CHAPTER 4

CONCEPTS OF STABILITY AND ROOT LOCUS

4.1 IMPULSE RESPONSE AND STABILITY

DEFINITIONS OF STABILITY

The term stability refers to the stable working condition of a control system. Every working system is designed to be stable. In a stable system, the response or output is predictable, finite and stable for a given input (or for any changes in input or for any changes in system parameters).

The different definitions of the stability are the following

1. A system is stable, if its output is bounded (finite) for any bounded (finite) input.
2. A system is asymptotically stable, if in the absence of the input, the output tends towards zero (or to the equilibrium state) irrespective of initial conditions.
3. A system is stable if for a bounded disturbing input signal the output vanishes ultimately as t approaches infinity.
4. A system is unstable if for a bounded disturbing input signal the output is of infinite amplitude or oscillatory.
5. For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called ***limitedly stable.***
6. If a system output is stable for all variations of its parameters, then the system is called ***absolutely stable system.***
7. If a system output is stable for a limited range of variations of its parameters, then the system is called ***conditionally stable system.***

IMPULSE RESPONSE OF A SYSTEM

Let, M(s) = Closed loop transfer function of a system.

C(s) = Output / Response in s-domain.

R(s) = Input in s-domain

Now, $M(s) = \frac{C(s)}{R(s)}$

$\therefore$ Response or Output in s-domain, $C(s) = M(s)\,R(s)$

Now, Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\}$

Input in time domain, $r(t) = \mathcal{L}^{-1}\{R(s)\}$

For an impulse input, $r(t) = \delta(t)$; $\therefore R(s) = \mathcal{L}[\delta(t)] = 1$

$$\therefore \text{ Impulse response } = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{M(s)\,R(s)\} = \mathcal{L}^{-1}\{M(s)\} = m(t) \qquad .....(4.1)$$

Hence, impulse response of a system is the inverse Laplace transform of system transfer function.

The importance of impulse response is that, the output of a system for any arbitrary input can be obtained by convolution of input and impulse response.

i.e., Response, $c(t) = m(t) * r(t)$

where $*$ is the symbol for convolution.

Mathematically the convolution operation is defined as,

$$c(t) = \int_{-\infty}^{+\infty} m(\tau)\; r(t-\tau)\; d\tau \qquad .....(4.2)$$

where t is the dummy variable used for integration.

BOUNDED - INPUT BOUNDED - OUTPUT (BIBO) STABILITY

A linear relaxed system is said to have BIBO stability if every bounded (finite) input results in a bounded (finite) output. A condition for BIBO stability can be obtained from convolution operation defined by equation (4.2).

For a relaxed system the equation (4.2) can be written as,

$$\text{Response, } c(t) = \int_{0}^{\infty} m(\tau)\; r(t-\tau)\; d\tau \qquad .....(4.3)$$

Note : *A relaxed system is one in which the initial conditions are zero. Hence the limits of integration is from 0 to ∞.*

If the input r(t) is bounded then there exists a constant A_1, such that $|r(t)| \le A_1 < \infty$. The condition for bounded output for this bounded input condition can be derived as follows.

On taking the absolute value on both sides of equation (4.3), we get,

$$|c(t)| = \left| \int_{0}^{\infty} m(\tau)\; r(t-\tau)\; d\tau \right| \qquad .....(4.4)$$

Since the absolute value of an integral is not greater than the integral of the absolute value of the integrand the equation (4.4) can be written as,

$$|c(t)| \le \int_{0}^{\infty} |m(\tau)\; r(t-\tau)|\; d\tau \;\Rightarrow\; |c(t)| \le \int_{0}^{\infty} |m(\tau)|\; |r(t-\tau)| d\tau \;\Rightarrow\; |c(t)| \le \int_{0}^{\infty} |m(\tau)|\; A_1 d\tau$$

$$\therefore |c(t)| \le A_1 \int_{0}^{\infty} |m(\tau)|\; d\tau$$

For bounded input, a constant exists such that, $|r(t-\tau)| \le A_1$.

