$$\therefore s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$$

$$\Downarrow$$

$$\underset{\text{Even polynomial}}{(s^4+s^2+1)}\ \underset{\text{Quotient polynomial}}{(s^3+9s^2+23s+15)}=0$$

The routh array is constructed for quotient polynomial as shown below.

$$\begin{array}{lcll} s^3 & : & 1 & 23 \\ s^2 & : & 9 & 15 \end{array}$$

Divide s^2 row by 3,

$$\begin{array}{lcll} s^3 & : & 1 & 23 \\ s^2 & : & 3 & 5 \\ s^1 & : & 21.33 & \\ s^0 & : & 5 & \end{array}$$

Column-1

$$s^1 : \frac{3\times 23-5\times 1}{3}$$

$$s^1 :\ 21.33$$

$$s^0 : \frac{21.33\times 5-0\times 3}{21.33}$$

$$s^0 :\ 5$$

$$\begin{array}{r|l} & s^3+9s^2+23s+15 \quad \text{(Quotient Polynomial)} \\ \hline \begin{array}{l} s^4+s^2+1 \\ \text{(Even Polynomial)} \end{array} & s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15 \\ & \underset{(-)}{s^7} \quad \underset{(-)}{+s^5} \quad \underset{(-)}{+s^3} \\ \hline & 9s^6+23s^5+24s^4+23s^3+24s^2+23s+15 \\ & \underset{(-)}{9s^6}+ \quad \underset{(-)}{+9s^4} \quad \underset{(-)}{+9s^2} \\ \hline & 23s^5+15s^4+23s^3+15s^2+23s+15 \\ & (-)23s^5 \quad (-)+23s^3 \quad (-)+23s \\ \hline & 15s^4 \quad +15s^2 \quad +15 \\ & (-)\,15s^4 \quad (-)+15s^2 \quad (-)+15 \\ \hline & 0 \end{array}$$

The elements of column-1 of quotient polynomial are all positive and there is no sign change. Hence all the roots of quotient polynomial are lying on the left half of s-plane. To determine the stability, the roots of auxiliary polynomial should be evaluated.

The auxiliary equation is, $s^4+s^2+1=0$.

Put, $s^2=x$ in the auxiliary equation. $s^4+s^2+1=x^2+x+1=0$

The roots of quadratic are, $x=\dfrac{-1\pm\sqrt{1-4}}{2}=-\dfrac{1}{2}\pm j\dfrac{\sqrt{3}}{2}=1\angle 120^\circ$ or $1\angle -120^\circ$

$$\begin{aligned} \text{But } s^2=x,\quad \therefore s=\pm\sqrt{x} &= \pm\sqrt{1\angle 120^\circ} && \text{or } \pm\sqrt{1\angle -120^\circ} \\ &= \pm\sqrt{1}\angle 120^\circ/2 && \text{or } \pm\sqrt{1}\angle -120^\circ/2 \\ &= \pm 1\angle 60^\circ && \text{or } \pm 1\angle -60^\circ \\ &= \pm(0.5+j0.866) && \text{or } \pm(0.5-j0.866) \end{aligned}$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by the roots of auxiliary polynomial and the roots of quotient polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so system is unstable. The remaining five roots are lying on left half of s-plane.

EXAMPLE 4.6

The characteristic polynomial of a system is $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$. Determine the location of roots on the s-plane and hence the stability of the system.

SOLUTION

The characteristic equation is, $s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$.

The given characteristic polynomial is 7^{th} order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row of array using the coefficients of even powers of s as shown below.

s^7	:	1	9	4	36	.... Row-1
s^6	:	5	9	20	36	.... Row-2

Divide s^6 row by 5 to simplify the computations.

s^7	:	1	9	4	36	.... Row-1
s^6	:	1	1.8	4	7.2	.... Row-2
s^5	:	1	0	4		.... Row-3
s^4	:	1	0	4		.... Row-4
s^3	:	0	0			.... Row-5

$$s^5 : \frac{1\times 9-1.8\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 36-7.2\times 1}{1}$$

$s^5 : 7.2 \quad 0 \quad 28.8$

Divide by 7.2

$s^5 : 1 \quad 0 \quad 4$

$$s^4 : \frac{1\times 1.8-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1} \quad \frac{1\times 7.2-0\times 1}{1}$$

$s^4 : 1.8 \quad 0 \quad 7.2$

Divide by 1.8

$s^4 : 1 \quad 0 \quad 4$

$$s^3 : \frac{1\times 0-0\times 1}{1} \quad \frac{1\times 4-4\times 1}{1}$$

$s^3 : 0 \quad 0$

$$\begin{array}{r|l} & s^3+5s^2+9s+9 \\ \hline s^4+4 & s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36 \\ & s^7 \qquad\qquad\qquad\quad +4s^3 \\ \hline & 5s^6+9s^5+9s^4 \qquad +20s^2+36s+36 \\ & 5s^6 \qquad\qquad\qquad\quad +20s^2 \\ \hline & 9s^5+9s^4 \qquad +36s+36 \\ & 9s^5 \qquad\qquad +36s \\ \hline & 9s^4 \qquad +36 \\ & 9s^4 \qquad +36 \\ \hline & 0 \end{array}$$

The row of all zeros indicate the existence of even polynomial, which is also the auxiliary polynomial. The auxiliary polynomial is, $s^4+4=0$. Divide the characteristic equation by auxiliary equation to get the quotient polynomial.

The characteristic equation can be expressed as a product of quotient polynomial and auxiliary equation.

$$\therefore s^7+5s^6+9s^5+9s^4+4s^3+20s^2+36s+36=0$$

$$\underbrace{(s^4+4)}_{\text{Even polynomial}} \quad \underbrace{(s^3+5s^2+9s+9)}_{\text{Quotient polynomial}}=0$$

The routh array is constructed for quotient polynomial as shown below.

s^3	:	1	9
s^2	:	5	9
s^1	:	7.2	
s^0	:	9	

Column-1

$$s^1 : \frac{5\times 9-9\times 1}{5}$$

$s^1 : 7.2$

$$s^0 : \frac{7.2\times 9-0\times 5}{7.2}$$

$s^0 : 9$

There is no sign change in the elements of first column of routh array of quotient polynomial. Hence all the roots of quotient polynomial are lying on the left half of s-plane.

