$$= \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2} \angle(-180 + \tan^{-1}0.5\omega)$$

$$\therefore \ |G(j\omega)H(j\omega)| = \frac{2\sqrt{1+0.25\omega^2}}{1+\omega^2}$$

$$\angle G(j\omega)H(j\omega) = -180° + \tan^{-1}0.5\omega$$

The exact shape of G(jω)H(jω) locus is determined by calculating the magnitude and phase of G(jω)H(jω) for various values of ω.

ω rad/sec	0	0.4	1.0	2.0	10.0	∞
|G(jω)H(jω)|	2	1.76	1.12	0.57	0.1	0
∠G(jω)H(jω) deg	−180	−168	−153	−135	−101	−90

From the above analysis, we can conclude that G(jω)H(jω) locus starts at −180° axis at a magnitude of −2 for ω = 0 and meets the origin along −90° axis when ω = +∞.

The section C_1 in s-plane and its corresponding mapping in G(s)H(s)-plane are shown in fig 4.18.2. and 4.18.3.

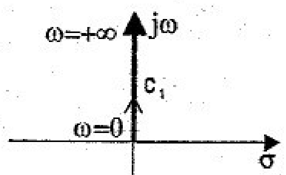


Fig 4.18.2 : Section C_1 in s-plane

⇒

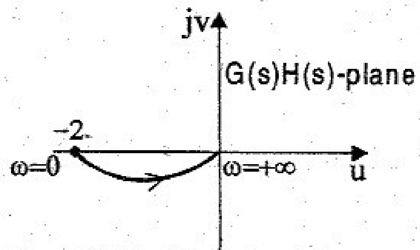


Fig 4.18.3 : Mapping of section C_1 in G(s)H(s)-plane

MAPPING OF SECTION C_2

The mapping of section C_2 from s-plane to G(s)H(s)-plane is obtained by letting $s = \underset{R\to\infty}{\text{Lt}} \ Re^{j\theta}$ in G(s)H(s) and varying θ from +π/2 to −π/2. Since $s \to Re^{j\theta}$ and $R \to \infty$, G(s) H(s) can be approximated as shown below, [i.e., $(1+sT) \approx sT$].

$$G(s)H(s) = \frac{2(1+0.5s)}{(1+s)\,(-1+s)} \approx \frac{2 \times 0.5s}{s \times s} = \frac{1}{s}$$

Let, $s = \underset{R\to\infty}{\text{Lt}} \ Re^{j\theta}$.

$$\therefore G(s)H(s)\Bigg|_{s=\underset{R\to\infty}{\text{Lt}} Re^{j\theta}} = \frac{1}{\underset{R\to\infty}{\text{Lt}} \ Re^{j\theta}} = 0e^{-j\theta}$$

When $\theta = \frac{\pi}{2}$, $G(s)H(s) = 0e^{-j\frac{\pi}{2}}$ .....(1)

When $\theta = -\frac{\pi}{2}$, $G(s)H(s) = 0e^{j\frac{\pi}{2}}$ .....(2)

From the equations (1) and (2) we can say that section C_2 in s-plane (fig 4.18.4) is mapped as circular arc of zero radius around origin in G(s)H(s)-plane with argument varying from −π/2 to +π/2 as shown in fig 4.18.5.

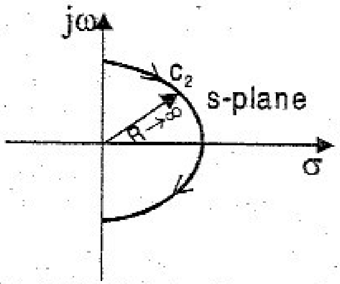


Fig 4.18.4 : *Section C_2 in s-plane*

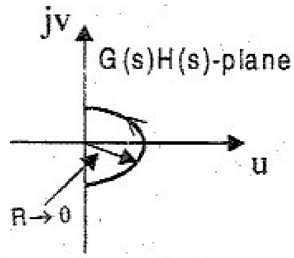


Fig 4.18.5 : *Mapping of section C_2 in G(s)H(s)-plane*

MAPPING OF SECTION C_3

In section C_3, ω varies from −∞ to 0. The mapping of section C_3 is given by the locus of G(jω)H(jω) as ω is varied from −∞ to 0. This locus is the inverse polar plot of G(jω)H(jω).

The inverse polar plot is given by the mirror image of polar plot with respect to real axis. The section C_3 in s-plane and its corresponding contour in G(s)H(s) plane are shown in fig 4.18.6 and fig 4.18.7.

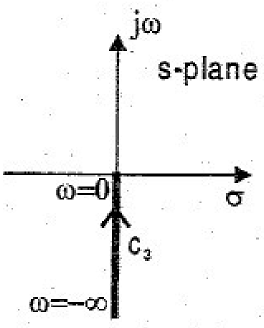


Fig 4.18.6 : *Section C_3 in s-plane*

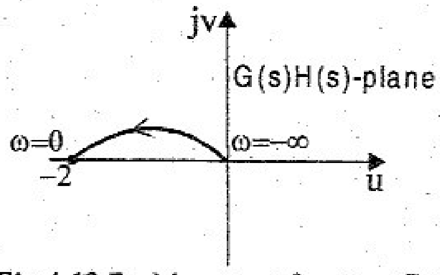


Fig 4.18.7 : *Mapping of section C_3 in G(s)H(s)-plane*

COMPLETE NYQUIST PLOT

The entire Nyquist plot in G(s)H(s)-plane can be obtained by combining the mappings of individual sections, as shown in fig 4.18.8.

STABILITY ANALYSIS

On travelling through Nyquist contour it is observed that −1+j0 point is encircled in anticlockwise direction one time. Also the open loop transfer function has one pole at right half s-plane. Since the number of anticlockwise encirclement is equal to number of open loop poles on right half s-plane, the closed loop system is stable.

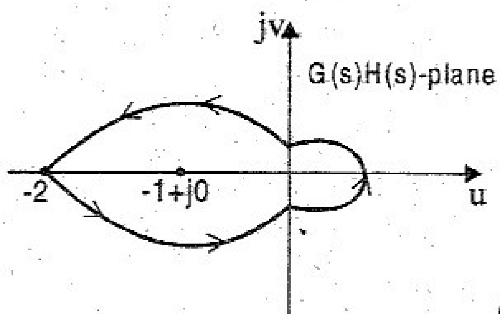


Fig 4.18.8 : Nyquist plot of $G(s)\,H(s) = \dfrac{(s+2)}{(s+1)(s-1)}$

RESULT

(a) Open loop system is unstable

(b) Closed loop system is stable.

4.6 RELATIVE STABILITY

The ***Relative stability*** indicates the closeness of the system to stable region. It is an indication of the strength or degree of stability.