If the output c(t) is bounded then there exists a constant A_2 such that $|c(t)| \le A_2 < \infty$.

$$\therefore A_1 \int_{0}^{\infty} |m(\tau)|\; d\tau \le A_2 < \infty \qquad .....(4.5)$$

The above condition is satisfied if, $\int_0^\infty |m(\tau)|\, d\tau < \infty$

τ is a dummy variable and so can be replaced by t

Hence for bounded output, $\int_0^\infty |m(t)|\, dt < \infty$ .....(4.6)

Therefore we can conclude that a system with impulse response m(t) is BIBO stable if and only if the impulse response is absolutely integrable (i.e., $\int_0^\infty |m(t)|\, dt$ is finite. This means that area under the absolute value curve of the impulse response m(t) evaluated from $t = 0$ to $t = \infty$ must be finite).

4.2 LOCATION OF POLES ON s-PLANE FOR STABILITY

The closed loop transfer function, M(s) can be expressed as a ratio of two polynomials in s. The denominator polynomial of closed loop transfer function is called characteristic equation. The roots of characteristic equation are poles of closed loop transfer function.

For BIBO stability the integral of impulse response should be finite, which implies that the impulse response should be finite as t tends to infinity. *[The impulse response is the inverse Laplace transform of the transfer function].* This requirement for stability can be linked to the location of roots of characteristic equation in the s-plane.

The closed loop transfer function M(s) can be expressed as a ratio of two polynomials in s as shown in equation (4.7).

$$M(s) = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + ... + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_{n-1} s + a_n} \qquad .....(4.7)$$

$$= \frac{(s+z_1)\,(s+z_2)\,(s+z_3)......(s+z_m)}{(s+p_1)\,(s+p_2)\,(s+p_3)\,......(s+p_n)} \qquad .....(4.8)$$

The roots of numerator polynomial z_1, z_2, ...z_n are zeros. The roots of denominator polynomial p_1, p_2, ...p_n are poles. The denominator polynomial is the characteristic equation and so the poles are roots of characteristic equation.

By partial fraction expansion we can write,

$$M(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \frac{A_3}{s+p_3} + ... + \frac{A_n}{s+p_n} \qquad ......(4.9)$$

