These functions are orthonormal and any of $s_1(t), s_2(t), s_3(t)$ and $s_4(t)$ can be expressed as a linear combination of above basis functions.

1.6 Bandwidth

1.6.1 Definitions of Bandwidth

There are different definitions of bandwidth. These are listed below :

i) Half power bandwidth : This is the bandwidth where PSD drops to half (or 3 dB) of its maximum value. It is also called 3-dB bandwidth. For most of the practical purposes, this bandwidth is considered. This is shown in Fig. 1.6.1.

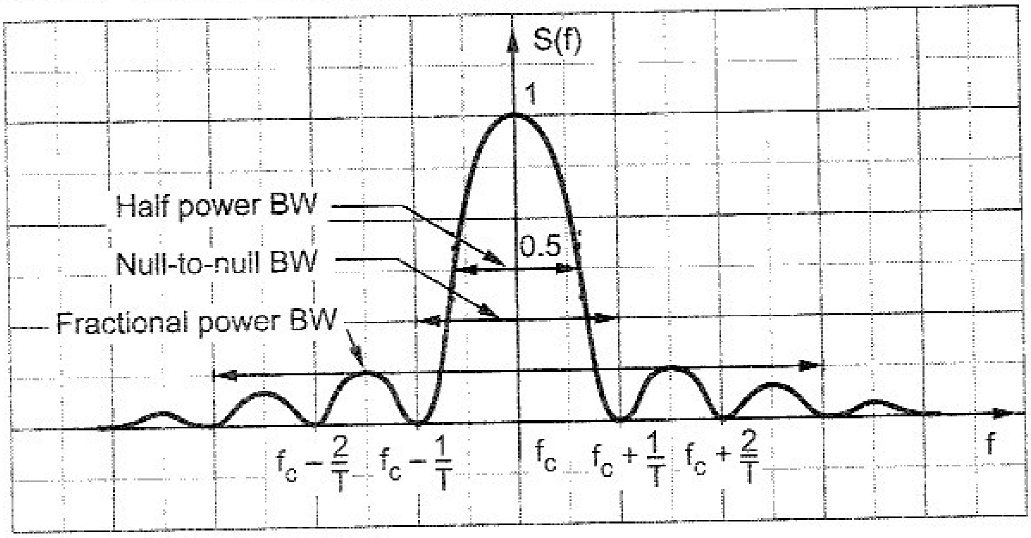

Fig. 1.6.1 Bandwidth of a signal

ii) Null-to-null bandwidth : This bandwidth defines width of the main lobe. In Fig. 1.6.1, the null-to-null bandwidth is $\frac{2}{T}$. This bandwidth is considered for most of the digital communication systems. It contains most of the power of the signal.

iii) Fractional power bandwidth : This bandwidth assures the specified power of the signal. For example if 95 % power bandwidth is 10 MHz. This means within the bandwidth of 10 MHz, the signal power will be 95 %.

iv) Absolute bandwidth : This is the total bandwidth of the signal. No frequency components are present outside the absolute bandwidth. In other words, absolute bandwidth is the bandwidth which contains all the 100 % power of the signal.

v) Noise equivalent bandwidth : Fig. 1.6.2 shows the response of ideal filter and practical filter. The bandwidth 'B_N' required to make the shaded areas of ideal and practical filters equal is called noise equivalent bandwidth.

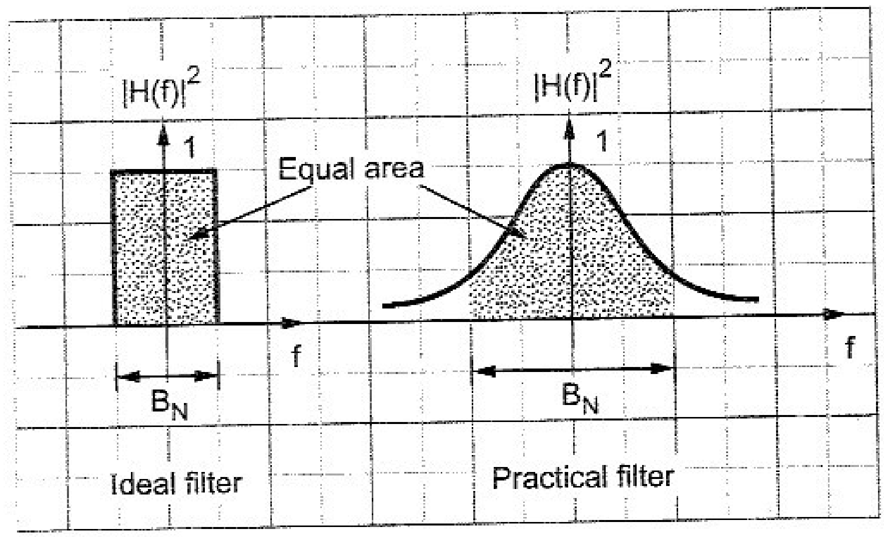

Fig. 1.6.2 Noise equivalent bandwidth

vi) **Bounded PSD :** Some times the bandwidth is specified as the range of frequencies after which PSD falls below certain attenuation level. For example bandwidth for 35 dB attenuation. This means range of frequencies after which, the psd achieves attenuation higher than 35 dB.

1.6.2 Dimensionality Theorem

The dimensionality theorem states that, a real waveform can be completely specified by 'N' independent pieces of information where N is given by,

$$N = 2BT_0 \qquad \dots (1.6.1)$$

Here N = Dimension of the waveform in signal space

B = Bandwidth of the signal

T_0 = Time over which the waveform is described (period)

Applications of dimensionality theorem

- Dimensionality theorem is used to calculate storage space required to store digital signal.
- It is used to estimate bandwidth of the signal from equation (1.6.1) we have

$$B = \frac{N}{2\,T_0} \qquad \dots (1.6.2)$$

The ratio $\frac{N}{T_0}$ gives number of independent pieces of infromation transmitted over interval T_0. It is also called symbols rate (r_s). i.e.,

$$r_s = \frac{N}{T_0} \qquad \dots (1.6.3)$$

$$\therefore \quad B = \frac{r_s}{2} \qquad \dots (1.6.4)$$

Thus bandwidth can be obained form dimensionality theorem.

1.7 Mathematical Models of Communication Channel

The communication channels are mathematically modelled to evaluate their effect on signal transmission.

Three models are described next.

- The samples $x_\delta(t)$ must represent all the information contained in $x(t)$.
- The sampled signal $x_\delta(t)$ is called Discrete Time (DT) signal. It is analyzed with the help of DTFT and z-transform.

2.1.2 Sampling Theorem for Low Pass (LP) Signals

A low pass or LP signal contains frequencies from 1 Hz to some higher value.

Statement of sampling theorem

1) *A bandlimited signal of finite energy, which has no frequency components higher than W hertz, is completely described by specifying the values of the signal at instants of time separated by* $\frac{1}{2W}$ *seconds and*
2) *A bandlimited signal of finite energy, which has no frequency components higher than W hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.*

The first part of above statement tells about sampling of the signal and second part tells about reconstruction of the signal. Above statement can be combined and stated alternately as follows :

A continuous time signal can be completely represented in its samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal. i.e.,

$$f_s \geq 2W$$

Here f_s is the sampling frequency and

W is the higher frequency content.

Proof of sampling theorem

There are two parts : (I) Representation of $x(t)$ in terms of its samples.

(II) Reconstruction of $x(t)$ from its samples.

Part I : Representation of $x(t)$ in its samples $x(nT_s)$

Step 1 : Define $x_\delta(t)$

Step 2 : Fourier transform of $x_\delta(t)$ i.e. $X_\delta(f)$

Step 3 : Relation between $X(f)$ and $X_\delta(f)$

Step 4 : Relation between $x(t)$ and $x(nT_s)$

Step 1 : Define $x_\delta(t)$

Refer Fig. 2.1.1. The sampled signal $x_\delta(t)$ is given as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s) \quad \text{... (2.1.1)}$$

Here observe that $x_\delta(t)$ is the product of x_δ and impulse train $\delta(t)$ as shown in Fig. 2.1.1. In the above equation $\delta(t-nT_s)$ indicates the samples placed at $\pm T_s$, $\pm 2T_s$, $\pm 3T_s$ ... and so on.