In time domain, the relative stability may be measured by relative settling times of each root or pair of roots. The settling time is inversely proportional to the location of roots of characteristic equation. If the root is located far away from the imaginary axis, then the transients dies out faster and so the relative stability of system will improve. The transient response and so the relative stability for various location of roots in s-plane are shown in fig 4.6.

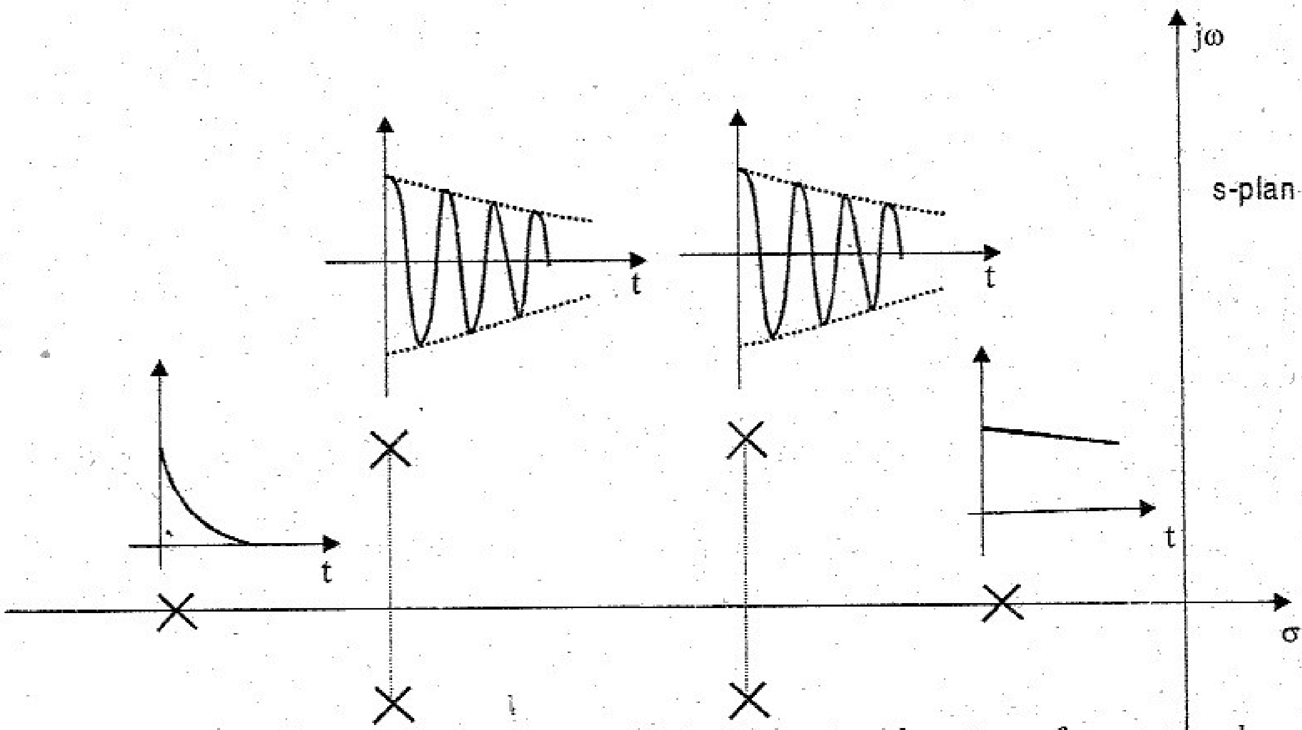


Fig 4.6 : Transient response and relative stability for various locations of roots on s-plane

In frequency domain the relative stability of a system can be studied from Nyquist plot. The relative stability of the system is given by closeness of polar plot to -1+j0 point. As the polar plot gets closer to -1+j0 point the system moves towards instability.

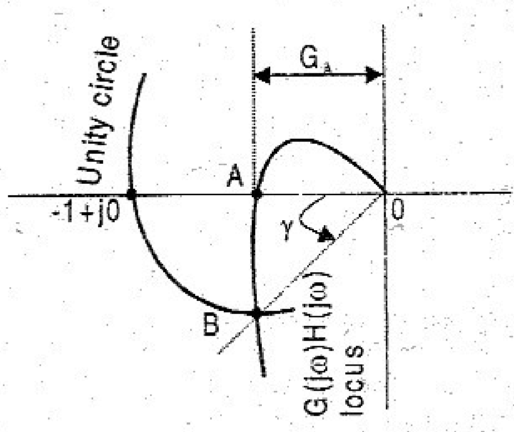


Fig : 4.7

The relative stability in frequency domain are quantitatively measured in terms of phase margin and gain margin. Consider a G(jω)H(jω) locus as shown in fig 4.7. Let this locus cross the real axis at point-**A** and a unit circle drawn with origin as centre cuts this locus at point-**B**. Let G_A be the magnitude of G(jω)H(jω) at point-A , and γ be the angle between negative real axis and line OB.

If the gain of the system is increased, then the locus will shift upwards and it may cross real axis at −1+j0 point. When the locus passes through −1+j0 point, $G_A \to 1$ and $\gamma \to 0$. Hence the closeness of G(jω) H(jω) locus to the critical point −1+j0 can be measured in terms of intercept G_A and angle γ. The value of G_A and γ are quantitative indications of relative stability. These values are used to define gain margin and phase margin as practical measures of relative stability.

The concepts of gain margin and phase margin are defined for open loop systems but from the values of gain margin and phase margin the stability of closed loop system can be judged.

4.7 GAIN MARGIN AND PHASE MARGIN

Gain margin is a factor by which the system gain can be increased to drive the system to the verge of instability. With reference to fig 4.7 the magnitude of G(jω)H(jω) is G_A when it crosses real axis and the phase corresponding to that point is −180°. The frequency corresponding to that point be ω_{pc}. If the gain of the system is increased by a factor $\frac{1}{G_A}$ then the magnitude at the frequency ω_{pc} will be, $G_A \times \frac{1}{G_A} = 1$. Now the G(jω)H(jω) locus will pass through −1+j0 point driving the system to the verge of instability. Hence the gain margin, K_g of the system may be defined as the reciprocal of the gain at which the phase angle is 180°. The frequency at which the phase angle is 180° is called phase crossover frequency.