To determine the stability, the roots of auxiliary polynomial should be evaluated.

The auxiliary polynomial is, $s^4+4=0$.

Put, $s^2=x$ in the auxiliary equation, $\therefore s^4+4=x^2+4=0$

$$\therefore x^2=-4 \quad\Rightarrow\quad x=\pm\sqrt{-4}=\pm j2=2\angle 90^\circ \text{ or } 2\angle-90^\circ$$

$$\text{But, } s=\pm\sqrt{x}=\pm\sqrt{2\angle 90^\circ} \text{ or } \pm\sqrt{2\angle-90^\circ}=\pm\sqrt{2}\angle 90^\circ/2 \text{ or } \pm\sqrt{2}\angle-90^\circ/2$$

$$=\pm\sqrt{2}\angle 45^\circ \text{ or } \pm\sqrt{2}\angle-45=\pm(1+j1) \text{ or } \pm(1-j1)$$

The roots of auxiliary equation are complex and has quadrantal symmetry. Two roots of auxiliary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

The roots of characteristic equation are given by roots of quotient polynomial and auxiliary polynomial. Hence we can conclude that two roots of characteristic equation are lying on the right half of s-plane and so the system is unstable. The remaining five roots are lying on the left half of s-plane.

RESULT

(a) The system is unstable.

(b) Two roots are lying on the right half of s-plane and five roots are lying on the left half of s-plane.

EXAMPLE 4.7

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

SOLUTION

The characteristic equation of the system is, $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

s^5 : 1 8 7 .... Row-1

s^4 : 4 8 4 .... Row-2

Divide s^4 row by 4 to simplify the calculations.

s^5 : 1 8 7 .... Row-1

s^4 : 1 2 1 .... Row-2

s^3 : 1 1 .... Row-3

s^2 : 1 1 .... Row-4

s^1 : $\in$ Row-5

s^0 : 1 .... Row-6

Column-1

When $\in \rightarrow 0$, there is no sign change in the first column of routh array. But we have a row of all zeros (s^1 row or row-5) and so there is a possibility of roots on imaginary axis. This can be found from the roots of auxiliary polynomial. Here the auxiliary polynomial is given by s^2 row.

$$s^3 : \frac{1\times 8-2\times 1}{1} \quad \frac{1\times 7-1\times 1}{1}$$

s^3 : 6 6

Divide by 6

s^3 : 1 1

$$s^2 : \frac{1\times 2-1\times 1}{1} \quad \frac{1\times 1-0\times 1}{1}$$

s^2 : 1 1

$$s^1 : \frac{1\times 1-1\times 1}{1}$$

s^1 : 0

Let $0 \rightarrow \in$

s^1 : $\in$

$$s^0 : \frac{\in\times 1-0\times 1}{\in}$$

s^0 : 1

The auxiliary polynomial is, $s^2 + 1 = 0$; $\therefore\ s^2 = -1$ or $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are +j1and –j1, lying on imaginary axis. The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots of characteristic equation are lying on imaginary axis and so the system is limitedly or marginally stable. The remaining three roots of characteristic equation are lying on the left half of s-plane.

RESULT

(a) The system is limitedly or marginally stable.

(b) Two roots are lying on imaginary axis and three roots are lying on left half of s-plane.

EXAMPLE 4.8

Use the routh stability criterion to determine the location of roots on the s-plane and hence the stability for the system represented by the characteristic equation, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

SOLUTION

The characteristic polynomial of the system is, $s^6 + s^5 + 3s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s as shown below.

s^6	:	1	3	3	1	.... Row-1
s^5	:	1	3	2		.... Row-2
s^4	:	$\in$	1	1		.... Row-3
s^3	:	$\frac{3\in-1}{\in}$	$\frac{2\in-1}{\in}$			.... Row-4
s^2	:	$\frac{-2\in^2+4\in-1}{3\in-1}$	1			.... Row-5
s^1	:	$\frac{4\in^2-\in}{2\in^2-4\in+1}$				.... Row-6
s^0	:	1				.... Row-7

$$s^4 : \frac{1\times3-3\times1}{1} \quad \frac{1\times3-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^4 : 0 \qquad 1 \qquad 1$$

let $0 \rightarrow \in$

$$s^4 : \in \qquad 1 \qquad 1$$

$$s^3 : \frac{\in\times3-1\times1}{\in} \quad \frac{\in\times2-1\times1}{\in}$$

$$s^3 : \frac{3\in-1}{\in} \quad \frac{2\in-1}{\in}$$

$$s^2 : \frac{\frac{3\in-1}{\in}-\frac{2\in-1}{\in}\times\in}{\frac{3\in-1}{\in}} \quad \frac{\frac{3\in-1}{\in}\times1-0\times\in}{\frac{3\in-1}{\in}}$$

$$s^2 : \frac{-2\in^2+4\in-1}{3\in-1} \qquad 1$$

On letting $\in \rightarrow 0$, we get,

	s^6	:	1	3	3	1	.... Row-1
	s^5	:	1	3	2		.... Row-2
	s^4	:	0	1	1		.... Row-3
	s^3	:	$-\infty$	$-\infty$			.... Row-4
+ to – ↓	s^2	:	1	1			.... Row-5
– to + ↓	s^1	:	0				.... Row-6
	s^0	:	1				.... Row-7

$$s^1 : \frac{\frac{-2\in^2+4\in-1}{3\in-1}\times\frac{2\in-1}{\in}-\frac{3\in-1}{\in}\times1}{\frac{-2\in^2+4\in-1}{3\in-1}}$$

$$s^1 : \frac{(-2\in^2+4\in-1)(2\in-1)-(3\in-1)(3\in-1)}{\in(-2\in^2+4\in-1)}$$

$$s^1 : \frac{-4\in^3+\in^2}{\in(-2\in^2+4\in-1)}=\frac{4\in^2-\in}{2\in^2-4\in+1}$$

$$s^0 : \frac{\frac{4\in^2-\in}{4\in^2-4\in+1}\times1-0\times\frac{-2\in^2+4\in-1}{3\in-1}}{(4\in^2-\in)/(4\in^2-4\in+1)}$$

$$s^0 : 1$$

Since there is a row of all zeros (s^1 row) there is a possibility of roots on imaginary axis. The auxiliary polynomial is $s^2 + 1 = 0$.