Step 2 : FT of $x_\delta(t)$ i.e. $X_\delta(f)$

Taking FT of equation (2.1.1).

$$X_\delta(f) = \text{FT}\left\{\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)\right\}$$

$$= \text{FT \{Product of } x(t) \text{ and impulse train\}}$$

We know that FT of product in time domain becomes convolution in frequency domain. i.e.,

$$X_\delta(f) = \text{FT}\,\{x(t)\} * FT\,\{\delta(t-nT_s)\} \quad \text{...(2.1.2)}$$

By definitions, $x(t) \xleftrightarrow{FT} X(f)$ and

$$\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Hence equation (2.1.2) becomes,

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Since convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \qquad \text{By shifting property of impulse function}$$

$$= \cdots f_s X(f-2f_s) + f_s X(f-f_s) + f_s X(f) + f_s X(f-f_s) + f_s X(f-2f_s) + \cdots$$

Comments

(i) The RHS of above equation shows that $X(f)$ is placed at $\pm f_s, \pm 2f_s, \pm 3f_s, \cdots$

(ii) This means $X(f)$ is periodic in f_s.

(iii) If sampling frequency is $f_s = 2W$, then the spectrums $X(f)$ just touch each other.

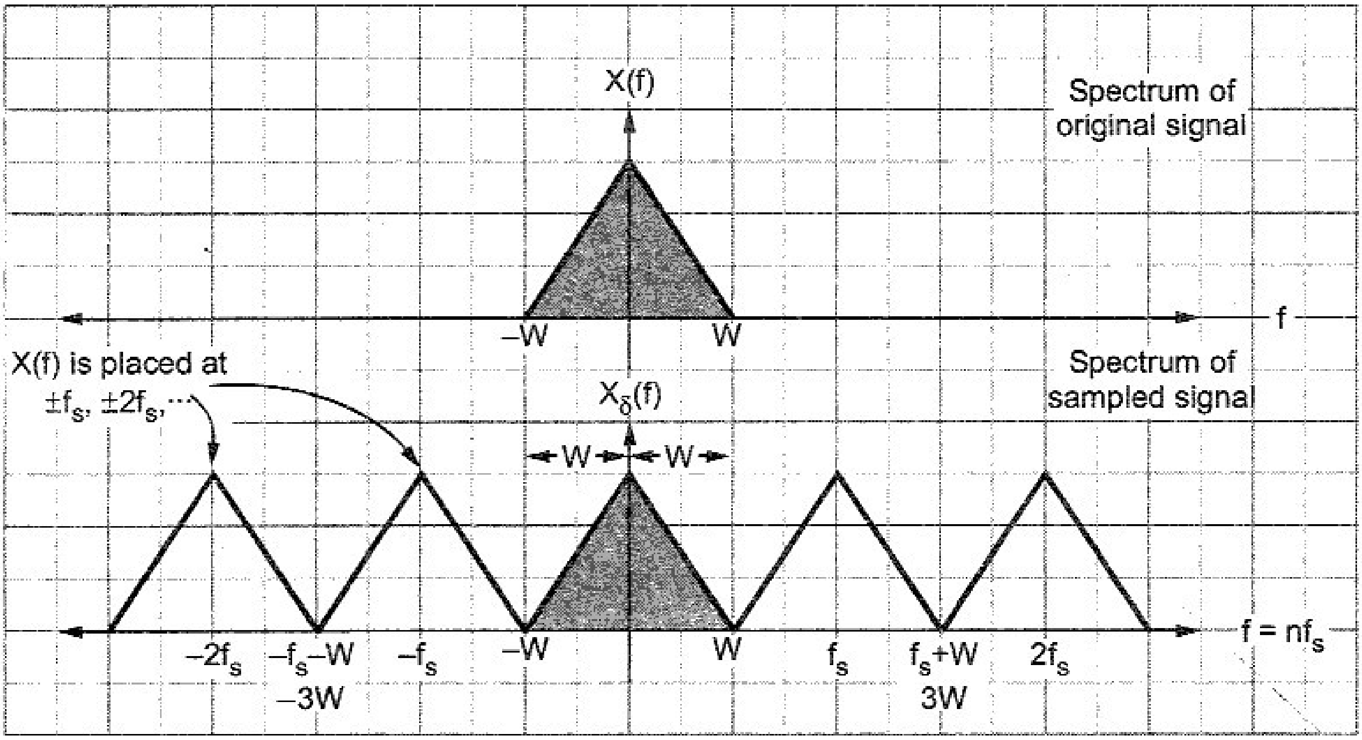

Fig. 2.1.2 Spectrum of original signal and sampled signal

Step 3 : Relation between $X(f)$ and $X_\delta(f)$

Important assumption : Let us assume that $f_s = 2W$, then as per above diagram,

$$X_\delta(f) = f_s X(f) \qquad \text{for } -W \le f \le W \text{ and } f_s = 2W$$

or

$$X(f) = \frac{1}{f_s} X_\delta(f) \qquad \dots (2.1.3)$$

Step 4 : Relation between $x(t)$ and $x(nT_s)$

DTFT is,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$\therefore \quad X(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi f n} \qquad \dots (2.1.4)$$

In above equation 'f' is the frequency of DT signal. If we replace $X(f)$ by $X_\delta(f)$, then 'f' becomes frequency of CT signal. i.e.,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi \frac{f}{f_s} n}$$

In above equation 'f' is frequency of CT signal. And $\frac{f}{f_s}$ = Frequency of DT signal in equation (2.1.4). Since $x(n) = x(nT_s)$, i.e. samples of $x(t)$, then we have,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \quad \text{since } \frac{1}{f_s} = T_s$$

Putting above expression in equation (2.1.3),

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}$$

Inverse Fourier Transform (IFT) of above equation gives $x(t)$ i.e.,

$$x(t) = IFT\left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}\right\} \qquad \ldots (2.1.5)$$

Comments :

i) Here $x(t)$ is represented completely in terms of $x(nT_s)$.

ii) Above equation holds for $f_s = 2W$. This means if the samples are taken at the rate of $2W$ or higher, $x(t)$ is completely represented by its samples.

iii) First part of the sampling theorem is proved by above two comments.

Part II : Reconstruction of $x(t)$ from its samples

> **Step 1 :** Take inverse Fourier transform of $X(f)$ which is in terms of $X_\delta(f)$.
>
> **Step 2 :** Show that $x(t)$ is obtained back with the help of interpolation function.

Step 1 : The IFT of equation (2.1.5) becomes,

$$x(t) = \int_{-\infty}^{\infty} \left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s}\right\} e^{j2\pi f t}\, df$$

Here the integration can be taken from $-W \leq f \leq W$. Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \leq f \leq W$. (See Fig. 2.1.2).

$$\therefore \qquad x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \cdot e^{j2\pi f t}\, df$$

Interchanging the order of summation and integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{1}{f_s}\int_{-W}^{W} e^{j2\pi f(t-nT_s)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\cdot\left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi(t-nT_s)}\right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\left\{\frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j2\pi(t-nT_s)}\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\frac{1}{f_s}\cdot\frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\pi(2Wt-2WnT_s)}{\pi(f_s t-f_s nT_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$. Simplifying above equation,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}(2Wt-n) \qquad \text{since } \frac{\sin\pi\theta}{\pi\theta} = \text{sinc}\,\theta \qquad ...(2.1.6)$$

Step 2 : Let us interpret the above equation. Expanding we get,

$$x(t) = \cdots + x(-2T_s)\text{sinc}(2Wt+2) + x(-T_s)\,\text{sinc}(2Wt+1) + x(0)\,\text{sinc}(2Wt) + x(T_s)\,\text{sinc}(2Wt-1) + \cdots$$

Comments :

(i) The samples $x(nT_s)$ are weighted by sinc functions.