Gain margin, $K_g = \left.\frac{1}{|G(j\omega)H(j\omega)|}\right|_{\omega=\omega_{pc}} = \frac{1}{G_A}$

Gain margin in db $= 20\log\frac{1}{|G(j\omega)H(j\omega)|} = 20\log\frac{1}{G_A} = -20\log G_A$

> ***Note :*** *Gain Margin in decibels is given by negative of db magnitude of $G(j\omega)$ at phase crossover frequency. Hence at ω_{pc}, if db magnitude is negative, then gain margin is positive and vice versa.*

The ***phase margin*** is defined as the amount of additional phase lag at gain crossover frequency required to bring the system to verge of instability. The frequency at which the magnitudes of $G(j\omega)H(j\omega)$ equals unity is called the gain crossover frequency, ω_{gc}. With reference to fig 4.7, the phase angle corresponding to the meeting point of unity circle and $G(j\omega)H(j\omega)$ locus is $-180° + \gamma$. Now with magnitude remaining unity, if an additional phase lag equal to γ is introduced then the net phase angle becomes $-180°$ and $G(j\omega)H(j\omega)$ locus will pass through $-1+j0$ point driving the system to the verge of instability. This additional phase lag γ is known as phase margin.

Let, $\phi_{gc} = \angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$; Now $-180°+\gamma = \phi_{gc}$

$\therefore$ Phase margin, $\gamma = 180° + \phi_{gc}$

For stability of closed loop system the gain margin of open loop system should be greater than 1 or if it is expressed in db it should be positive and phase margin of open loop system should be positive.

EXAMPLE 4.19

The open loop transfer function of a unity feedback system is given by, $G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$ Derive an expression for gain K in terms of T_1, T_2 and specified gain margin, K_g.

SOLUTION

Given that, $G(s) = \frac{K}{s(1+sT_1)(1+sT_2)}$

Let $s = j\omega$.

$$G(j\omega) = \frac{K}{j\omega(1+j\omega T_1)(1+j\omega T_2)} = \frac{K}{j\omega(1+j\omega T_2 + j\omega T_1 - \omega^2 T_1 T_2)}$$

$$= \frac{K}{j\omega\left[1+j\omega(T_1+T_2) - \omega^2 T_1 T_2\right]} = \frac{K}{j\omega - \omega^2(T_1+T_2) - j\omega^3 T_1 T_2}$$

$$= \frac{K}{-\omega^2(T_1+T_2) + j\omega(1-\omega^2 T_1 T_2)}$$

The gain margin, K_g is defined as the reciprocal of the magnitude of G(jω) at phase crossover frequency. At phase crossover frequency the magnitude is purely real. Hence at phase crossover frequency, ω_{pc}, the imaginary part of G(jω) is zero.

$$\therefore \text{At } \omega = \omega_{pc}, \quad \omega_{pc}\left(1-\omega_{pc}^2 T_1T_2\right) = 0 \quad \Rightarrow \quad 1-\omega_{pc}^2 T_1T_2 = 0 \quad \Rightarrow \quad -\omega_{pc}^2 T_1T_2 = -1$$

$$\therefore \omega_{pc} = \frac{1}{\sqrt{T_1T_2}}$$

At $\omega = \omega_{pc}$, the imaginary part is zero,

$$\therefore |G(j\omega)| = \left|\frac{K}{-\omega^2(T_1+T_2)}\right| = \frac{K}{\omega^2(T_1+T_2)}$$

$$\therefore \text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{K/\omega_{pc}^2(T_1+T_2)} = \frac{\omega_{pc}^2(T_1+T_2)}{K}$$

Put, $\omega_{pc}^2 = \frac{1}{T_1T_2}$ in the above equation.

$$\therefore K_g = \frac{\left(\frac{1}{T_1T_2}\right)(T_1+T_2)}{K} \quad \Rightarrow \quad K = \frac{1}{K_g}\frac{T_1+T_2}{T_1T_2} = \frac{1}{K_g}\left(\frac{T_1}{T_1T_2}+\frac{T_2}{T_1T_2}\right) \quad \Rightarrow \quad K = \frac{1}{K_g}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)$$

RESULT

The expression for gain K in terms of K_g, T_1 and T_2 is, $K = \frac{1}{K_g}\left(\frac{1}{T_1}+\frac{1}{T_2}\right)$

EXAMPLE 4.20

Determine the Gain crossover frequency, Phase crossover frequency, Gain margin and Phase margin of a system with open loop transfer function, $G(s) = \frac{1}{s(1+2s)(1+s)}$.

SOLUTION

(i) To find phase crossover frequency and gain margin

Given that, $G(s) = \frac{1}{s\,(1+2s)\,(1+s)}$

Let $s = j\omega$.

$$\therefore G(j\omega) = \frac{1}{j\omega\,(1+j2\omega)\,(1+j\omega)} = \frac{1}{j\omega\left(1+j\omega+j2\omega-2\omega^2\right)} = \frac{1}{j\omega\,(j3\omega+1-2\omega^2)} = \frac{1}{-3\omega^2+j\omega\,(1-2\omega^2)}$$

At phase crossover frequency the imaginary part of G(jω) is zero. Hence put $\omega = \omega_{pc}$ in imaginary part and equate to zero to solve for ω_{pc}.

$$\therefore \omega_{pc}\left(1-2\omega_{pc}^2\right) = 0$$

$$\text{since } \omega_{pc} \neq 0,\ 1-2\omega_{pc}^2 = 0 \ \Rightarrow\ -2\omega_{pc}^2 = -1 \ \Rightarrow\ \omega_{pc}^2 = \frac{1}{2} \ \Rightarrow\ \omega_{pc} = \frac{1}{\sqrt{2}} = 0.707 \text{ rad/sec}$$

The gain margin, K_g is defined as reciprocal of magnitude of G(jω) at phase cross over frequency.

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{|1/-3\omega^2|_{\omega=\omega_{pc}}} = 3\omega_{pc}^2 = 3\times 0.707^2 = 1.5$$

Gain margin in db = 20 log K_g = 20 log 1.5 = 3.5 db

(ii) To find gain crossover frequency and phase margin

Given that, $G(s) = \dfrac{1}{s\,(1+2s)\,(1+s)}$

Let $s = j\omega$.

$$\therefore\ G(j\omega) = \frac{1}{j\omega(1+j2\omega)(1+j\omega)} = \frac{1}{\omega\angle 90^\circ \sqrt{1+4\omega^2}\angle\tan^{-1}2\omega \sqrt{1+\omega^2}\angle\tan^{-1}\omega}$$

$$\therefore |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}2\omega - \tan^{-1}\omega$$

At gain crossover frequency, ω_{gc} the magnitude of $G(j\omega)$ is unity.

$$\therefore \text{At } \omega = \omega_{gc}, \qquad |G(j\omega)| = \frac{1}{\omega_{gc}\sqrt{1+4\omega_{gc}^2}\sqrt{1+\omega_{gc}^2}} = 1$$