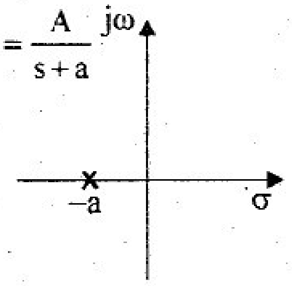
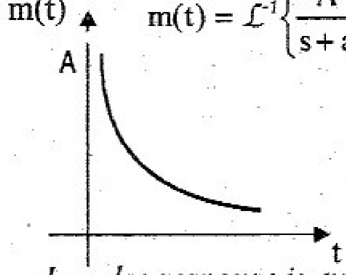
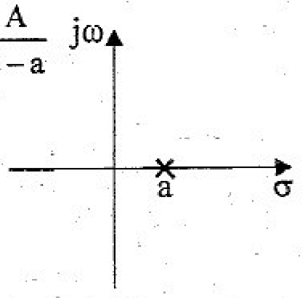
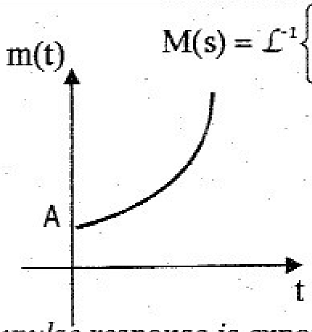
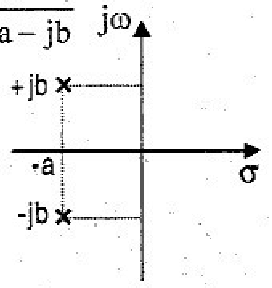
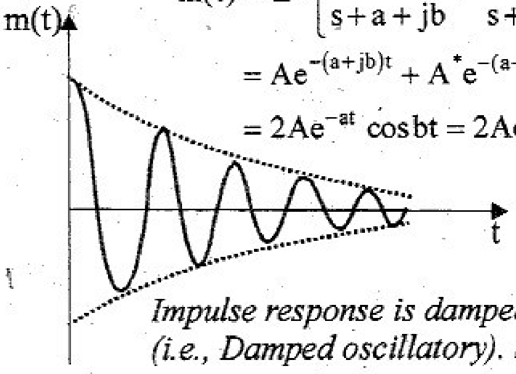
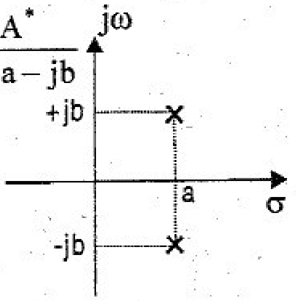
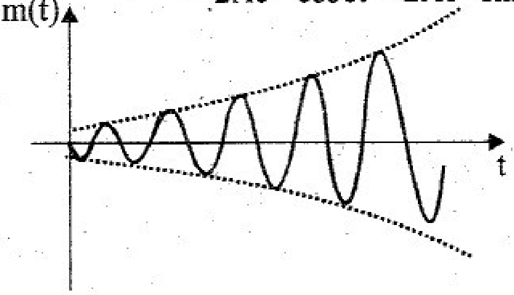
The roots (or poles) p_1, p_2, p_3, ... p_n may be at origin or lying on imaginary axis or lying on right or left half of s-plane. The impulse response is given by inverse Laplace transform of M(s). The inverse Laplace transform of each term of M(s) depends on the location of roots (or poles) in s-plane. The impulse response of various types of M(s) are shown in table-4.1.

From table 4.1, the following conclusions are drawn based on the location of roots of characteristic equation.

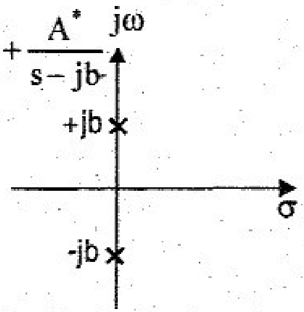
1. If all the roots of characteristic equation have negative real parts (i.e., lying on left half s-plane) then the impulse response is bounded (i.e., it decreases to zero as t tends to ∞).

   Hence $\int_0^\infty |m(t)|\, dt$ is finite and the system is bounded-input bounded-output stable.

TABLE-4.1

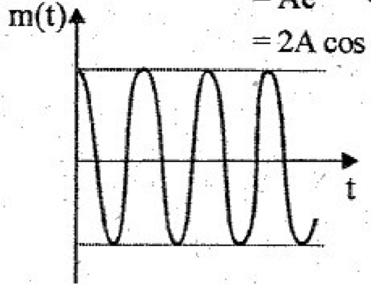
Transfer function, M(s) and location of roots on s-plane	**Impulse response, m(t)**
$M(s) = \frac{A}{s+a}$  *Root on negative real axis*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a}\right\} = Ae^{-at}$  *Impulse response is exponentially decaying. Stable system.*
$M(s) = \frac{A}{s-a}$  *Root on positive real axis*	$M(s) = \mathcal{L}^{-1}\left\{\frac{A}{s-a}\right\} = Ae^{+at}$  *Impulse response is exponentially increasing. Unstable system.*
$M(s) = \frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}$  *Complex conjugate roots on left half of s-plane*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s+a+jb} + \frac{A^*}{s+a-jb}\right\}$ $= Ae^{-(a+jb)t} + A^*e^{-(a-jb)t}$ $= 2Ae^{-at}\cos bt = 2Ae^{-at}\sin\left(bt + 90^\circ\right)$  *Impulse response is damped sinusoidal (i.e., Damped oscillatory). Stable system*
$M(s) = \frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}$  *Complex conjugate roots on right half of s-plane*	$m(t) = \mathcal{L}^{-1}\left\{\frac{A}{s-a+jb} + \frac{A^*}{s-a-jb}\right\}$ $= Ae^{-(-a+jb)t} + A^*e^{-(-a-jb)t}$ $= 2Ae^{at}\cos bt = 2Ae^{at}\sin(bt + 90^\circ)$  *Impulse response is exponentially increasing sinusoidal (i.e., Amplitude of oscillations exponentially increases with time). Unstable system.*