(ii) The sinc function is the interpolating function. Fig. 2.1.3 shows, how $x(t)$ is interpolated.

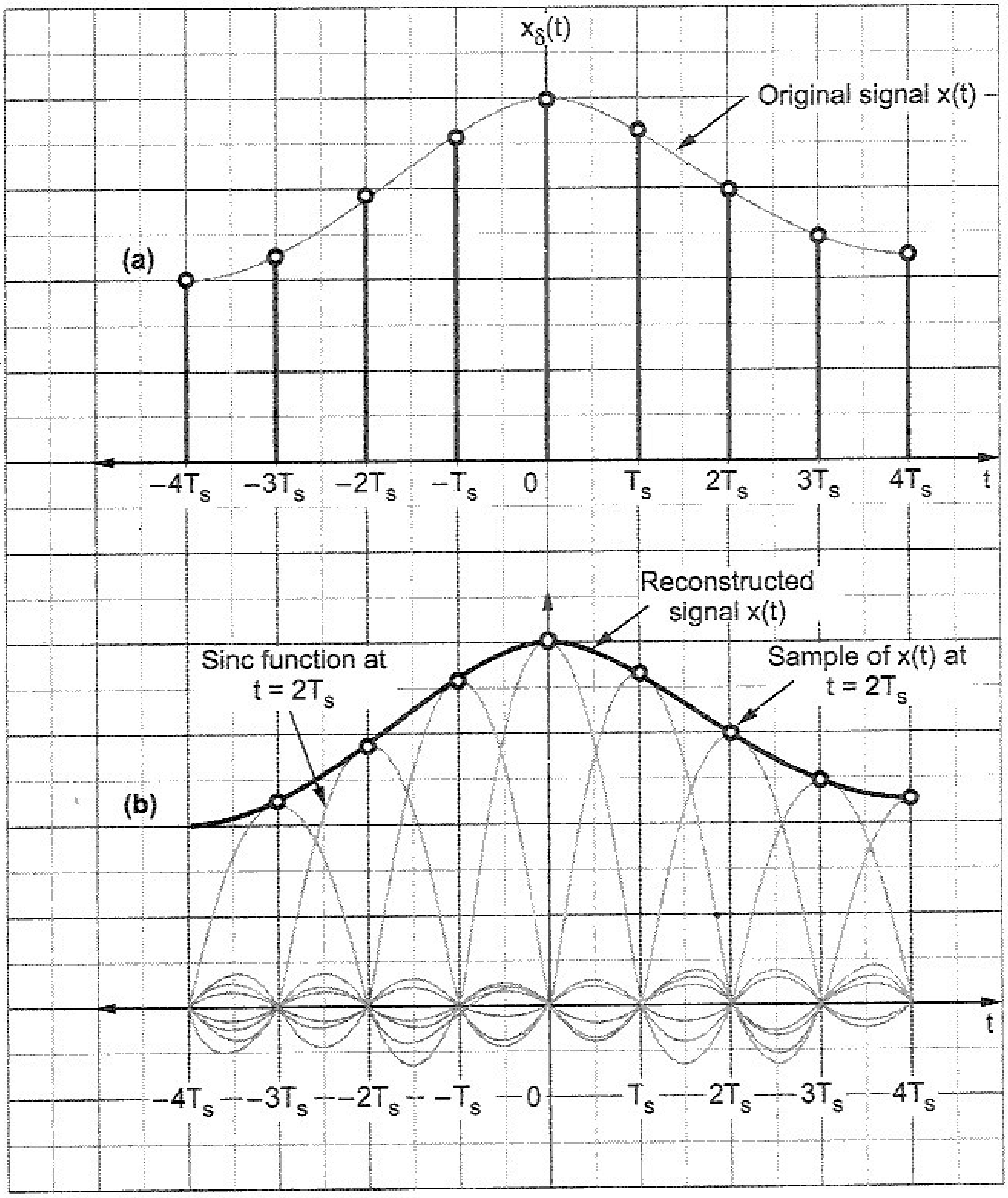

Fig. 2.1.3 (a) Sampled version of signal $x(t)$

(b) Reconstruction of $x(t)$ from its samples

Step 3 : Reconstruction of $x(t)$ by low pass filter

When the interpolated signal of equation (2.1.6) is passed through the low pass filter of bandwidth $-W \leq f \leq W$, then the reconstructed waveform shown in above Fig. 2.1.3 (b) is obtained. The individual sinc functions are interpolated to get smooth $x(t)$.

2.1.3 Effects of Undersampling (Aliasing)

While proving sampling theorem we considered that $f_s = 2W$. Consider the case of $f_s < 2W$. Then the spectrum of $X_\delta(f)$ shown in Fig. 2.1.4 will be modified as follows :

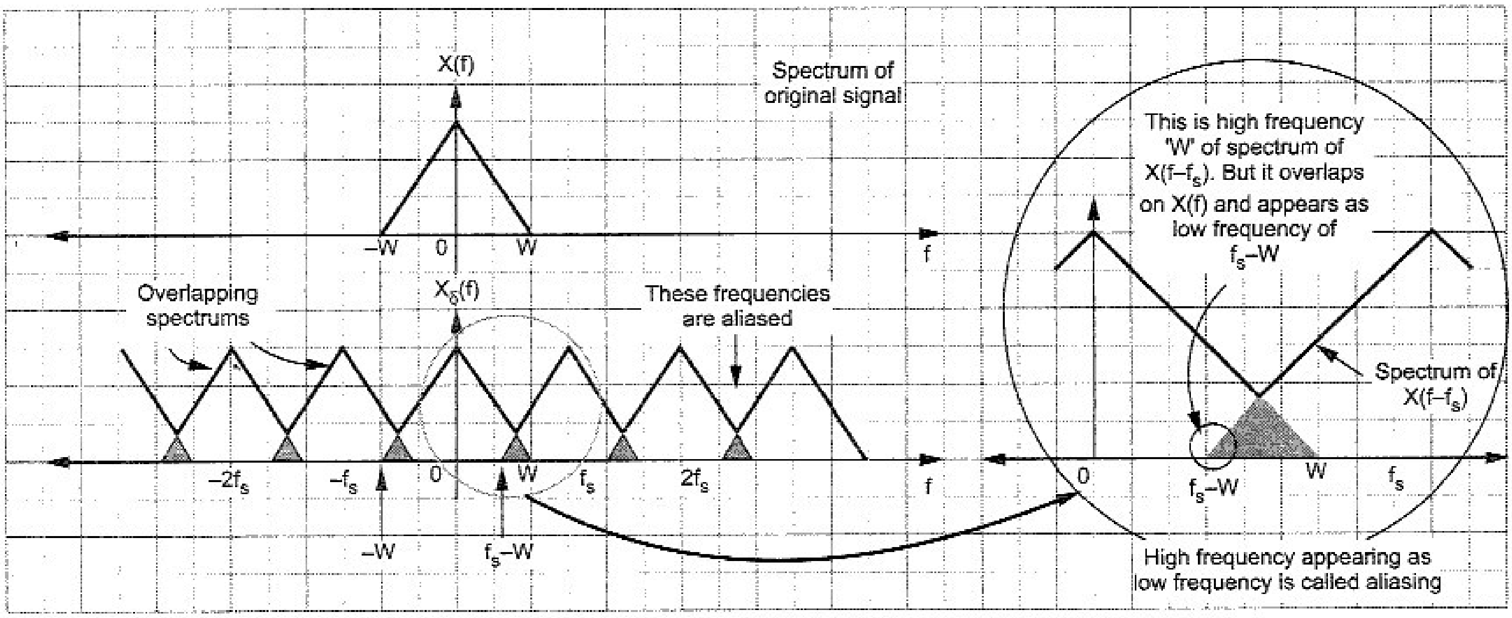

Fig. 2.1.4 Effects of undersampling or aliasing

Comments :

i) The spectrums located at $X(f)$, $X(f-f_s)$, $X(f-2f_s)$,... overlap on each other.

ii) Consider the spectrums of $X(f)$ and $X(f-f_s)$ shown as magnified in above figure. The frequencies from $(f_s - W)$ to W are overlapping in these spectrums.

iii) The high frequencies near 'ω' in $X(f-f_s)$ overlap with low frequencies $(f_s - W)$ in $X(f)$.