Solving the above equation for ω_{gc} will be tedious. Hence by trial and error find the root of the above equation.

$$\text{When } \omega = 1, \quad |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}} = \frac{1}{1\sqrt{1+4}\sqrt{1+1}} = 0.3$$

$$\text{When } \omega = 0.5, \quad |G(j\omega)| = \frac{1}{\omega\sqrt{1+4\omega^2}\sqrt{1+\omega^2}} = \frac{1}{0.5\sqrt{1+4\times 0.5^2}\sqrt{1+0.5^2}} = 1.26$$

From the two calculations shown above, we can conclude that the unity magnitude will occur for a frequency between 0.5 and 1.0.

$$\text{When } \omega = 0.6, \quad |G(j\omega)| = \frac{1}{0.6\sqrt{1+4\times 0.6^2}\sqrt{1+0.6^2}} = 0.915$$

$$\text{When } \omega = 0.57, \quad |G(j\omega)| = \frac{1}{0.57\sqrt{1+4\times 0.57^2}\sqrt{1+0.57^2}} = 1.005$$

Let $\omega = 0.57$ be the gain crossover frequency, since for this value of ω the magnitude of $G(j\omega)$ is approximately equal to one.

$\therefore$ Gain crossover frequency, $\omega_{gc} = 0.57$ rad/sec.

Let the phase of $G(j\omega)$ at ω_{gc} be ϕ_{gc}.

$$\text{At } \omega = \omega_{gc} = 0.57, \qquad \phi_{gc} = -90^\circ - \tan^{-1}2\omega - \tan^{-1}\omega$$

$$= -90^\circ - \tan^{-1}(2\times 0.57) - \tan^{-1}0.57 = -168^\circ$$

$\therefore$ Phase margin, $\gamma = 180^\circ + \phi_{gc} = 180^\circ - 168^\circ = 12^\circ$

RESULT

(a) The phase crossover frequency, $\omega_{pc} = 0.707$ rad/sec

(b) The gain crossover frequency, $\omega_{gc} = 0.57$ rad/sec

(c) The gain margin, $K_g = 1.5$

The gain margin in db = 3.5 db

(d) The phase margin, $\gamma = 12^\circ$

EXAMPLE 4.21

The open loop transfer function of a system is $G(s) = \frac{K}{s(1+0.1s)(1+s)}$.

(i) Determine the value of K so that gain margin is 6 db.

(ii) Determine the value of K so that phase margin is 40°.

SOLUTION

(i) To find K for specified gain margin

Given that, $G(s) = \frac{K}{s(1+0.1s)(1+s)}$

Let $s = j\omega$.

$$\therefore G(j\omega) = \frac{K}{j\omega(1+j0.1\omega)(1+j\omega)} = \frac{K}{j\omega(1+j1.1\omega - 0.1\omega^2)} = \frac{K}{-1.1\omega^2 + j\omega(1-0.1\omega^2)}$$

At phase crossover frequency ω_{pc}, the $G(j\omega)$ is real and so equate the imaginary part to zero to solve for ω_{pc}.

$$\text{At } \omega = \omega_{pc}, \quad \omega_{pc}(1-0.1\omega_{pc}^2) = 0 \quad \Rightarrow \quad 1-0.1\omega_{pc}^2 = 0 \quad \Rightarrow \quad -0.1\omega_{pc}^2 = -1$$

$$\therefore \omega_{pc} = \frac{1}{\sqrt{0.1}} = 3.162 \text{ rad/sec}$$

$$\therefore |G(j\omega)|_{\omega=\omega_{pc}} = \left|\frac{K}{-1.1\omega^2}\right|_{\omega=\omega_{pc}} = \frac{K}{1.1 \times 3.162^2} = 0.0909K$$

Given that gain margin = 6db, $\therefore 20 \log K_g = 6 \Rightarrow \log K_g = \frac{6}{20}$

$$\therefore \text{ Gain margin, } K_g = 10^{\frac{6}{20}} = 1.9953$$

By definition of gain margin,

$$\text{Gain margin, } K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$$

$$\therefore 1.9953 = \frac{1}{0.0909K}$$

$$\therefore K = \frac{1}{0.0909 \times 1.9953} = 5.5135$$

(ii) To find K for specified phase margin

Given that, $G(s) = \frac{K}{s(1+0.1s)(1+s)}$. Let $s = j\omega$.

$$\therefore G(j\omega) = \frac{K}{j\omega(1+j0.1\omega)(1+j\omega)} = \frac{K}{\omega\angle 90^\circ \sqrt{1+(0.1\omega)^2}\angle\tan^{-1}0.1\omega \sqrt{1+\omega^2}\angle\tan^{-1}\omega}$$

$$|G(j\omega)| = \frac{K}{\omega\sqrt{1+0.01\omega^2}\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.1\omega - \tan^{-1}\omega$$

Let, ω_{gc} = Gain crossover frequency

$\phi_{gc} = \angle G(j\omega)$ at $\omega = \omega_{gc}$.

At $\omega = \omega_{gc}$, $\phi_{gc} = \angle G(j\omega)|_{\omega=\omega_{gc}} = -90 - \tan^{-1}0.1\,\omega_{gc} - \tan^{-1}\omega_{gc}$

By definition of phase margin,

Phase margin, $\gamma = 180° + \phi_{gc}$

The required phase margin is 40°, $\therefore \gamma = 40°$

$$\therefore\ 40° = 180° - 90° - \tan^{-1}0.1\omega_{gc} - \tan^{-1}\omega_{gc} \quad\Rightarrow\quad \tan^{-1}0.1\,\omega_{gc} + \tan^{-1}\omega_{gc} = 180° - 90° - 40°$$

$$\therefore\ \tan^{-1}0.1\,\omega_{gc} + \tan^{-1}\omega_{gc} = 50°$$

On taking tan on either side we get,

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\tan\left[\tan^{-1}0.1\omega_{gc} + \tan^{-1}\omega_{gc}\right] = \tan 50°$$

$$\frac{\tan\tan^{-1}0.1\omega_{gc} + \tan\tan^{-1}\omega_{gc}}{1 - \tan\tan^{-1}0.1\omega_{gc} \times \tan\tan^{-1}\omega_{gc}} = \tan 50° \Rightarrow \frac{0.1\omega_{gc} + \omega_{gc}}{1 - 0.1\omega_{gc} \times \omega_{gc}} = 1.192 \Rightarrow \frac{1.1\omega_{gc}}{1 - 0.1\omega_{gc}^2} = 1.192$$