The roots of auxiliary polynomial are, $s = \pm\sqrt{-1} = \pm j1$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence two roots are lying on imaginary axis. Therefore divide the characteristic polynomial by auxiliary equation and construct the routh array for quotient polynomial to find the roots lying on right half of s-plane.

The characteristic polynomial can be expressed as a product of auxiliary polynomial and quotient polynomial.

$$\therefore s^6+s^5+3s^4+3s^3+3s^2+2s+1=0 \Rightarrow \underset{\text{Even polynomial}}{(s^2+1)}\ \underset{\text{Quotient polynomial}}{(s^4+s^3+2s^2+2s+1)}=0$$

The routh array for quotient polynomial is constructed as shown below.

s^4	:	1	2	1	.... Row-1
s^3	:	1	2		.... Row-2
s^2	:	$\in$	1		.... Row-3
s^1	:	$\frac{2\in-1}{\in}$			.... Row-4
s^0	:	1			.... Row-5

$$s^2 : \frac{1\times2-2\times1}{1} \quad \frac{1\times1-0\times1}{1}$$

$$s^2 : 0 \qquad 1$$

let $0 \rightarrow \in$

$$s^2 : \in \qquad 1$$

$$s^1 : \frac{\in\times2-1\times1}{\in}$$

$$s^1 : \frac{2\in-1}{\in}$$

$$s^0 : \frac{\frac{2\in-1}{\in}\times1-0\times\in}{(2\in-1)/\in}$$

$$s^0 : 1$$

On letting $\in \rightarrow 0$, we get

$$\begin{array}{llllll} & s^4 & : & 1 & 2 & 1 & \text{.... Row-1} \\ & s^3 & : & 1 & 2 & & \text{.... Row-2} \\ +\text{ to }- & s^2 & : & 0 & 1 & & \text{.... Row-3} \\ -\text{ to }+ & s^1 & : & -\infty & & & \text{.... Row-4} \\ & s^0 & : & 1 & & & \text{.... Row-5} \end{array}$$

Column-1

On examining the first column of the routh array of quotient polynomial, we found that there are two sign changes. Hence two roots are lying on the right half of s-plane and other two roots of quotient polynomial are lying on the left half of s-plane.

The roots of characteristic equation are given by roots of auxiliary polynomial and quotient polynomial. Hence two roots are lying on imaginary axis, two roots are lying on right half of s-plane and the remaining two roots are lying on left half of s-plane. Hence the system is unstable.

$$\begin{array}{r|l} & s^4+s^3+2s^2+2s+1 \quad \text{(Quotient polynomial)} \\ \hline s^2+1 & s^6+s^5+3s^4+3s^3+3s^2+2s+1 \\ & \underset{(-)}{s^6} \quad \underset{(-)}{+} s^4 \\ \hline & s^5+2s^4+3s^3+3s^2+2s+1 \\ & (-)s^5 \quad (-)+s^3 \\ \hline & 2s^4+2s^3+3s^2+2s+1 \\ & (-)2s^4 \quad (-)+2s^2 \\ \hline & 2s^3+s^2+2s+1 \\ & (-)2s^3 \quad (-)+2s \\ \hline & s^2 \quad +1 \\ & (-)s^2 \quad (-)+1 \\ \hline & 0 \end{array}$$

(Even polynomial)

RESULT

(a) The system is unstable.

(b) Two roots are lying on imaginary axis, two roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 4.9

Determine the range of K for stability of unity feedback system whose open loop transfer function is $G(s)=\dfrac{K}{s(s+1)(s+2)}$.

SOLUTION

The closed loop transfer function, $\dfrac{C(s)}{R(s)}=\dfrac{G(s)}{1+G(s)}=\dfrac{\dfrac{K}{s(s+1)(s+2)}}{1+\dfrac{K}{s(s+1)(s+2)}}=\dfrac{K}{s(s+1)(s+2)+K}$

The characteristic equation is, $\quad s(s+1)(s+2)+K=0$

$$\therefore s(s^2+3s+2)+K=0 \quad \Rightarrow \quad s^3+3s^2+2s+K=0$$

The routh array is constructed as shown below.

The highest power of s in the characteristic polynomial is odd number. Hence form the first row using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$\begin{array}{llll} s^3 & : & 1 & 2 \\ s^2 & : & 3 & K \\ s^1 & : & \dfrac{6-K}{3} & \\ s^0 & : & K & \end{array}$$

Column-1

$$s^1 : \frac{3\times 2-K\times 1}{3}$$

$$s^1 : \frac{6-K}{3}$$

$$s^0 : \frac{\dfrac{6-K}{3}\times K-0\times 3}{(6-K)/3}$$

$$s^0 : K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^0 row, for the system to be stable, $K > 0$

From s^1 row, for the system to be stable, $\frac{6-K}{3} > 0$

For $\frac{6-K}{3} > 0$, the value of K should be less than 6.

$\therefore$ The range of K for the system to be stable is $0<K<6$.

RESULT

The value of K is in the range $0<K<6$ for the system to be stable.

EXAMPLE 4.10

The open loop transfer function of a unity feedback control system is given by,

$$G(s) = \frac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}.$$

By applying the routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the value of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillating frequencies?