Definition of aliasing : When the high frequency interferes with low frequency and appears as low frequency, then the phenomenon is called aliasing.

Effects of aliasing : i) Since high and low frequencies interfere with each other, distortion is generated.

ii) The data is lost and it cannot be recovered.

Different ways to avoid aliasing :

Aliasing can be avoided by two methods :

i) Sampling rate $f_s \geq 2W$.

ii) Strictly bandlimit the signal to 'W'.

i) Sampling rate $f_s \geq 2W$

When the sampling rate is made higher than $2W$, then the spectrums will not overlap and there will be sufficient gap between the individual spectrums. This is shown in Fig. 2.1.5.

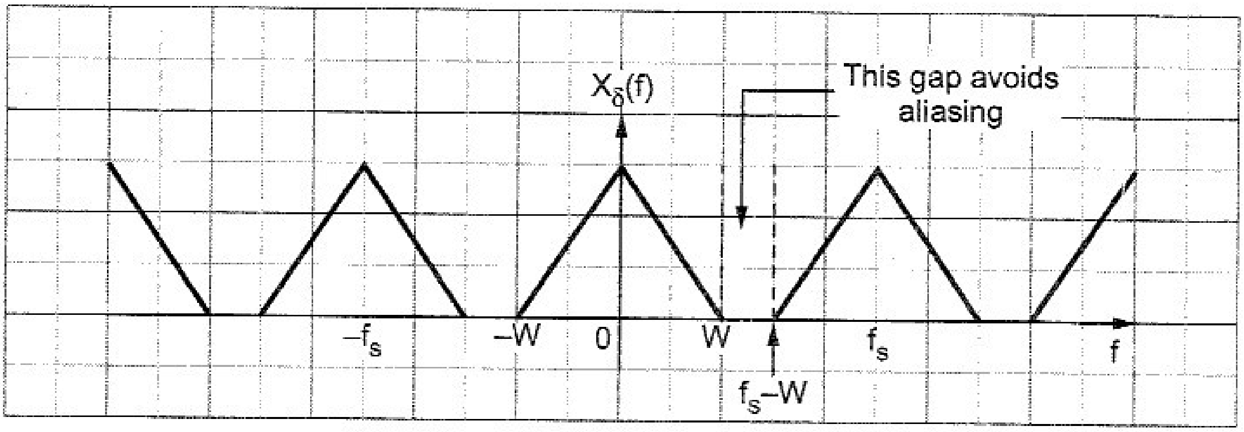

Fig. 2.1.5 $f_s \geq 2W$ avoids aliasing by creating a bandgap

ii) Bandlimiting the signal

The sampling rate is, $f_s = 2W$. Ideally speaking there should be no aliasing. But there can be few components higher than $2W$. These components create aliasing. Hence a low pass filter is used before sampling the signals as shown in Fig. 2.1.6. Thus the output of low pass filter is strictly bandlimited and there are no frequency components higher than 'W'. Then there will be no aliasing.

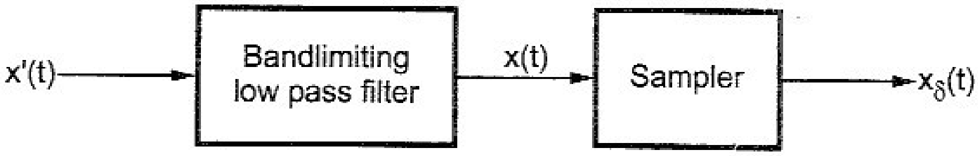

Fig. 2.1.6 Bandlimiting the signal. The bandlimiting LPF is called prealias filter

2.1.4 Nyquist Rate and Nyquist Interval

Nyquist rate : When the sampling rate becomes exactly equal to '$2W$' samples/sec, for a given bandwidth of W hertz, then it is called Nyquist rate.

Nyquist interval : It is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

$$\text{Nyquist rate} = 2W \text{ Hz} \quad \dots (2.1.7)$$

$$\text{Nyquist interval} = \frac{1}{2W} \text{ seconds} \quad \dots (2.1.8)$$

2.1.5 Reconstruction Filter (Interpolation Filter)

Definition

In section 2.1.2 we have shown that the reconstructed signal is the succession of sinc pulses weighted by $x(nT_s)$. These pulses are interpolated with the help of a low pass filter. It is also called *reconstruction* filter or *interpolation* filter.

Ideal filter

Fig. 2.1.7 shows the spectrum of sampled signal and frequency response of required filter. When the sampling frequency is exactly $2W$, then the spectrums just touch each other as shown in Fig. 2.1.7. The spectrum of original signal, $X(f)$ can be filtered by an ideal filter having passband from $-W \leq f \leq W$.

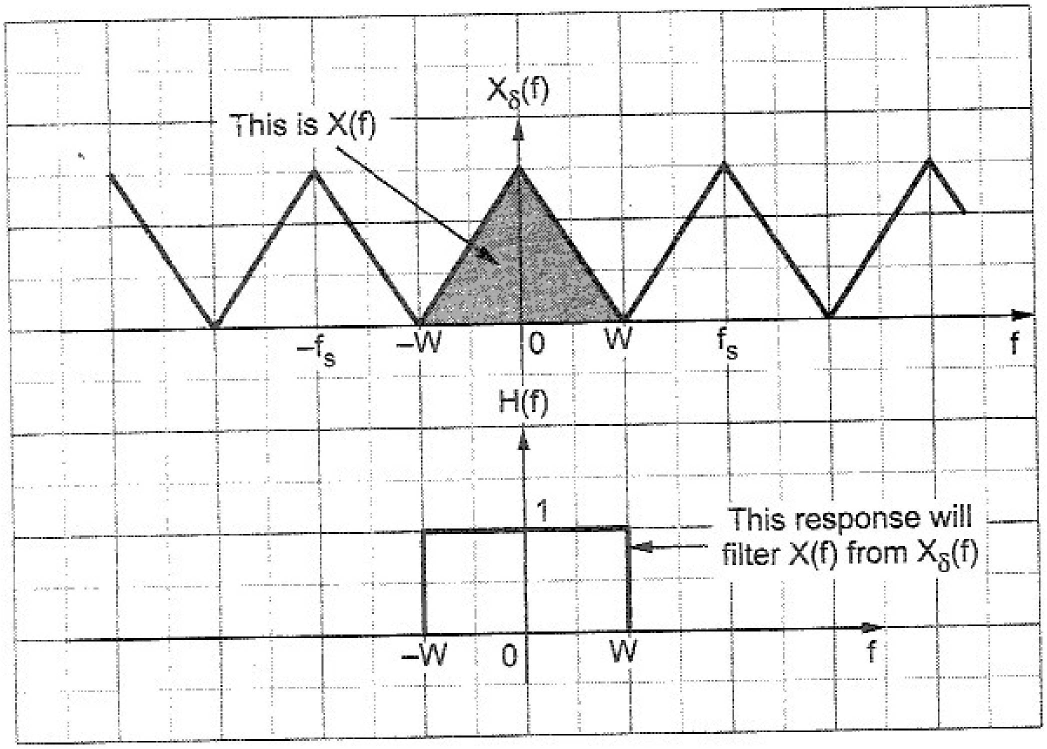

Fig. 2.1.7 Ideal reconstruction filter

Non-ideal filter

As discussed above, an ideal filter of bandwidth 'W' filters out an original signal. But practically ideal filter is not realizable. It requires some transition band. Hence f_s must be greater than $2W$. It creates the gap between adjacent spectrums of $X_\delta(f)$. This gap can be used for the transition band of the reconstruction filter. The spectrum $X(f)$ is then properly filtered out from $X_\delta(f)$. Hence the sampling frequency must be greater than '$2W$' to ensure sufficient gap for transition band.