$$M(s)=\frac{A}{s+jb}+\frac{A^*}{s-jb}$$



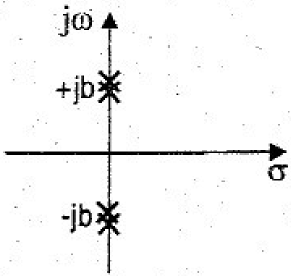
Single pair of roots on imaginary axis

$$m(t)=\mathcal{L}^{-1}\left\{\frac{A}{s+jb}+\frac{A^*}{s-jb}\right\}$$
$$=Ae^{-jbt}+A^*e^{+jbt}$$
$$=2A\cos bt=2A\sin(bt+90°)$$



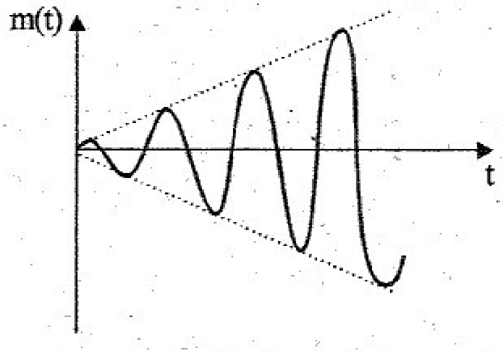
Impulse response is oscillatory
Marginally stable

$$M(s)=\frac{A}{(s+jb)^2}+\frac{A^*}{(s-jb)^2}$$



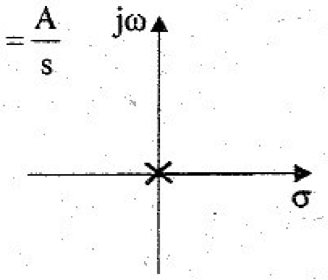
Double pair of roots on imaginary axis

$$m(t)=\mathcal{L}^{-1}\left\{\frac{A}{(s+jb)^2}+\frac{A^*}{(s-jb)^2}\right\}$$
$$=At\,e^{-jbt}+A^*\,t\,e^{+jbt}$$
$$=2At\cos bt=2At\sin(bt+90°)$$

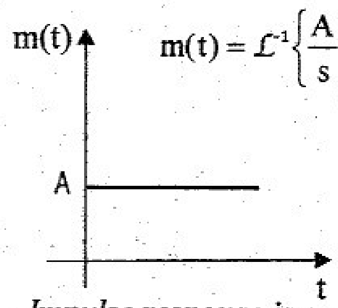


Impulse response is linearly increasing sinusoidal (i.e., amplitude of oscillations linearly increases with time). Unstable system.

$$M(s)=\frac{A}{s}$$

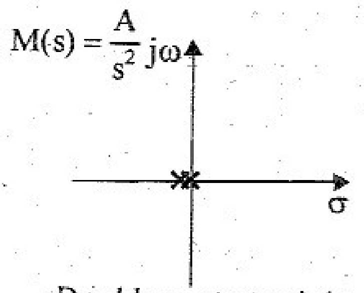


Single root at origin

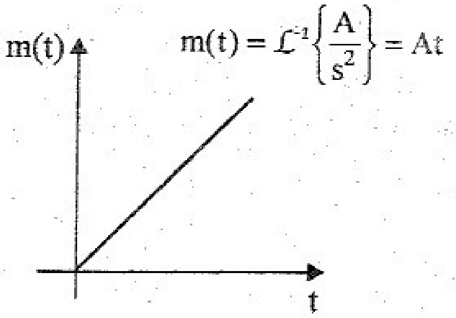


$$m(t)=\mathcal{L}^{-1}\left\{\frac{A}{s}\right\}=A$$

Impulse response is constant.
Marginally stable system.