SOLUTION

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}}{1+\dfrac{K}{(s+2)\,(s+4)\,(s^2+6s+25)}} = \frac{K}{(s+2)\,(s+4)\,(s^2+6s+25)+K}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

The characteristic equation is, $(s+2)\,(s+4)\,(s^2+6s+25)+K=0$.

$$\therefore (s^2+6s+8)\,(s^2+6s+25)+K=0 \qquad \Rightarrow \qquad s^4+12s^3+69s^2+198s+200+K=0$$

The routh array is constructed as shown below. The highest power of s in the characteristic equation is even number. Hence form the first row using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

s^4	:	1	69	200+K	..... Row-1
s^3	:	12	198		..... Row-2

Divide s^3 row by 12 to simplify the calculations

s^4	:	1	69	200+K	..... Row-1
s^3	:	1	16.5		..... Row-2
s^2	:	52.5	200+K		..... Row-3
s^1	:	$\frac{666.25-K}{52.5}$			..... Row-4
s^0	:	200+K			..... Row-5

Column-1

$$s^2: \quad \frac{1\times 69-16.5\times 1}{1} \qquad \frac{1\times(200+K)}{1}$$

$$s^2: 52.5 \qquad 200+K$$

$$s^1: \quad \frac{52.5\times 16.5-(200+K)\times 1}{52.5}$$

$$s^1: \quad \frac{666.25-K}{52.5}$$

$$s^0: \quad \frac{\dfrac{666.25-K}{52.5}\times(200+K)}{(666.25-K)\,/\,52.5}$$

$$s^0: 200+K$$

For the system to be stable there should not be any sign change in the elements of first column. Hence choose the value of K so that the first column elements are positive.

From s^1 row, for the system to be stable, (666.25-K) > 0.

Since (666.25-K) > 0, should be less than 666.25.

From s^0 row, for the system to be stable, (200+K) > 0

Since (200+K) > 0, K should be greater than -200, but practical values of K starts from 0. Hence K should be greater than zero.

∴ The range of K for the system to be stable is $0<K<666.25$.

When K= 666.25 the s^1 row becomes zero, which indicates the possibility of roots on imaginary axis. A system will oscillate if it has roots on imaginary axis and no roots on right half of s-plane.

When K=666.25, the coefficients of auxiliary equation are given by the s^2 row.

∴ The auxiliary equation is, $52.5s^2+200+K=0$

$$52.5s^2+200+666.25=0$$

$$s^2=\frac{-200-666.25}{52.5}=-16.5$$

$$s=\pm\sqrt{-16.5}=\pm j\sqrt{16.5}=\pm j4.06$$

When K = 666.25, the system has roots on imaginary axis and so it oscillates. The frequency of oscillation is given by the value of root on imaginary axis.

∴ The frequency of oscillation, $\omega = 4.06$ rad/sec.

RESULT

(a) The range of K for stability is $0<K<666.25$

(b) The system oscillates when K = 666.25

(c) The frequency of oscillation, $\omega = 4.06$ rad/sec. (When K = 666.25)

EXAMPLE 4.11

The open loop transfer function of a unity feedback system is given by, $G(s)=\frac{K(s+1)}{s^3+as^2+2s+1}$. Determine the value of K and a so that the system oscillates at a frequency of 2 rad/sec.

SOLUTION

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\}\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\frac{K(s+1)}{s^3+as^2+2s+1}}{1+\frac{K(s+1)}{s^3+as^2+2s+1}}=\frac{K(s+1)}{s^3+as^2+2s+1+K(s+1)}$$

The characteristic equation is, $s^3+as^2+2s+1+K(s+1)=0$.

$$s^3+as^2+2s+1+Ks+K=0 \quad\Rightarrow\quad s^3+as^2+(2+K)s+1+K=0$$

The routh array of characteristic polynomial is constructed as shown below. The maximum power of s is odd, hence the first row of routh array is formed using coefficients of odd powers of s and the second row of routh array is formed using coefficients of even powers of s.

If the elements of s^1 row are all zeros then there exists an even polynomial (or auxiliary polynomial). If the roots of the auxiliary polynomial are purely imaginary then the roots are lying on imaginary axis and the system oscillates. The frequency of oscillation is the root of auxiliary polynomial.

Routh array

$$\begin{array}{lll}
s^3: & 1 & 2+K \\
s^2: & a & 1+K \\
s^1: & \dfrac{a(2+K)-(1+K)}{a} & \\
s^0: & 1+K &
\end{array}$$

From s^2 row, the auxiliary polynomial is,

$$as^2+(1+K)=0 \quad \Rightarrow \quad as^2=-(1+K) \quad \Rightarrow \quad s=\pm j\sqrt{\frac{1+K}{a}}$$

Given that, $s=\pm j2$, $\quad \therefore \sqrt{\dfrac{1+K}{a}}=2 \quad \Rightarrow \quad \dfrac{1+K}{a}=4 \quad \Rightarrow \quad K=4a-1$

From s^1 row, $\dfrac{a(2+K)-(1+K)}{a}=0 \Rightarrow a(2+K)-(1+K)=0 \Rightarrow 2a+Ka-1-K=0$

$$\therefore \quad 2a-1+K(a-1)=0$$

Put, $K = 4a-1$

$$\therefore 2a-1+(4a-1)(a-1)=0 \Rightarrow 2a-1+4a^2-4a-a+1=0 \Rightarrow 4a^2-3a=0 \text{ (or) } a(4a-3)=0$$

Since $a \neq 0$, $\quad 4a-3=0, \quad \therefore a=3/4$

When $a=(3/4)$, $\quad K=4a-1=4\times(3/4)-1=2$

RESULT

When the system oscillates at a frequency of 2 rad/sec, $K=2$ and $a=3/4$.

EXAMPLE 4.12

A feedback system has open loop transfer function of $G(s)=\dfrac{Ke^{-s}}{s(s^2+5s+9)}$. Determine the maximum value of K for stability of closed loop system.

SOLUTION

Generally control systems have very low bandwidth which implies that it has very low frequency range of operation. Hence for low frequency ranges the term e^{-s} can be replaced by, $1-s$, (i.e., $e^{s-T} \approx 1-sT$).

$$\therefore G(s)=\frac{Ke^{-s}}{s(s^2+5s+9)} \approx \frac{K(1-s)}{s(s^2+5s+9)}$$

$$\left.\begin{array}{l}\text{The closed loop}\\ \text{transfer function}\end{array}\right\} \frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\dfrac{K(1-s)}{s(s^2+5s+9)}}{1+\dfrac{K(1-s)}{s(s^2+5s+9)}}=\frac{K(1-s)}{s(s^2+5s+9)+K(1-s)}$$

The characteristic equation is given by the denominator polynomial of closed loop transfer function.