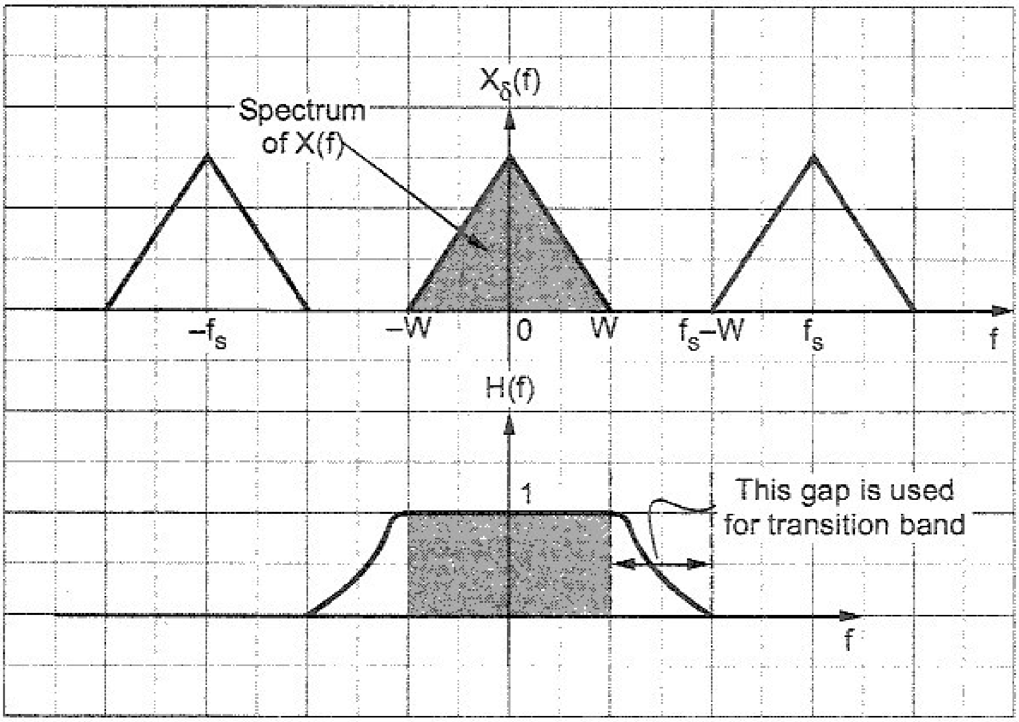

Fig. 2.1.8 Practical reconstruction filter

2.1.6 Zero-Order Hold for Practical Reconstruction

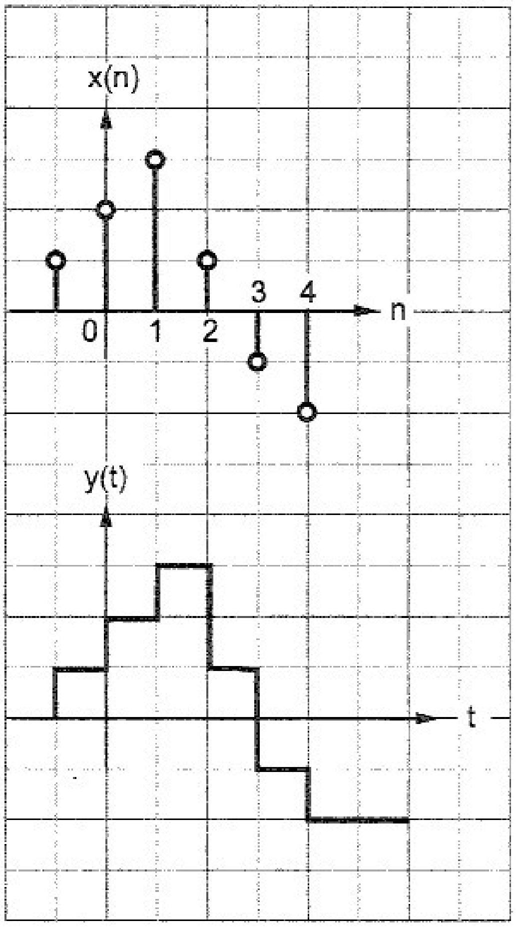

Fig. 2.1.9 Input and output of zero order hold

- The zero-order hold circuit is used for practical reconstruction. It simply hold the value $x(n)$ for 'T' seconds. Here 'T' is the sampling period .
- The output of the zero-order hold is staircase signal that approximates $x(n)$. This is shown in Fig. 2.1.9.
- Let the impulse response of zero-order hold be represented as,

$$h(t) = \begin{cases} 1, & 0 \le t \le T \\ 0 & \text{Otherwise} \end{cases}$$

Then the output $y(t)$ of the zero-order hold will be convolution of $h(t)$ and sampled input $x_\delta(t)$. i.e,

$$y(t) = h(t) * \sum_{n=-\infty}^{\infty} x(n)\ \delta(t - nT)$$

$$= h(t) * x_\delta(t)$$

$$\therefore \quad Y(\omega) = H(\omega) \cdot X(\omega)$$

Here $$H(\omega) = 2e^{-j\omega T/2} \frac{\sin \frac{\omega T}{2}}{\omega}$$

$$\therefore \quad Y(\omega) = 2e^{-j\frac{\omega T}{2}} \frac{\sin \frac{\omega T}{2}}{\omega} \cdot X(\omega)$$

The above equation shows that spectrum $X(\omega)$ is changed due to convolution or passing through zero order hold. These changes are,

(i) There is linear phase shift corresponding to time delay of $\frac{T}{2}$ sec.

(ii) The main lobe of $\frac{\sin \frac{\omega T}{2}}{\omega}$ modifies the shape of $X(\omega)$.

- The above modifications can be reduced by increasing the sampling frequency ω_s or reducing the time 'T'.
- Sometimes anti-imaging filter is used for compensating the modifications. It's spectrum is given as,

$$H_c(\omega) = \begin{cases} \frac{\omega T}{2 \sin \frac{\omega T}{2}}, & |\omega| \le \omega_m \\ 0, & |\omega| > \omega_s - \omega_m \end{cases}$$

This filter provides reverse action to that of zero-order hold. Fig. 2.1.10 shows the block diagram with anit-imaging filter.

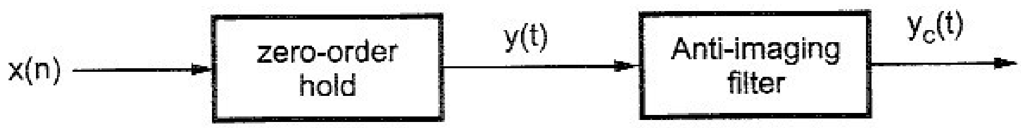

Fig. 2.1.10 Block diagram of practical reconstruction

2.1.7 Sampling Theorem in Frequency Domain

Statement :

We have seen that if the bandlimited signal is sampled at the rate of ($f_s > 2W$) in time domain, then it can be fully recovered from its samples. This is sampling theorem in time domain. A dual of this also exists and it is called sampling theorem in frequency domain. It states that, *"A timelimited signal which is zero for $|t|>T$ is uniquely determined by the samples of its frequency spectrum at intervals less than $\frac{1}{2T}$ hertz apart".*

Explanation :

- Thus the spectrum is sampled at $f_s < \frac{1}{2T}$ in the frequency domain. T is the maximum time limit above which signal $x(t)$ goes to zero. 'f_s' represents the sampling frequency interval in the frequency spectrum of the signal. Note that here f_s does not represent number of samples taken per second. But it represents the frequency interval at which the samples are separated in frequency domain.
- Fig. 2.1.11 illustrates the sampling theorem in frequency domain. We can see from Fig. 2.1.11 (a) that a rectangular pulse is time limited to $\pm\frac{T}{2}$ seconds i.e., $x(t) = A$ for $-\frac{T}{2} \leq t < \frac{T}{2}$. The spectrum of rectangular pulse is shown in Fig. 2.1.11 (b). This spectrum $X(f)$ of Fig. 2.1.11 (b) is sampled at the uniform intervals less than $\frac{1}{2T}$ Hz. The sampled version of this spectrum is shown in Fig. 2.1.11 (c) and called $X_\delta(f)$. Thus each frequency sample of $X_\delta(f)$ is separated by 'f_s' Hz with respect to the neighbouring frequency samples.