Double root at origin



Impulse reponse linearly increases with time. Unstable system

2. If any root of the characteristic equation has a positive real part (i.e., lying on right half s-plane) then impulse response is unbounded, (i.e., it increases to ∞ as t tends to ∞). Hence $\int_0^{\infty} |m(t)|\, dt$ is infinite and so system is unstable.
3. If the characteristic equation has repeated roots on the imaginary axis then impulse response is unbounded (i.e., it increases to ∞ as t tends to ∞).

   Hence $\int_0^{\infty} |m(t)|\, dt$ is infinite and so the system is unstable.
4. If one or more non - repeated roots of the characteristic equation are lying on the imaginary axis, then impulse response is bounded (i.e., it has constant amplitude oscillations) but is infinite and so the system is unstable.
5. If the characteristic equation has single root at origin then the impulse response is bounded (i.e, it has constant amplitude) but $\int_0^{\infty} |m(t)|\, dt$ is infinite and so the system is unstable.
6. If the characteristic equation has repeated roots at origin then the impulse response is unbounded (i.e., it linearly increases to infinity as t tends to ∞) and so the system is unstable.
7. In system with one or more non-repeated roots on imaginary axis or with single root at origin, the output is bounded for bounded inputs except for the inputs having poles matching the system poles. These cases may be treated as acceptable or non-acceptable. Hence when the system has non repeated poles on imaginary axis or single pole at origin, it is referred as limitedly or marginally stable system.

In summary, the following three points may be stated regarding the stability of the system depending on the location of roots of characteristic equation.

1. ***If all the roots of characteristic equation has negative real parts, then the system is stable.***
2. ***If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis then the system is unstable.***
3. ***If the condition (i) is satisfied except for the presence of one or more non repeated roots on the imaginary axis, then the system is limitedly or marginally stable.***

In order to ascertain the stability of a system, it is necessary to determine if any of the roots of the characteristic equation lie in the right half s-plane. The characteristic equation is given by the denominator polynomial of closed loop transfer function, [equation (4.7)].

Consider the n^{th} order characteristic equation shown below.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_{n-1} s + a_n = 0 \quad .....(4.10)$$

Let the roots of n^{th} order characteristic equation [equation (4.10)] be $s = r_1, r_2, ...r_n$. These roots are functions of the coefficients $a_0, a_1, a_2, ... a_{n-1}, a_n$.

Consider a second order polynomial,

$$\begin{aligned} a_0 s^2 + a_1 s + a_2 &= a_0 \left(s^2 + \frac{a_1}{a_0} s + \frac{a_2}{a_0} \right) \\ &= a_0 (s - r_1)(s - r_2) \\ &= a_0 s^2 - a_0 (r_1 + r_2) s + a_0 r_1 r_2 \end{aligned} \quad .....(4.11)$$

Consider a third order polynomial

$$\begin{aligned} a_0 s^3 + a_1 s^2 + a_2 s + a_3 &= a_0 \left(s^3 + \frac{a_1}{a_0} s^2 + \frac{a_2}{a_0} s + \frac{a_3}{a_0} \right) \\ &= a_0 (s - r_1)(s - r_2)(s - r_3) \\ &= a_0 s^3 - a_0 (r_1 + r_2 + r_3) s^2 \\ &\quad + a_0 (r_1 r_2 + r_1 r_3 + r_2 r_3) s - a_0 r_1 r_2 r_3 \end{aligned} \quad .....(4.12)$$

On extending this expansion to the n^{th} order polynomial, we get.