On cross multipling the above equation we get,

$$1.1\,\omega_{gc} = 1.192\,(1 - 0.1\omega_{gc}^2) \quad\Rightarrow\quad 0.1192\omega_{gc}^2 + 1.1\omega_{gc} - 1.192 = 0$$

$$\therefore\ \omega_{gc}^2 + \frac{1.1}{0.1192}\omega_{gc} - \frac{1.192}{0.1192} = 0 \quad\Rightarrow\quad \omega_{gc}^2 + 9.228\,\omega_{gc} - 10 = 0$$

$$\therefore \omega_{gc} = \frac{-9.228 \pm \sqrt{9.228^2 + 4 \times 10}}{2} = \frac{-9.228 \pm 11.1873}{2}$$

On taking positive value we get,

$$\omega_{gc} = \frac{-9.228 + 11.1873}{2} = 0.98 \text{ rad/sec.}$$

$$\text{At } \omega = \omega_{gc},\quad |G(j\omega)| = 1\ ;\quad |G(j\omega)|_{\omega=\omega_{gc}} = \frac{K}{\omega_{gc}\sqrt{1 + 0.01\omega_{gc}^2}\sqrt{1 + \omega_{gc}^2}} = 1$$

$$\therefore\ K = \omega_{gc}\sqrt{1 + 0.01\omega_{gc}^2}\sqrt{1 + \omega_{gc}^2} = 0.98\sqrt{1 + 0.01 \times 0.98^2}\sqrt{1 + 0.98^2} = 1.3787$$

RESULT

For a gain margin of 6 db, K = 5.5135

For a phase margin of 40°, K = 1.3787

4.8 ROOT LOCUS

The root locus technique was introduced by **W.R.Evans** in 1948 for the analysis of control systems. The root locus technique is a powerful tool for adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system parameters.

Consider the open loop transfer function of system $G(s) = \dfrac{K}{s\,(s+p_1)\,(s+p_2)}$

The closed loop transfer function of the system with unity feedback is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s\,(s+p_1)\,(s+p_2)}}{1 + \dfrac{K}{s\,(s+p_1)\,(s+p_2)}} = \frac{K}{s\,(s+p_1)\,(s+p_2) + K}$$

The denominator polynomial of C(s)/R(s) is the characteristic equation of the system. The characteristic equation is given by,

$$s\,(s + p_1)\,(s + p_2) + K = 0\,.$$

The roots of characteristic equation is a function of open loop gain K. [In other words the roots of characteristic equation depend on open loop gain K]. When the gain K is varied from 0 to ∞, the roots of characteristic equation will take different values. When K = 0, the roots are given by open loop poles. When K→ ∞ , the roots will take the value of open loop zeros.

The path taken by the roots of characteristic equation when open loop gain K is varied from 0 to ∞ are called ***root loci*** (or the path taken by a root of characteristic equation when open loop gain K is varied from 0 to ∞ is called root locus).

> ***Note :*** *In general the roots of characteristic equation can be varied by varying any other system parameter other than gain.*

In general the closed loop transfer function of system with multiple loops is obtained from the signal flow graph of the system using Mason's gain formula.

$$\frac{C(s)}{R(s)} = T(s) = \frac{1}{\Delta}\sum_k P_k\Delta_k$$ *(Refer chapter 1 section 1.12)*

The determinant, Δ is the denominator polynominal of C(s)/R(s). The characteristic equation of the system is given by , Δ = 0

For the single loop system shown in fig 4.8

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

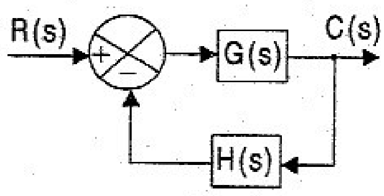


Fig 4.8

The Characteristic equation is,

$$1+G(s)\,H(s) = 0$$

$$\therefore\ G(s)\,H(s) = -1 \qquad .....(4.22)$$

From equation (4.22) it can be concluded that the roots of the characteristic equation occur only for those values of s for which, G(s)H(s) = –1.

The equation (4.22) can be converted to two Evans conditions given below,

$$|G(s)H(s)| = 1 \qquad .....(4.23)$$

$$\angle G(s)H(s) = \pm 180^\circ(2q+1), \qquad \text{where } q = 0,1,2,3,\ ..... \qquad .....(4.24)$$

The equation (4.23) is called magnitude criterion and equation (4.24) is called angle criterion.

The magnitude criterion states that $s = s_a$ will be a point on root locus if for that value of s,

$|G(s)\,H(s)| = 1$.

The angle criterion states that $s = s_a$ will be a point on root locus if for that value of s, $\angle G(s)\,H(s)$ is equal to an odd multiple of 180°.

The function G(s)H(s) can be expressed as a ratio of two polynomials in s as shown below.

$$G(s)H(s) = K\,\frac{(s+z_1)\,(s+z_2)\,(s+z_3).....}{(s+p_1)\,(s+p_2)\,(s+p_3).....} \qquad .....(4.25)$$

$$\therefore |G(s)H(s)| = K\,\frac{|s+z_1|\times|s+z_2|\times|s+z_3|.....}{|s+p_1|\times|s+p_2|\times|s+p_3|....} = K\,\frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|}$$

where, m = Number of zeros of loop transfer function.

n = Number of poles of loop transfer function.

The magnitude criterion states that $|G(s)H(s)| = 1$.

$$\therefore K\,\frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|} = 1 \quad \text{or} \quad K = \frac{\prod_{i=1}^{n}|s+p_i|}{\prod_{i=1}^{m}|s+z_i|} \qquad .....(4.26)$$

The open-loop gain K corresponding to a point $s = s_a$ on root locus can be calculated using equation (4.26). It can be shown that $|s + p_i|$ is equal to the length of vector drawn from $s = p_i$ to $s = s_a$ and $|s + z_i|$ is equal to the length of vector drawn from $s = z_i$ to $s = s_a$. Hence the equation K can be written as,

$$K = \frac{\text{Product of length of vector from open loop poles to the point } s = s_a}{\text{Product of length of vectors from open loop zeros to the point } s = s_a}$$

From equation (4.25),

$$\angle G(s)H(s) = \angle(s+z_1) + \angle(s+z_2) + \angle(s+z_3) + ... - \angle(s+p_1) - \angle(s+p_2) - \angle(s+p_3)....$$

$$= \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$$

where, m = Number of zeros of loop transfer function.

n = Number of poles of loop transfer function.