As shown in Fig. 2.1.11 (c); since $X(f)$ is sampled in frequency domain, this is called sampling theorem in frequency domain. Therefore now we can state sampling theorem in frequency domain as,

"A timelimited spectrum $X(f)$ can be represented and recovered fully from its samples if the samples are taken at the intervals less than $\frac{1}{2T}$ Hz apart".

We have represented $x(t)$ in terms of its samples $x(nT_s)$ and an interpolation function (sinc function) in equation (2.1.6) for time domain sampling theorem. Similarly a continuous frequency spectrum $X(f)$ can be represented in terms of its samples $X(kf_s)$ and an interpolation function (sinc function) as follows :

$$X(f) = \sum_{k=-\infty}^{\infty} X(kf_s)\,\text{sinc}[T(f-kf_s)] \quad \ldots (2.1.9)$$

Aliasing can also occur in frequency domain sampling. The replicas are in the time domain and are located at multiple of

$$T_s = \frac{1}{f_s}$$

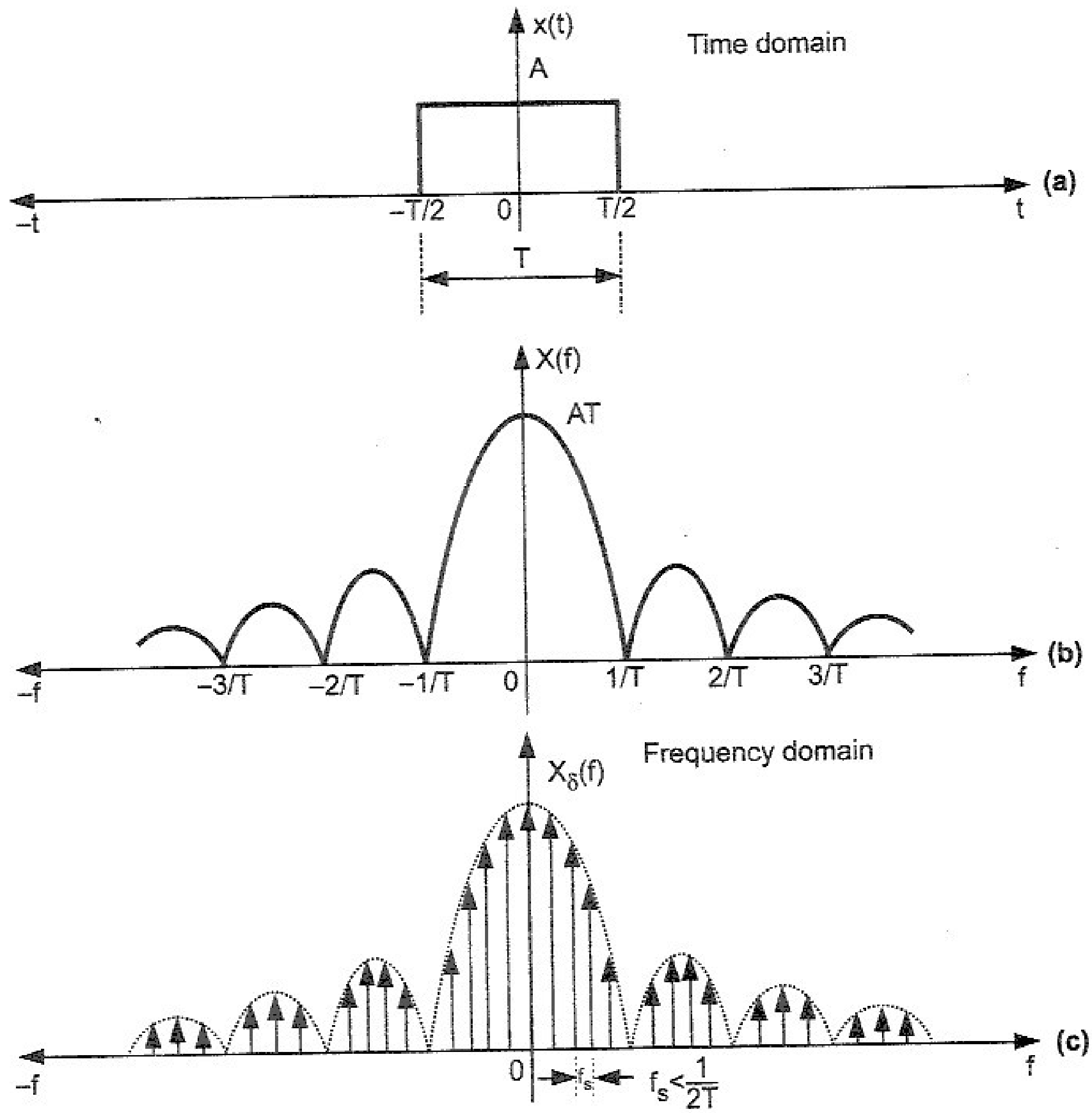

Fig. 2.1.11 (a) Signal x(t) time limited to $\pm\frac{T}{2}$

(b) Continuous spectrum of $x(t)$

(c) Sampled spectrum X_δ (f)

2.1.8 Sampling of Bandpass Signals

April/May-2004, May/June 2006

Statement :

In the last sections we discussed sampling theorem for low pass signals. However when the signal is bandpass in nature, then different criteria should be used to sample the signal. The sampling theorem for bandpass signals can be written as follows.

The bandpass signal x(t) whose maximum bandwidth is 2W can be completely represented into and recovered from its samples if it is sampled at the minimum rate of twice the bandwidth.

- **Spectrum of bandpass signal :** Thus if bandwidth is $2W$, then minimum sampling rate for bandpass signal should be $4W$ samples per second. Fig. 2.1.12 shows the spectrum of bandpass signal.

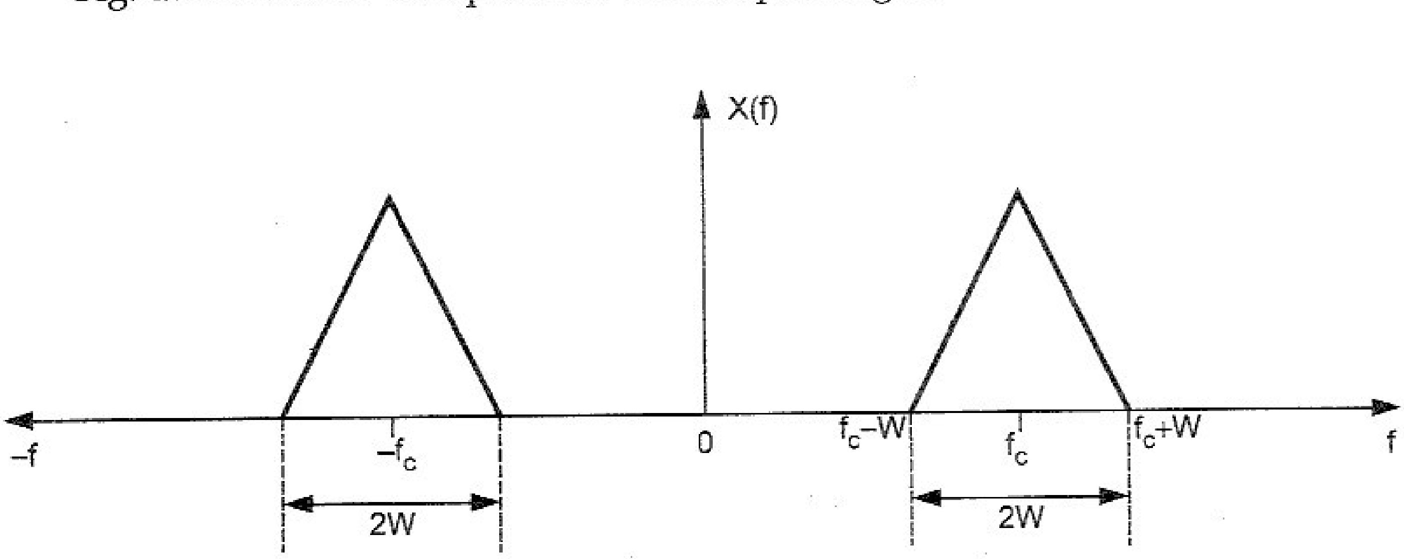

Fig. 2.1.12 Spectrum of bandpass signal

The spectrum is centred around frequency f_c. The bandwidth is $2W$. Thus the frequencies in the bandpass signal are from $f_c - W$ to $f_c + W$. That is the highest frequency present in the bandpass signal is $f_c + W$. Normally the centre frequency $f_c > W$.