$$\begin{aligned} a_0 s^n + a_1 s^{n-1} + ...+ a_{n-1} s + a_n &= a_0 s^n - a_0 (\text{sum of all the roots}) s^{n-1} \\ &\quad + a_0 \left(\begin{matrix} \text{sum of the products of the roots} \\ \text{taken 2 at a time} \end{matrix} \right) s^{n-2} \\ &\quad - a_0 \left(\begin{matrix} \text{sum of the products of the roots} \\ \text{taken 3 at a time} \end{matrix} \right) s^{n-3} \\ &\quad + ... + a_0 (-1)^n \ (\text{Product of all the n roots}) \end{aligned} \quad .....(4.13)$$

If all the roots of a polynomial are real and in the left half of s-plane, then all r_i in equations (4.11) and (4.12) are real and negative. Therefore all polynomial coefficients are positive. This characteristic also applies to the general case of equation (4.13). If atleast one root is in the right half of s-plane then some of the coefficients will be negative. Also, it can be observed that if all the roots are in the left half of s-plane, no coefficient can be zero.

Since the characteristic polynomial coefficients are real, the complex roots should occur as conjugate pairs. From equation (4.13) it can be inferred that when polynomial coefficients are formed, the imaginary parts of roots/products of roots will cancel. Therefore, if all roots occur in the left half plane, (whether it is complex or real) then all coefficients of the general polynomial of equation (4.13) will be positive. Presence of a negative coefficient implies that there is atleast one root in the right half of s- plane.

A zero coefficient indicates presence of complex-conjugate roots on the imaginary axis and/or one or more roots in the right half of s- plane.

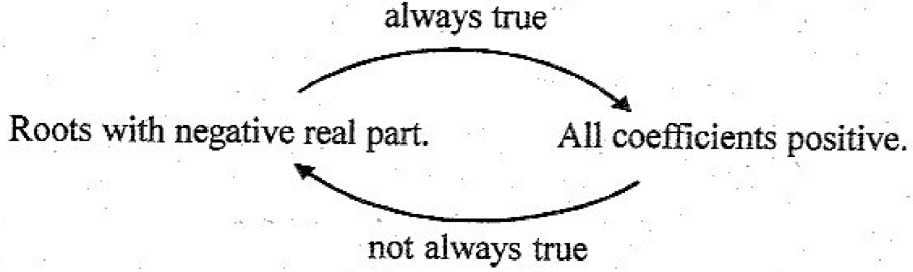
In summary, following conclusions can be made about coefficients of characteristic polynomial.

1. ***If all the coefficients are positive and if no coefficient is zero, then all the roots are in the left half of s- plane.***
2. ***If any coefficient a_i is equal to zero then, some of the roots may be on the imaginary axis or on the right half of s- plane.***

3. ***If any coefficient a_i is negative then atleast one root is in the right half of s- plane.***

It can be concluded that the absence or negativeness of any of the coefficients of a characteristic polynomial indicates that the system is either unstable or at most marginally stable. Thus ***the necessary condition for stability of the system is that all the coefficients of its characteristic polynomial be positive.*** If any coefficient is zero/negative, we can immediately say that the system is unstable.

In order for all the roots to have negative real parts, it is necessary that all of the coefficients of characteristic equation be positive, but it is not sufficient, because there may be roots in the right half plane and/or on the imaginary axis, even when coefficients are positive.(i.e., when roots have negative real part, then all the coefficients of characteristic polynomial will be positive, but the reverse condition is not true always).



Hence, when all the coefficients are positive, the system may or may not be stable, because there may be roots in the right half plane and/or on the imaginary axis.

