geq \frac{x_{01}(T) + x_{02}(T)}{2} - x_{02}(T)$$

$$\therefore \quad n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2} \quad \text{... (4.5.1)}$$

- Similarly, let $x_1(t)$ is transmitted, but $x_{02}(T)$ is greater than $x_{01}(T)$. Then incorrect decision will be taken if noise $n_0(T)$ is less than $\frac{1}{2}[x_{01}(T) + x_{02}(T)] - x_{01}(T)$. i.e., error will be generated if,

$$n_0(T) \leq -\frac{x_{01}(T) + x_{02}(T)}{2} - x_{01}(T)$$

$$\therefore \quad n_0(T) \leq -\frac{x_{02}(T) - x_{01}(T)}{2} \quad \text{... (4.5.2)}$$

The above two error conditions are summarized in the following table :

Input x(t)	Value of $n_0(t)$ for error in the output	Probability of error
$x_2(t)$	$n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}$	Probability of error will be obtained by evaluating the probability that $n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}$
$x_1(t)$	$n_0(T) \leq -\frac{x_{02}(T) - x_{01}(T)}{2}$	Probability of error can be obtained by evaluating the probability that $n_0(T) \leq -\frac{x_{02}(T) - x_{01}(T)}{2}$

Table 4.5.1 : Error conditions in a matched filter