- **Inphase and quadrature components :** This bandpass signal is first represented in terms of its inphase and quadrature components.

Let $x_I(t)$ = Inphase component of $x(t)$

and $x_Q(t)$ = Quadrature component of $x(t)$

Then we can write $x(t)$ in terms of inphase and quadrature components as,

$$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) \qquad \text{... (2.1.10)}$$

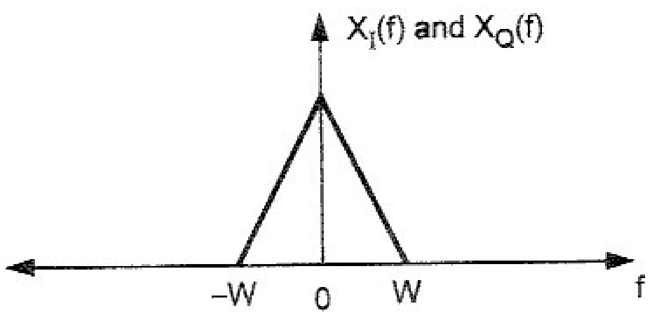

Fig. 2.1.13 Spectrum of inphase and quadrature components of bandpass signal x(t)

The inphase and quadrature components are obtained by multiplying $x(t)$ by $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ and then suppressing the sum frequencies by means of low pass filters. Thus inphase $x_I(t)$ and quadrature $x_Q(t)$ components contain only low frequency components. The spectrum of these components is limited between $-W$ to $+W$. This is shown in Fig. 2.1.13.

- **Representation in terms of samples** : After some mathematical manipulations, on equation (2.1.10) we obtain the reconstruction formula as,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4W}\right) \operatorname{sinc}\left(2Wt - \frac{n}{2}\right) \cos\left[2\pi f_c\left(t - \frac{n}{4W}\right)\right] \quad \ldots (2.1.11)$$

Compare this reconstruction formula with that of low pass signals given by equation (2.1.6). It is clear that $x(t)$ is represented by $x\left(\frac{n}{4W}\right)$ completely.

Here, $x\left(\frac{n}{4W}\right) = x(nT_s)$ = Sampled version of bandpass signal

and $T_s = \frac{1}{4W}$

- Thus if $4W$ samples per second are taken, then the bandpass signal of band width $2W$ can be completely recovered from its samples.

Thus, for bandpass signals of bandwidth $2W$,

Minimum sampling rate = Twice of bandwidth

= $4W$ samples per second

Example 2.1.1 : *Show that a bandlimited signal of finite energy which has no frequency components higher than W Hz is completely described by specifying values of the signals at instants of time separated by 1 / 2W seconds and also show that if the instantaneous values of the signal are separated by intervals larger than* $\frac{1}{2W}$ *seconds, they fail to describe the signal. A bandpass signal has spectral range that extends from 20 to 82 kHz. Find the acceptable range of sampling frequency* f_s.

April/May-2005, May/June-2006, 8 Marks

Solution :

Step 1 : Define $x_\delta(t)$.

Let $x(t)$ be the bandlimited signal which has no frequency components higher than W Hz. Let it be sampled by a sampling function,

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The sampling function is the train of impulses with T_s as distance between successive impulses. Let $x(nT_s)$ be the instantaneous amplitude of signal $x(t)$ at instant $t = T_s$. The sampled version of $x(t)$ can be represented as multiplication of $x(nT_s)$ and $\delta(t)$ i.e.

$$X_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - nT_s) \quad \text{... (2.1.12)}$$

Step 2 : Fourier transform of $x_\delta(t)$ i.e. $X_\delta(f)$

Fourier transform of this sampled signal can be obtained as,

$$X_\delta(f) = FT\{x_\delta(t)\}$$

$$\therefore \quad X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad \text{... (2.1.13)}$$

Here f_s is the sampling rate which is given as $f_s = \frac{1}{T_s}$.

And $X(f - nf_s) = X(f)$ at $nf_s = 0, \ \pm f_s, \pm 2f_s, \pm 3f_s \ldots$

Thus the same spectrum $X(f)$ appears at $f = 0, f = \pm f_s, f = \pm 2f_s$ etc. This means that a periodic spectrum with period equal to f_s is generated in frequency domain because of sampling $x(t)$ in time domain. Therefore equation (2.1.13) can be written as,

$$X_\delta(f) = f_s\,X(f) + f_s\,X(f \pm f_s) + f_s\,X(f \pm 2f_s) + f_s\,X(f \pm 3f_s) + f_s\,X(f \pm 4f_s) + \ldots.. \quad \text{... (2.1.14)}$$

or

$$X_\delta(f) = f_s\,X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f_s\,X(f - nf_s) \quad \text{... (2.1.15)}$$

Step 3 : Relation between $X(f)$ and $X_\delta(f)$.

By definition of Fourier transform, $X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$

For sampled version of $x(t)$, we have $t = nT_s$. Then above equation becomes,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\,e^{-j2\pi fnT_s} \quad \text{... (2.1.16)}$$

It is given that the signal is bandlimited to W Hz and,

$$T_s = \frac{1}{2W} \text{ seconds}, \quad \therefore \ f_s = \frac{1}{T_s} = 2W \quad \text{... (2.1.17)}$$

From equation (2.1.14) we know that $X_\delta(f)$ is periodic in f_s. The spectrum $X(f)$ and $X_\delta(f)$ are shown in Fig. 2.1.14.