For example, consider the characteristic polynomial with all positive coefficients,

$$s^3 + s^2 + 2s + 8 = 0.$$

The characteristic polynomial can be written as,

$$(s^3 + s^2 + 2s + 8) = (s+2)\left(s - \frac{1}{2} - j\frac{\sqrt{15}}{2}\right)\left(s - \frac{1}{2} + j\frac{\sqrt{15}}{2}\right) = 0$$

Now the roots are,

$$s = -2, \quad +\frac{1}{2} + j\frac{\sqrt{15}}{2}, \quad +\frac{1}{2} - j\frac{\sqrt{15}}{2}$$

The coefficients of the polynomial are all positive, but two roots have positive real part and so will lie on on right half of s-plane, therefore the system is unstable.

4.3 ROUTH HURWITZ CRITERION

The Routh-Hurwitz stability criterion is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not.

The first step in analysing the stability of a system is to examine its characteristic equation. The necessary condition for stability is that all the coefficients of the polynomial be positive. If some of the coefficients are zero or negative it can be concluded that the system is not stable.

When all the coefficients are positive, the system is not necessarily stable. Eventhough the coefficient are positive, some of the roots may lie on the right half of s-plane or on the imaginary axis. In order for all the roots to have negative real parts, it is necessary but not sufficient that all coefficients of the characteristic equation be positive. If all the coefficients of the characteristic equation are positive, then the system may be stable and one should proceed further to examine the sufficient conditions of stability.

A. Hurwitz and E.J. Routh independently published the method of investigating the sufficient conditions of stability of a system. The Hurwitz criterion is in terms of determinants and Routh criterion is in terms of array formulation. The Routh stability criterion is presented here.

The Routh stability criterion is based on ordering the coefficients of the characteristic equation, into a schedule, called the Routh array as shown below.

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + ... + a_{n-1}s + a_n = 0, \text{ where } a_0 > 0 ,$$

$$\begin{array}{llllllll}
s^n & : & a_0 & a_2 & a_4 & a_6 & a_8 & ...... \\
s^{n-1} & : & a_1 & a_3 & a_5 & a_7 & a_9 & ...... \\
s^{n-2} & : & b_0 & b_1 & b_2 & b_3 & b_4 & ...... \\
s^{n-3} & : & c_0 & c_1 & c_2 & c_3 & c_4 & ...... \\
 & & & & & & & \\
s^1 & : & g_0 & & & & & \\
s_0 & : & h_0 & & & & &
\end{array}$$

The Routh stability criterion can be stated as follows.

"The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of the s-plane".

Note : If the order of sign of first column element is +, +, –, + and +. Then + to – is considered as one sign change and – to + as another sign change.

CONSTRUCTION OF ROUTH ARRAY

Let the characteristic polynomial be,

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + a_3s^{n-3} + ...a_{n-1}s^1 + a_ns^0$$

The coefficients of the polynomial are arranged in two rows as shown below.

$$\begin{array}{llllll}
s^n & : a_0 & a_2 & a_4 & a_6 & ..... \\
s^{n-1} & : a_1 & a_3 & a_5 & a_7 & .....
\end{array}$$

When n is even, the s^n row is formed by coefficients of even order terms (i.e., coefficients of even powers of s) and s^{n-1} row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s).

When n is odd, the s^n row is formed by coefficients of odd order terms (i.e., coefficients of odd powers of s) and s^{n-1} row is formed by coefficients of even order terms(i.e., coefficients of even powers of s) .

The other rows of routh array upto s^0 row can be formed by the following procedure. Each row of Routh array is constructed by using the elements of previous two rows.

Consider two consecutive rows of Routh array as shown below.

$$\begin{array}{lllllll} s^{n-x} & : x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \ldots\ldots \\ s^{n-x-1} & : y_0 & y_1 & y_2 & y_3 & y_4 & y_5 \ldots\ldots \end{array}$$

Let the next row be,

$$s^{n-x-2} : z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \ldots\ldots$$