$\therefore$ The characteristic equation is, $s(s^2+5s+9)+K(1-s)=0$

$\therefore s(s^2+5s+9)+K(1-s)=s^3+5s^2+9s+K-Ks=0 \Rightarrow s^3+5s^2+(9-K)s+K=0$

The routh array of characteristic polynomial is constructed as shown below.

The maximum power of s in the characteristic polynomial is odd, hence form the first row of routh array using coefficients of odd powers of s and second row of routh array using coefficients of even powers of s.

$$\begin{array}{lccc} s^3 & : & 1 & 9-K \\ s^2 & : & 5 & K \\ s^1 & : & 9-1.2K & \\ s^0 & : & K & \end{array}$$

$$s^1 : \frac{5\times(9-K)-K\times 1}{5}$$

$$s^1 : \frac{45-5K-K}{5}$$

$$s^1 : \frac{45-6K}{5} \approx 9-12K$$

$$s^0 : \frac{(9-12K)\times K}{(9-12K)}$$

$$s^0 : K$$

From s^1 row, for stability of the system, (9-1.2K)>0

$$\text{If} \quad (9-1.2K)>0 \quad \text{then} \quad 1.2K<9 \; ; \; \therefore K<\frac{9}{12}=7.5$$

From s^0 row, for stability of the system, K>0

Finally we can conclude that for stability of the system K should be in the range of 0<K<7.5

RESULT

For stability of the system K should be in the range of, $0<K<7.5$.

4.4 MATHEMATICAL PRELIMINARIES FOR NYQUIST STABILITY CRITERION

Let F(s) be a function of s, which is expressed as a ratio of two polynomials in s, as shown in equation (4.14), (the polynomials are expressed in the factored form).

$$F(s)=\frac{(s-z_1)\,(s-z_2)\,......\,(s-z_m)}{(s-p_1)\,(s-p_2)\,......\,(s-p_n)} \qquad .....(4.14)$$

The roots of numerator polynomial are zeros and the roots of denominator polynomial are poles. The function has m number of zeros and n number of poles.

Here, s is a complex variable expressed as, $s=\sigma+j\omega$, where s is real part of σ and ω is imaginary part of s. (The s is also called complex frequency). For a particular value of σ and ω, the s will represent a point in the s-plane.

Since s is a complex variable, the function F(s) will also be a complex quantity for any value of s. Hence, F(s) can also be expressed as, F(s) = u+jv, where u is real part of F(s) and v is imaginary part of F(s). Let us define another complex plane called F(s)-plane, with coordinates u and v. For a particular value of s, the F(s) will represent a point in F(s)-plane.

Therefore, for every point s in the s-plane at which F(s) is analytic, there exists a corresponding point F(s) in the F(s)-plane. Hence it can be concluded that the function F(s) maps the points in the s-plane into the F(s)-plane.

Note : A function is analytic in the s-plane provided the function and all its derivatives exist. The points in the s-plane where the function (or it derivatives) does not exist are called singular points.

Since any number of points of analycity in the s-plane can be mapped into the F(s)- plane it can be concluded that for a contour in the s-plane which does not go through any singular point, there exists a corresponding contour in the F(s)-plane as shown in fig 4.2.

The table 4.2 shows examples of arbitrary s-plane contours and their corresponding F(s)-plane contours (exact shape is not shown).

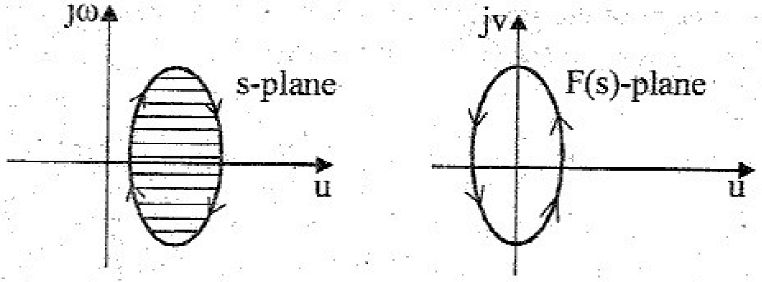


Fig 4.2 : *An arbitrary contour in s-plane and its corresponding contour in F(s)-plane*

Normally the direction of arbitrary contour in s-plane is chosen as clockwise. Here zeros are marked by small circles (o) and poles by (X).

On observing the s-plane contours and the corresponding F(s)-plane contours shown in table-4.2, it can be proved that there exists a relationship between the enclosure of poles and zeros by the s-plane closed contour and number of encirclements of the origin of F(s)- plane by the corresponding F(s)-plane contour.

> ***Note :*** *For the development of Nyquist criterion, the exact shape of the contour is not required but only the number of encirclements of the origin of the F(s) - plane is essential.*

Concept of encircled and enclosed

It is important to distinguish between the concept of encircled and enclosed which are frequently used to apply Nyquist criterion.