The angle criterion states that $\angle G(s)H(s) = \pm 180°\,(2q + 1)$

$$\therefore \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = \pm 180°(2q+1) \qquad .....(4.27)$$

The equations (4.27) can be used to check whether a point $s = s_a$ is a point on root locus or not. It can be shown that $\angle(s+p_i)$ is equal to the angle of vector drawn from $s = p_i$ to $s = s_a$ and $\angle(s+z_i)$ is equal to the angle of vector drawn from $s = z_i$ to $s = s_a$. Hence equation (5.27) can be written as

$$\begin{pmatrix}\text{Sum of angles of vector}\\ \text{from open loop zeros}\\ \text{to the point } s = s_a\end{pmatrix} - \begin{pmatrix}\text{Sum of angles of vector}\\ \text{from open loop poles}\\ \text{to the point } s = s_a\end{pmatrix} = \pm 180°\ (2q+1)$$

CONSTRUCTION OF ROOT LOCUS

The exact root locus is sketched by trial and error procedure. In this method, the poles and zeros of G(s)H(s) are located on the s-plane on a graph sheet and a trial point $s = s_a$ is selected. Determine the angles of vectors drawn from poles and zeros to the trial point. From the angle criterion, determine the angle to be contributed by these vectors to make the trial point as a point on root locus. Shift the trial point suitably so that the angle criterion is satisfied.

A number of points are determined using the above procedure. Join the points by a smooth curve which is the root locus. The value of K for a particular root can be obtained from the magnitude criterion.

The trial and error procedure for sketching root locus is tedious. A set of rules have been developed to reduce the task involved in sketching root locus and to develop a quick approximate sketch. From the approximate sketch, a more accurate root locus can be obtained by a few trials.

RULES FOR CONSTRUCTION OF ROOT LOCUS

Rule 1 : The root locus is symmetrical about the real axis.

Rule 2 : Each branch of the root locus originates from an open-loop pole corresponding to K = 0 and terminates at either on a finite open loop zero (or open loop zero at infinity) corresponding to $K = \infty$. The number of branches of the root locus terminating on infinity is equal to n−m, (i.e., the number of open loop poles minus the number of finite zeros)

Rule 3 : Segments of the real axis having an odd number of real axis open-loop poles plus zeros to their right are parts of the root locus.

Rule 4 : The n−m root locus branches that tend to infinity, do so along straight line asymptotes making angles with the real axis given by,

$$\phi_A = \frac{180°(2q+1)}{n-m} \; ; \quad q = 0,1,2,\ldots, n-m.$$

Rule 5 : The point of intersection of the asymptotes with the real axis is at $s = \sigma_A$ where,

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Rule 6 : The breakaway and breakin points of the root locus are determined from the roots of the equation dK/ds = 0. If r numbers of branches of root locus meet at a point, then they break away at an angle of ±180°/r.

Rule 7 : The angle of departure from a complex open-loop pole is given by,

$$\phi_p = \pm 180° \, (2q + 1) + \phi \; ; \qquad q = 0, 1, 2, \ldots$$

where ϕ is the net angle contribution at the pole by all other open loop poles and zeros. Similarly the angle of arrival at a complex open loop zero is given by,

$$\phi_z = \pm 180° \, (2q + 1) + \phi \; ; \qquad q = 0, 1, 2, \ldots$$

where ϕ is the net angle contribution at the zero by all other open-loop poles and zeros.

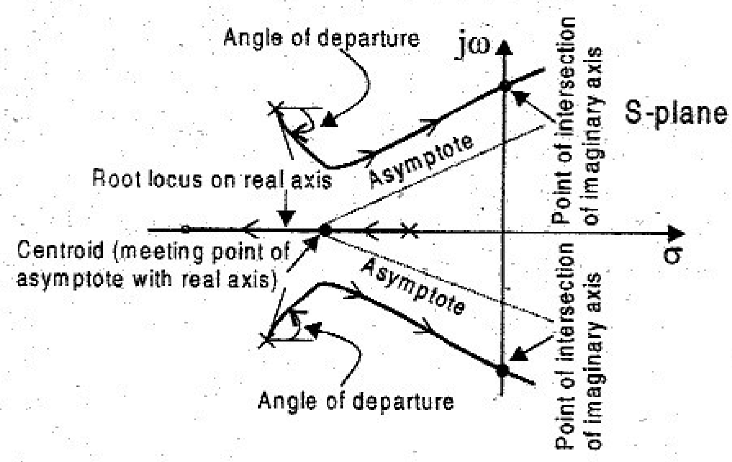
Rule 8 : The points of intersection of root locus branches with the imaginary axis can be determined by use of the Routh criterion. Alternatively they can be evaluated by letting $s = j\omega$ in the characteristic equation and equating the real part and imaginary part to zero, to solve for ω and K. The values of ω are the intersection points on imaginary axis and K is the value of gain at the intersection points.

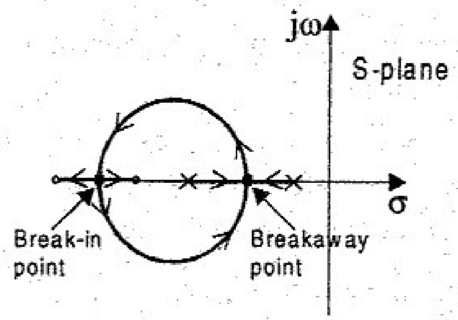
Rule 9 : The open-loop gain K at any point $s = s_a$ on the root locus is given by,

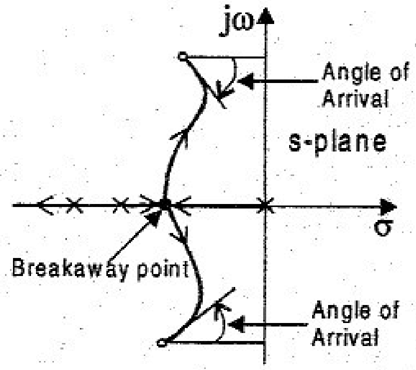
$$K = \frac{\prod_{i=1}^{n} |s_a + p_i|}{\prod_{i=1}^{m} |s_a + z_i|} = \frac{\text{Product of vector lengths from open loop poles to the point } s_a}{\text{Product of vector lengths from open loop zeros to the point } s_a}$$

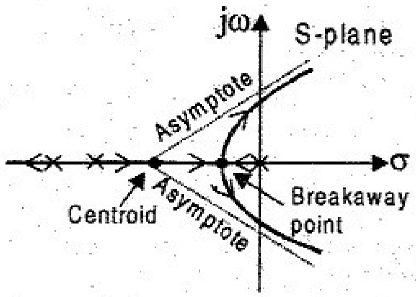
> ***Note :*** *The length of vector should be measured to scale. If there is no finite zero then the product of vector lengths from zeros is equal to 1.*

TYPICAL SKETCHES OF ROOT LOCUS PLOTS









PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Locate the poles and zeros of G(s)H(s) on the s-plane. The root locus branch starts from open loop poles and terminates at zeros.