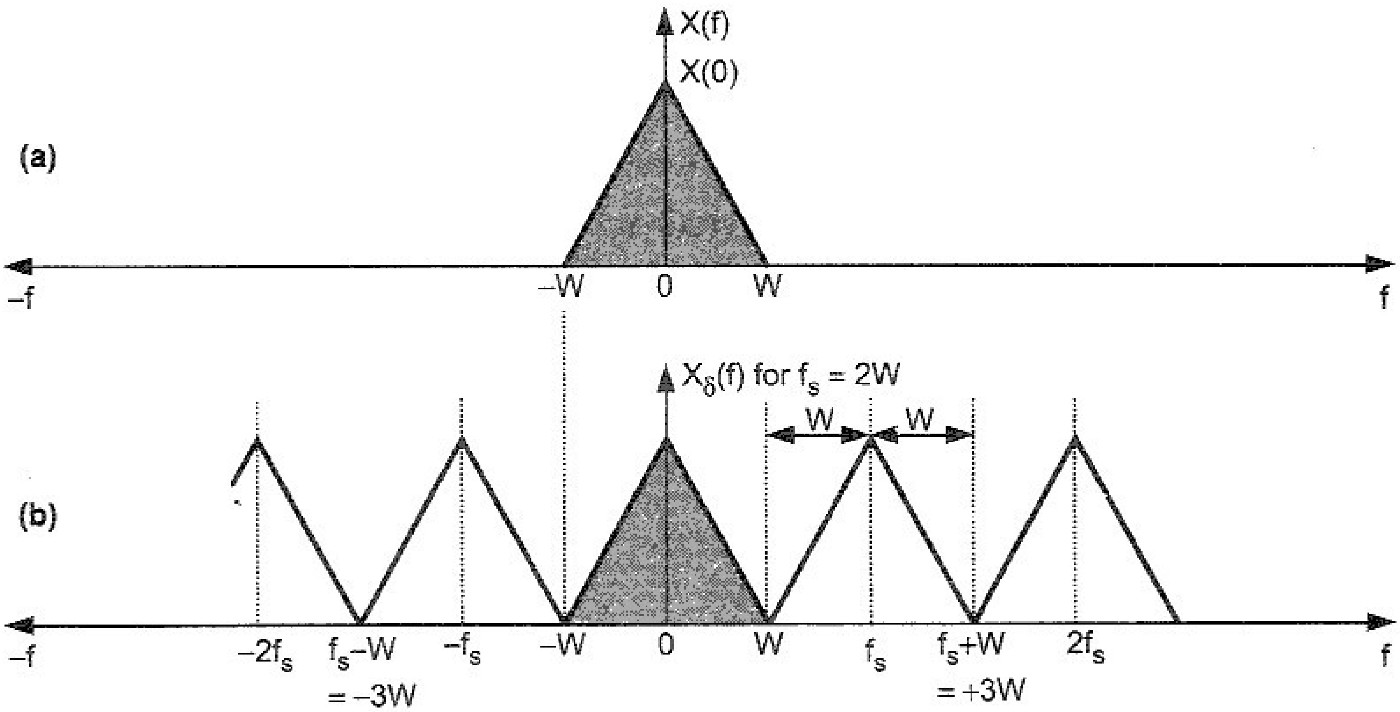

Fig. 2.1.14 (a) Spectrum of $x(t)$

(b) Spectrum of $X_\delta(f)$ with $f_s = 2W$

Since $f_s = 2W$; $f_s - W = W$ and $f_s + W = 3W$

Thus the periodic spectrums $X(f)$ just touch $\pm W, \pm 3W, \pm 5W$ .... etc.

Thus there is no aliasing. From equation (2.1.15) we can write,

$$X(f) = \frac{1}{f_s} X_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_s) \quad \text{... (2.1.18)}$$

With $f_s = 2W$ in above equation,

$$X(f) = \frac{1}{2W} X_\delta(f) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} X(f - nf_s)$$

i.e.

$$X(f) = \frac{1}{2W} X_\delta(f) \qquad \text{For } -W \le f \le W \quad \text{... (2.1.19)}$$

Step 4 : Relation between $x(t)$ and $x(nT_s)$ or $x\left(\frac{n}{2W}\right)$.

Putting the value of $X_\delta(f)$ from equation (2.1.16) in the above equation,

$$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n \cdot T_s}$$

Since

$$T_s = \frac{1}{2W}$$

i.e. $$X(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W} \qquad ...(2.1.20)$$

$x(t)$ can be recovered from $X(f)$ by taking Inverse Fourier Transform of above equation.

i.e. $$x(t) = IFT\left\{\frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W}\right\} \qquad ... (2.1.21)$$

This equation shows that $x(t)$ is represented completely by its samples $x\left(\frac{n}{2W}\right)$ for $-\infty < n < \infty$. That is the sequence $x\left(\frac{n}{2W}\right)$ has all the information contained in $x(t)$.

Reconstruction of signal from samples :

Consider equation (2.1.21),

$$x(t) = IFT\left\{\frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W}\right\}$$

- By definition of Inverse Fourier Transform (IFT) the above equation becomes,

$$x(t) = \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi f n/W} e^{j2\pi f t}\, dt$$

Interchanging the order of summation of integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{1}{2W} \int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)} dt$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

Since $\text{sinc}\,\theta = \frac{\sin \pi\theta}{\pi\theta}$, above equation becomes,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \text{sinc}(2Wt - n) \qquad -\infty < n < \infty$$

This is interpolation formula to reconstruct $x(t)$ from its samples $x(nT_s)$.

- Thus the above discussion shows that the signal can be completely represented into and recovered from its samples if the spacing between the successive samples is $\frac{1}{2W}$ seconds. i.e. $f_s = 2W$ samples per second.

Sampling frequency for bandpass signal :

The spectral range of the bandpass signal is 20 to 82 kHz.

$$\text{Bandwidth} = 2W = 82 \text{ kHz} - 20 \text{ kHz} = 62 \text{ kHz}$$

$$\begin{aligned} \text{Minimum sampling rate} &= 2\times \text{ Bandwidth} \\ &= 2\times 62 \text{ kHz} \\ &= 124 \text{ kHz} \end{aligned}$$

Normally the range of minimum sampling frequencies is specified for bandpass signals. It lies between 4 W to 8 W samples per second.

∴ Range of minimum sampling frequencies

$$\begin{aligned} &= (2 \times \text{Bandwidth}) \text{ to } (4 \times \text{Bandwidth}) \\ &= 2 \times 62 \text{ kHz to } 4 \times 62 \text{ kHz} \\ &= 124 \text{ kHz to } 248 \text{ kHz} \end{aligned}$$

Example 2.1.2 : *Find the Nyquist rate and Nyquist interval for following signals.*

i) $m(t) = \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t)$

ii) $m(t) = \frac{\sin 500\pi t}{\pi t}$

Solution : i)

$$\begin{aligned} m_1(t) &= \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t) \\ &= \frac{1}{2\pi} \left\{ \frac{1}{2} [\cos(4000\pi t - 1000\pi t) + \cos(4000\pi t + 1000\pi t)] \right\} \\ &= \frac{1}{4\pi} [\cos 3000\pi t + \cos 5000\pi t] \\ &= \frac{1}{4\pi} [\cos 2\pi f_1 t + \cos 2\pi f_2 t] \end{aligned}$$

Comparing, we get, $f_1 = 1500$ Hz and $f_2 = 2500$ Hz

Here highest frequency $W = f_2 = 2500$ Hz

$\therefore$ Nyquist rate $= 2W = 2 \times 2500 = 5000$ Hz

$$\text{Nyquist interval} = \frac{1}{2W} = \frac{1}{2 \times 2500} = 0.2 \text{ msec}$$

ii)

$$m_2(t) = \frac{\sin 500 \pi t}{\pi t}$$

$$= \frac{\sin 2\pi f t}{\pi t}$$

Comparing, we get, $f = 250$ Hz or $W = 250$ Hz

$\therefore$ Nyquist rate $= 2W = 2 \times 250 = 500$ Hz

$$\text{Nyquist interval} = \frac{1}{2W} = \frac{1}{2 \times 250} = 2 \text{ msec}$$

Example 2.1.3 : *Fig. 2.1.15 shows the spectrum of a message signal. The signal is sampled at the rate of $f_s = 1.5 f_{max}$, where $f_{max} = 1$ Hz, is maximum signal frequency. Sketch the spectrum of the sampled version of the signal. If the sampled signal is passed through an ideal low pass filter of bandwidth f_{max}, sketch the spectrum of the output signal from this filter.*

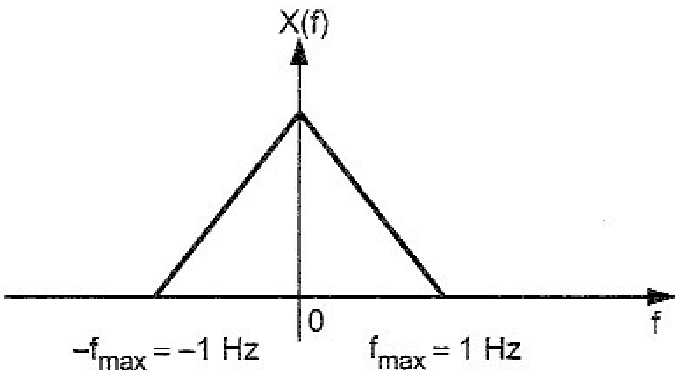

Fig. 2.1.15 Spectrum of message signal *X* (*f*)

Solution : When $x(t)$ is sampled instantaneously its spectrum is given as,

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - n f_s)$$