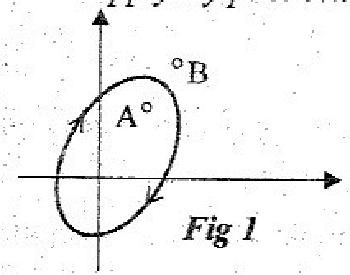


Fig 1

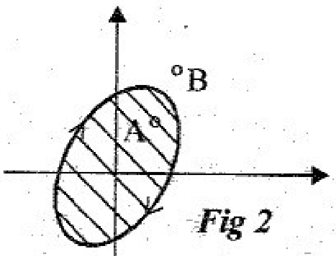


Fig 2

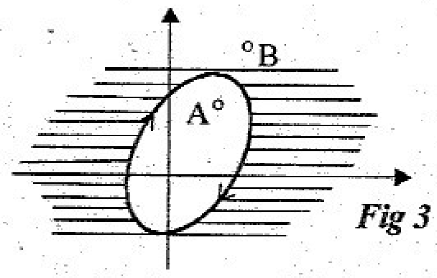
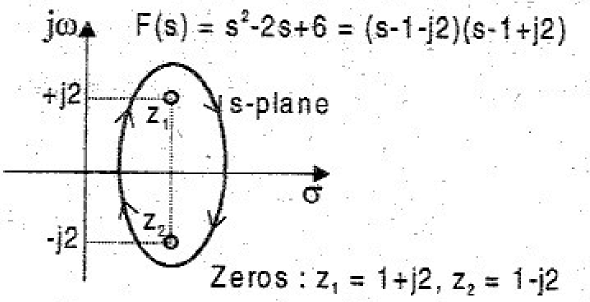
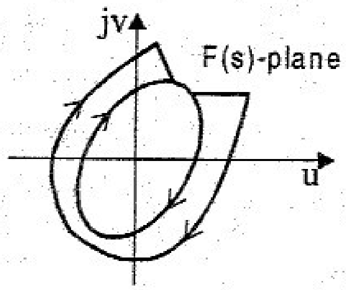


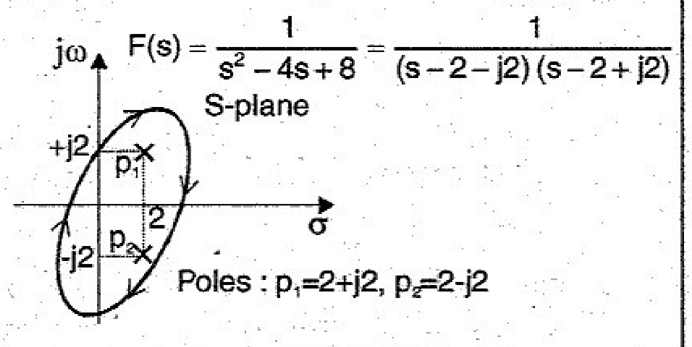
Fig 3

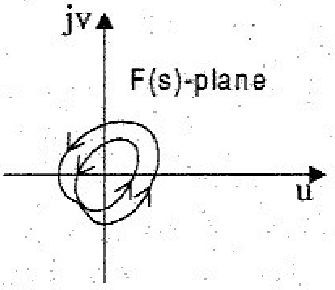
Encircled : *A point is said to be encircled by a closed path if it is found inside the path. With reference to fig 1, the point A is encircled in the clockwise direction and the point B is not encircled.*

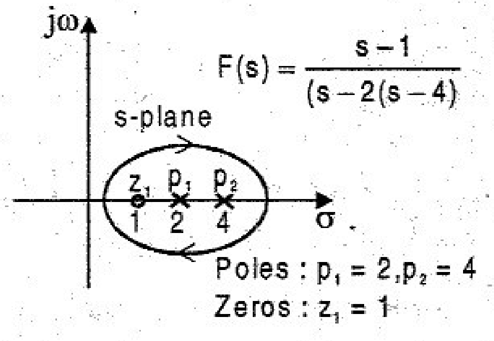
Enclosed : *Any point or region is said to be enclosed by a closed path, if it is found to lie to the right of the path when the path is traversed in the prescribed direction. The shaded regions in fig 2 and 3 are the regions enclosed by the closed path. With reference to fig 2, the point A is enclosed by closed path and the point B is not enclosed. With reference to fig 3 the point A is not enclosed by closed path but point B is enclosed.*

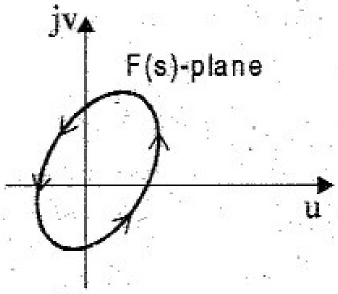
TABLE-4.2

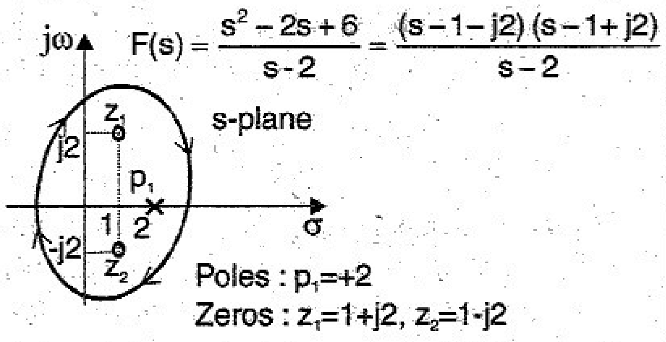
The function, F(s) and s-plane contour	**F(s)-plane contour**
	