Step 2 : Determine the root locus on real axis.

Step 3 : Determine the asymptotes of root locus branches and meeting point of asymptotes with real axis.

Step 4 : Find the breakaway and breakin points.

Step 5 : If there is a complex pole then determine the angle of departure from the complex pole. If there is a complex zero then determine the angle of arrival at the complex zero.

Step 6 : Find the points where the root loci may cross the imaginary axis.

Step 7 : Take a series of test points in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy angle criterion. Sketch the root locus by joining the test points by smooth curve.

Step 8 : The value of gain K at any point on the locus can be determined from magnitude condition.

The value of K at a point $s = s_a$, is given by,

$$K = \frac{\text{product of length of vectors from poles to the point, } s = s_a}{\text{product of length of vectors from finite zeros to the point, } s = s_a}$$

> *Note : When there is no finite zero, the denominator is taken as unity. The length of vectors should be measured to scale.*

EXPLANATION FOR THE VARIOUS STEPS IN THE PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Location of poles and zeros

Draw the real and imaginary axis on an ordinary graph sheet and choose same scales both on real and imaginary axis.

The poles are marked by cross **"X"** and zeros are marked by small circle **"o"**. The number of root locus branches is equal to number of poles of open loop transfer function. The origin of a root locus is at a pole and the end is at a zero.

Let, n = number of poles
m = number of finite zeros

Now, **m** root locus branches ends at finite zeros. The remaining **n–m** root locus branches will end at zeros at infinity.

Step 2 : Root locus on real axis

In order to determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd number, then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

Step 3 : Angles of asymptotes and centroid

If n is number of poles and m is number of finite zeros, then n – m root locus branches will terminate at zeros at infinity.

These n–m root locus branches will go along an asymptotic path and meets the asymptotes at infinity. Hence number of asymptotes is equal to number of root locus branches going to infinity. The angles of asymptotes and the centroid are given by the following formulae.

$$\text{Angles of asymptotes} = \frac{\pm 180\,(2q+1)}{n-m}$$

where, q = 0, 1, 2, 3, ...(n–m)

$$\text{Centroid (meeting point of asymptote with real axis)} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

Step 4 : Breakaway and Breakin points

The breakaway or breakin points either lie on real axis or exist as complex conjugate pairs. If there is a root locus on real axis between 2 poles then there exist a breakaway point. If there is a root locus on real axis between 2 zeros then there exist a breakin point. If there is a root locus on real axis between pole and zero then there may be or may not be breakaway or breakin point.

Let the characteristic equation be in the form,

$$B(s) + K\,A(s) = 0$$

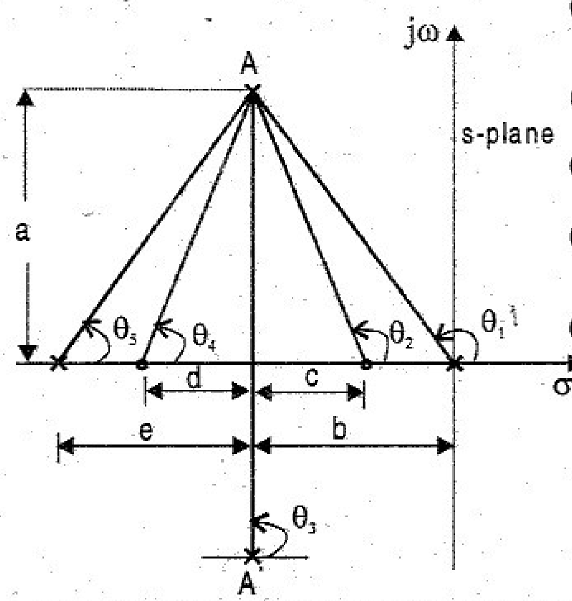
$$\therefore K = \frac{-B(s)}{A(s)}$$

The breakaway and breakin point is given by roots of the equation dK/ds = 0. The roots of dK/ds = 0 are actual breakaway or breakin point provided for this value of root, the gain K should be positive and real.

Step 5 : Angle of Departure and angle of arrival

$$\left.\begin{array}{l}\text{Angle of Departure}\\ \text{(from a complex pole A)}\end{array}\right\} = 180^\circ - \left(\begin{array}{l}\text{Sum of angles of vector to the}\\ \text{complex pole A from other poles}\end{array}\right) + \left(\begin{array}{l}\text{Sum of angles of vectors to the}\\ \text{complex pole A from zeros}\end{array}\right)$$

Note : *The angles can be calculated as shown in fig 4.9 or they can be measured using protractor.*



$$\theta_1 = 180^\circ - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90^\circ$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig 4.9 : Calculation of angle of departure

Example:

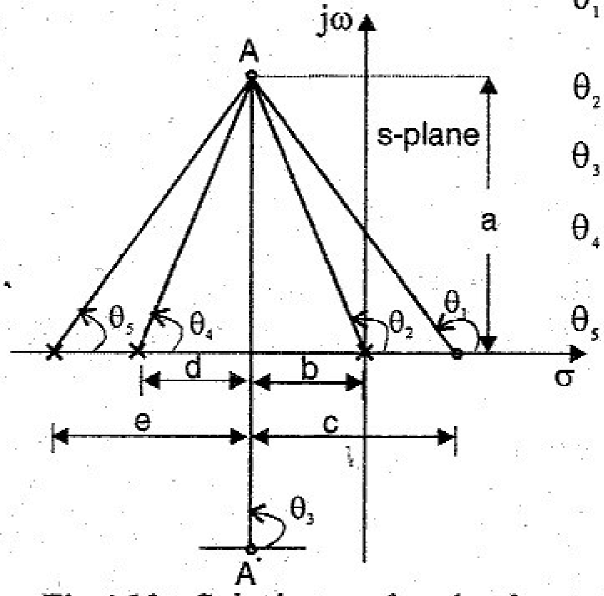
Consider the two complex conjugate poles A and A* shown in fig 4.9.(If poles are complex then they exist only as conjugate pairs)

$$\left.\begin{array}{c}\text{Angle of departure}\\ \text{at pole A}\end{array}\right\} = 180^\circ - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$$

$$\left.\begin{array}{c}\text{Angle of departure}\\ \text{at pole A *}\end{array}\right\} = -[\text{Angle of departure at pole A}]$$

$$\left.\begin{array}{l}\text{Angle of arrival at a}\\ \text{complex zero A}\end{array}\right\} = 180^\circ - \left(\begin{array}{l}\text{Sum of angles of vectors to the}\\ \text{complex zero A from all other zeros}\end{array}\right) + \left(\begin{array}{l}\text{Sum of angles of vectors to the}\\ \text{complex zero A from poles}\end{array}\right)$$