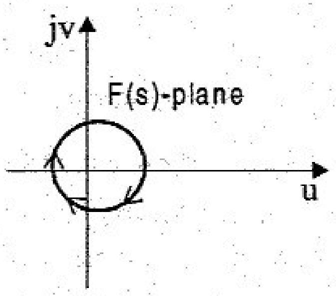


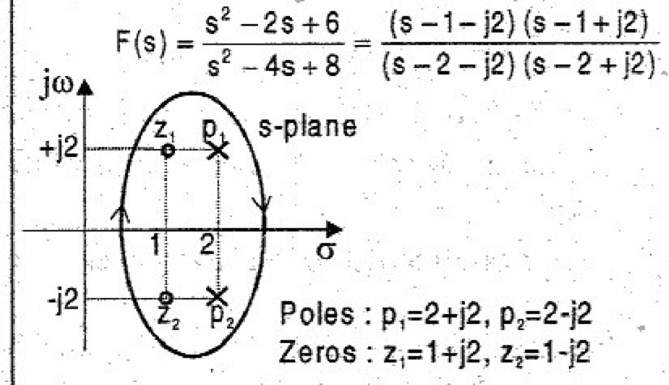


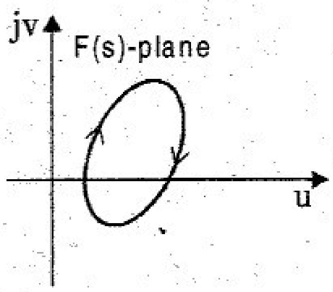


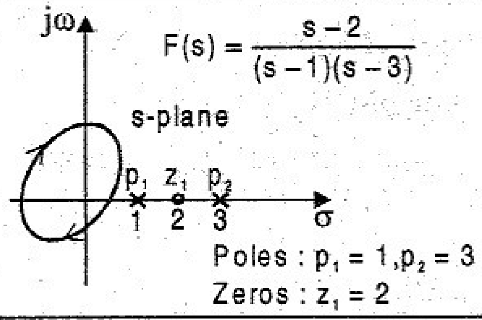


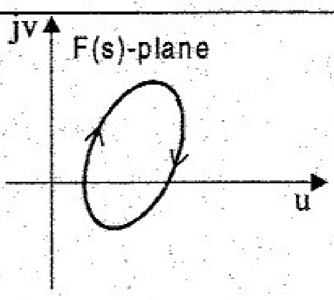












The summary of relationship between the enclosure of poles and zeros by the s-plane closed contour and number of encirclements of the origin of F(s)-plane by the corresponding F(s)-plane contour, are given below.

1. If s-plane closed contour encloses Z number of zeros in the right half of s-plane then the corresponding contour in F(s)-plane will encircle, the origin of F(s)-plane Z times in the clockwise direction.
2. If s-plane closed contour encloses P number of poles in the right half of s-plane then the corresponding contour in F(s)-plane will encircle the origin of F(s)- plane P times in anticlockwise direction.
3. If the s-plane closed contour encloses Z zeros and P poles in the right half of s-plane and if $P > Z$, then the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane $(P-Z)$ times in the anti-clockwise direction.
4. If the s-plane closed contour encloses Z zeros and P poles in the right half of s-plane and if $P < Z$ then the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane $(Z-P)$ times in the clockwise direction.
5. If s-plane closed contour encloses Z zeros and P poles in right half of s-plane and if $P = Z$, then corresponding contour in F(s)-plane will not encircle the origin of F(s)-plane.
6. If the s-plane closed contour does not enclose any pole or zero, then the corresponding contour in F(s)-plane will not encircle the origin of F(s)-plane.

The relation between the enclosure of poles and zeros of F(s) lying on the right half of s-plane by the s-plane contour and the encirclements of the origin of F(s)-plane by the corresponding F(s)-plane contour is called ***principle of argument.***

The principle of argument is stated as follows.

Let F(s) is a single valued rational function and is analytic in a given region in the s-plane except at some points. Now, if an arbitrary closed contour is chosen in the s-plane, so that F(s) is analytic at every point on the closed contour in s-plane then the corresponding F(s)-plane contour mapped in the F(s)-plane will encircle the origin N times in anticlockwise direction where N is the difference between the number of poles and number of zeros of F(s) that are encircled by the chosen closed contour in s-plane.

Mathematically, it can be expressed as, $N = P - Z$.

where, N = Number of encirclement of origin of F(s)-plane, made by F(s)-contour.

Z = Number of zeros of F(s) lying on right half of s-plane and enclosed by the s-plane closed contour.

P = Number of poles of F(s) lying on right half of s-plane and enclosed by the s-plane closed contour.

The value of N can be positive, zero or negative. Based on the sign of N, following conclusions can be made, provided the arbitrary s-plane contour is chosen in the clockwise direction.

1. If N is positive, then direction of encirclement of origin of F(s)-plane will be anticlockwise.

2. If N is zero, then there will be no encirclement of origin of F(s)-plane.

3. If N is negative, then direction of encirclement of origin of F(s)-plane will be clockwise.

4.5 NYQUIST STABILITY CRITERION

Consider the closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

The characteristic equation of the system is given by the condition, $1+ G(s)\,H(s)=0$.

Let, F(s) = 1 + G(s)H(s).

The loop transfer function G(s)H(s) can be expressed as,

$$G(s)H(s)=\frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}\text{, where } m\leq n \qquad .....(4.15)$$

$$\therefore F(s)=1+G(s)H(s)=1+\frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}$$

$$=\frac{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)+K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.\;.(s+p_n)}$$

$$=\frac{(s+z_1')\,(s+z_2')\,.....\,(s+z_n')}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)} \qquad .....(4.16)$$

In equation (4.16), z_1', z_2' ..... z_n', are zeros of F(s), which are obtained by combining the numerator and denominator polynomial of G(s)H(s).

For the condition F(s) = 0, the numerator of F(s) should be equal to zero.

$$\therefore (s+z_1')\,(s+z_2')\,.......\,(s+z_n')=0 \qquad .....(4.17)$$

We can say that equation (4.17) is the characteristic equation of the system. For the stability of the system the roots of the characteristic equation should not lie on the right half s-plane. The roots of characteristic equation are zeros of F(s) and also they are poles of closed loop transfer function.

Hence we can conclude that for the stability of closed loop system the zeros of F(s) should not lie on the right half s-plane.

Note : *For a unity feedback system.*

$$G(s)H(s)=G(s)=\frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}$$

$$\therefore\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{\dfrac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}}{1+\dfrac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)}}$$

$$=\frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+p_1)\,(s+p_2)\,.....\,(s+p_n)+K(s+z_1)\,(s+z_2)\,.....(s+z_m)}$$

$$=\frac{K(s+z_1)\,(s+z_2)\,.....\,(s+z_m)}{(s+z_1')\,(s+z_2')\,.....\,(s+z_n')}$$

From the above equation we can say that the poles of closed loop transfer function are z_1', z_2', ... z_n'.

Let us choose an arbitrary contour in the s-plane which encircles the right half zeros and poles of F(s) (equation (4.16)). The principle of argument (explained in section 4.4) states that the corresponding contour in F(s)-plane will encircle the origin of F(s)-plane, N times in the anticlockwise direction.

Let, N = Number of anticlockwise encirclement

Now, N = P − Z .....(4.18)

where, P = Number of poles of F(s) (or poles of loop transfer function) lying on right half s-plane

Z = Number of zeros of F(s) (or poles of closed loop transfer function) lying on right half s-plane

> ***Note :*** *The stability is related to poles lying on right half s-plane and so, while applying principle of argument only poles and zeros lying on right half s-plane alone are considered.*

For the stability of the system the roots of characteristic equation and so the zeros of F(s) should not lie on the right half of s-plane. Hence for a stable system Z = 0. Hence from equation (4.18) we get,

When $Z = 0, \quad N = P$ .....(4.19)

When $Z \neq 0, \quad N \neq P$ .....(4.20)

From equation (4.15) and (4.16) we can say that the poles of F(s) are also poles of loop transfer function. Hence for the stability of the system, (with reference to equation (4.19) and equation (4.20)) number of poles of loop transfer function lying on right of s-plane should be equal to anticlockwise encirclement of the origin of F(s)-plane. If this condition is not met the system is unstable.

> The principle of argument can also be used to find the number of poles of closed loop transfer function lying on right half of s-plane.
>
> Let, M = Number of clockwise encirclement
>
> Now, $M = Z - P$
>
> When $P = 0, \quad M = Z$
>
> Therefore, when there is no right half open loop poles, number of clockwise encirclement of origin of F(s)-plane gives number of poles of closed loop transfer function lying on right half s-plane.

The loop transfer function, G(s)H(s) can be expressed as,

$$G(s)H(s) = [1+G(s)\,H(s)]-1 = F(s) - 1 \qquad .....(4.21)$$

From equation (4.21) it can be concluded that the contour of F(s) drawn with respect to origin of F(s)-plane is same as the contour of F(s)–1 drawn with respect to −1+j0 of F(s)-plane as shown in fig 4.3.

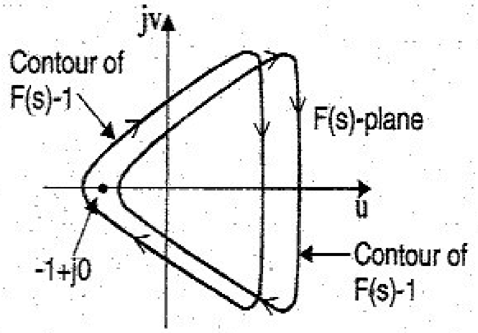


Fig 4.3

Thus the encirclement of the origin of F(s)-plane by the contour of F(s) is equivalent to the encirclement of the point −1 + j0 by the contour of F(s) − 1.

From equation (4.21), we can say that F(s) − 1, represents loop transfer function G(s)H(s). Hence contour of F(s) − 1 is same as contour of G(s)H(s), and F(s)-plane is G(s)H(s)-plane.

Therefore, the encirclement of −1 + j0 point of G(s)H(s)-contour in the G(s)H(s)-plane can be used to determine the stability of closed loop system. The Nyquist stability criterion have been proposed based on this concept.

In order to investigate the presence of poles of G(s)H(s) on the right half s-plane a contour, C is chosen such that it encloses the entire right half s-plane as shown in fig 4.4, such a contour C is called ***Nyquist contour***.

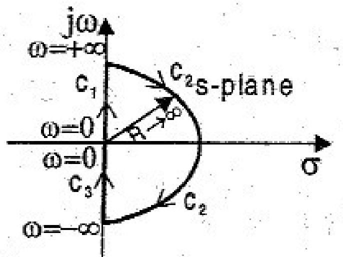


Fig 4.4 : *Nyquist Contour*

The Nyquist contour is directed clockwise and comprises of three segments,

1. An infinite line segment C_1 along the positive imaginary axis.
2. An arc, C_2 of infinite radius, enclosing the entire right half of s-plane.
3. An infinite line segment C_3 along the negative imaginary axis.

Along C_1, $s = j\omega$, with ω varying from 0 to $+\infty$.

Along C_2, $s = \underset{R\to\infty}{\text{Lt}}\ R\,e^{j\theta}$, with θ varying from $+\frac{\pi}{2}$ to $-\frac{\pi}{2}$.

Along C_3, $s = j\omega$, with ω varying from $-\infty$ to 0.

Using the loop transfer function G(s)H(s), the Nyquist contour-C of s-plane, is mapped to G(s)H(s)-plane. The mapped contour in G(s)H(s)-plane is called G(s)H(s)-contour.

Note : *The s-plane is a complex plane. Any point on a complex plane can be expressed by the complex number in polar form, $Re^{j\theta}$, where R is the magnitude and θ is the argument (or phase).*

Now the Nyquist stability criterion can be stated as follows.

"If the G(s)H(s) contour in the G(s)H(s)-plane corresponding to Nyquist contour in the s-plane encircles the point -1 + j0 in the anticlockwise direction as many times as the number of right half s-plane poles of G(s)H(s), then the closed loop system is stable".

In examining the stability of linear control systems using the Nyquist stability criterion, we come across the following three situations.

1. **No encirclement of –1 + j0 point :** This implies that the system is stable if there are no poles of G(s)H(s) in the right half s-plane. If there are poles on right half s-plane then the system is unstable.

2. **Anticlockwise encirclements of –1 + j0 point :** In this case the system is stable if the number of anticlockwise encirclements is same as the number of poles of G(s)H(s) in the right half s-plane. If the number of anticlockwise encirclements is not equal to number of poles on right half s-plane then the system is unstable.

3. **Clockwise encirclements of the –1 + j0 point :** In this case the system is always unstable. Also in this case, if no poles of G(s)H(s) in right half s-plane, then the number of clockwise encirclement is equal to number of poles of closed loop system on right half s-plane.

PROCEDURE FOR INVESTIGATING THE STABILITY USING NYQUIST CRITERION

The following procedure can be followed to investigate the stability of closed loop system from the knowledge of open loop system, using Nyquist stability criterion.

1. Choose a Nyquist contour as shown in fig 4.5, which encloses the entire right half s-plane except the singular points. The Nyquist contour encloses all the right half s-plane poles and zeros of G(s)H(s). [The poles on imaginary axis are singular points and so they are avoided by taking a detour around it as shown in fig 4.5 b and c].

Note : *For mapping a contour from s-plane to G(s)H(s) plane the Nyquist contour in s-plane should be analytic at every point. At singular points it is not analytic.*