Note : *The angles can be calculated as shown in fig 4.10 or they can be measured using protractor.*



$$\theta_1 = 180^\circ - \tan^{-1}\frac{a}{b}$$

$$\theta_2 = 180^\circ - \tan^{-1}\frac{a}{c}$$

$$\theta_3 = 90^\circ$$

$$\theta_4 = \tan^{-1}\frac{a}{d}$$

$$\theta_5 = \tan^{-1}\frac{a}{e}$$

Fig 4.10 : Calculation of angle of arrival

Example:

Consider the two complex conjugate zeros B and B* as shown in fig 4.10.(If zeros are complex then they exist only as conjugate pairs)

$$\left.\begin{array}{c}\text{Angle of arrival}\\ \text{at zero B}\end{array}\right\} = 180^\circ - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$$

$$\left.\begin{array}{c}\text{Angle of arrival}\\ \text{at zero B*}\end{array}\right\} = -[\text{Angle of arrival at zero B}]$$

Step 6 : Point of intersection of root locus with imaginary axis

The point where the root loci intersects the imaginary axis can be found by following three methods.

1. By Routh Hurwitz array.
2. By trial and error approach.
3. Letting $s = j\omega$ in the characteristic equation and separate the real part and imaginary part. Two equations are obtained : *one by equating real part to zero and the other by equating imaginary part to zero.* Solve the two equations for ω and **K**. The values of ω gives the points where the root locus crosses imaginary axis. The value of K gives the value of gain K at there crossing points. Also this value of K is the limiting value of K for stability of the system.

Step 7 : Test points and root locus

Choose a test point. Using a protractor roughly estimate the angles of vectors drawn to this point and adjust the point to satisfy angle criterion. Repeat the procedure for few more test points. Sketch the root locus from the knowledge of typical sketches and the informations obtained in steps 1 through 6.

> *Note : In practice the approximate root locus can be sketched from the informations obtained in steps 1 through 6 and from the knowledge of typical sketches of root locus.*

DETERMINATION OF OPEN LOOP GAIN FOR A SPECIFIED DAMPING OF THE DOMINANT ROOTS

The dominant pole is a pair of complex conjugate pole which decides the transient response of the system. In higher order systems the dominant poles are given by the poles which are very close to origin, provided all other poles are lying far away from the dominant poles. The poles which are far away from the origin will have less effect on the transient response of the system.

The transfer function of higher order systems can be approximated to a second order transfer function. The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The dominant poles, s_d and s_d^*, are given by the roots of quadratic factor, $s^2+2\zeta\omega_n s+\omega_n^2 = 0$.

$$\therefore s_d = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The dominant pole can be plotted on the s-plane as shown in fig 4.11.

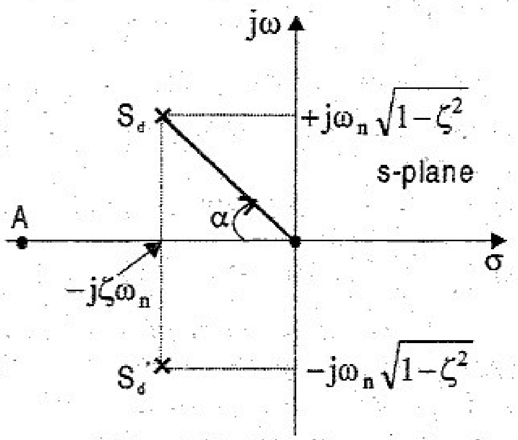


Fig 4.11 : Dominant pole, s_d

In fig 4.11, the right angle triangle OAP,

$$\cos\alpha = \frac{\zeta\omega_n}{\omega_n} = \zeta, \qquad \therefore \alpha = \cos^{-1}\zeta$$

To fix a dominant pole on root locus, draw a line at an angle of $\cos^{-1}\zeta$ with respect to negative real axis. The meeting point of this line with root locus will give the location of dominant pole. The value of K corresponding to dominant pole can be obtained from magnitude condition.

Let, K_{sd} be the value of gain at dominant pole s_d,

$$\text{Now,}\quad K_{sd} = \frac{\text{Product of length of vectors from open loop poles to dominant pole}}{\text{Product of length of vectors from open loop zeros to dominant pole}}$$

Importance of root locus

The root locus technique is an important tool in designing control systems with desired performance characteristics. The desired performance of the system can be achieved by adjusting the location of its closed loop poles in the s-plane by varying one or more system parameters.

The root locus can be plotted in the s-plane by varying a system parameter (usually gain, K) over the complete range of values. The roots corresponding to a particular value of the system parameter can then be located on the locus or the value of the parameter for a desired root location can be determined from the locus.

The root locus technique is also used for stability analysis. Using root locus the range of values of K, for a stable system can be determined. It is also easier to study the relative stability of the system from the knowledge of location of closed loop poles. The dominant roots are used to estimate the damping ratio and natural frequency of oscillation of the system. From ζ and ω_n the time domain specifications can be calculated.

EXAMPLE 4.22

A unity feedback control system has an open loop transfer function, $G(s) = \frac{K}{s\,(s^2+4s+13)}$. Sketch the root locus.

SOLUTION

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s\,(s^2+4s+13)=0$.

The roots of the quadratic are, $s = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j3$

$\therefore$ The poles are lying at s= 0, $-2+j3$ and $-2-j3$.

Let us denote the poles as P_1, P_2, and P_3.

Here, $P_1 = 0$, $P_2 = -2+j3$ and $P_3 = -2-j3$.

The poles are marked by X (cross) as shown in fig 4.22.1.

Step 2 : To find the root locus on real axis

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line in fig 4.22.1.

> ***Note :*** *For the given transfer function one root locus branch will start at the pole at the origin and meet the zero at infinity through the negative real axis.*

Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ\,(2q+1)}{n-m} \quad ; \quad q = 0,\ 1, \ldots\ldots\ldots n-m$$

Here n = 3, and m = 0. $\therefore$ q = 0, 1, 2, 3 .

When q = 0, $\text{Angles} = \pm\frac{180^\circ}{3} = \pm 60^\circ$

When q = 1, $\text{Angles} = \pm\frac{180^\circ \times 3}{3} = \pm 180^\circ$

When q = 2, $\text{Angles} = \pm\frac{180^\circ \times 5}{3} = \pm 300^\circ = \mp 60^\circ$

When q = 3, $\text{Angles} = \pm\frac{180^\circ \times 7}{3} = \pm 420^\circ = \pm 60^\circ$

> ***Note :*** *It is enough if you calculate the required number of angles. Here it is given by first three values of angles. The remaining values will be repetitions of the previous values.*

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3} = -1.33$$