The elements of s^{n-x-2} row are given by,

$$z_0 = \frac{(-1)\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}}{y_0} = \frac{y_0x_1 - y_1x_0}{y_0}$$

$$z_1 = \frac{(-1)\begin{vmatrix} x_0 & x_2 \\ y_0 & y_2 \end{vmatrix}}{y_0} = \frac{y_0x_2 - y_2x_0}{y_0}$$

$$z_2 = \frac{(-1)\begin{vmatrix} x_0 & x_3 \\ y_0 & y_3 \end{vmatrix}}{y_0} = \frac{y_0x_3 - y_3x_0}{y_0}$$

$$z_3 = \frac{(-1)\begin{vmatrix} x_0 & x_4 \\ y_0 & y_4 \end{vmatrix}}{y_0} = \frac{y_0x_4 - y_4x_0}{y_0}$$

$$z_4 = \frac{(-1)\begin{vmatrix} x_0 & x_5 \\ y_0 & y_5 \end{vmatrix}}{y_0} = \frac{y_0x_5 - y_5x_0}{y_0} \quad \text{and so on.}$$

The elements z_0, z_1, z_2, z_3,.... are computed for all possible computations as shown above.

In the process of constructing Routh array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied or divided by a positive constant to simplify the computational work.

In the construction of Routh array one may come across the following three cases.

Case-I : Normal Routh array (Non-zero elements in the first column of routh array).

Case-II : A row of all zeros.

Case-III : First element of a row is zero but some or other elements are not zero.

Case-I : Normal routh array

In this case, there is no difficulty in forming routh array. The routh array can be constructed as explained above. The sign changes are noted to find the number of roots lying on the right half of s-plane and the stability of the system can be estimated.

In this case,

1. If there is no sign change in the first column of Routh array then all the roots are lying on left half of s-plane and the system is stable.

2. If there is sign change in the first column of routh array, then the system is unstable and the number of roots lying on the right half of s-plane is equal to number of sign changes. The remaining roots are lying on the left half of s-plane.

Case-II : A row of all zeros

An all zero row indicates the existence of an even polynomial as a factor of the given characteristic equation. In an even polynomial the exponents of s are even integers or zero only. This even polynomial factor is also called ***auxiliary polynomial.*** The coefficients of the auxiliary polynomial will always be the elements of the row directly above the row of zeros in the array.

The roots of an even polynomial occur in pairs that are equal in magnitude and opposite in sign. Hence, these roots can be purely imaginary, purely real or complex. The purely imaginary and purely real roots occur in pairs. The complex roots occur in groups of four and the complex roots have quadrantal symmetry, that is the roots are symmetrical with respect to both the real and imaginary axes. The fig 4.1 shows the roots of an even polynomial.

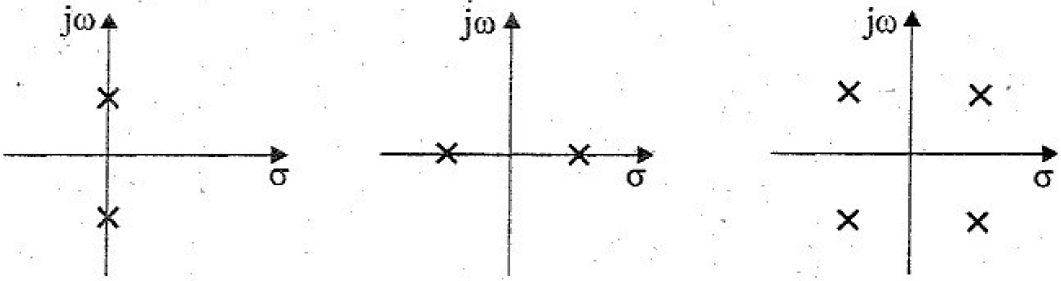


Fig 4.1 : *The roots of an even polynomial.*

The case-II polynomial can be analyzed by any one of the following two methods.

METHOD-1

1. Determine the auxiliary polynomial, A(s)
2. Differentiate the auxiliary polynomial with respect to s, to get d A(s)/ds
3. The row of zeros is replaced with coefficients of dA(s)/ds.
4. Continue the construction of the array in the usual manner (as that of case-I) and the array is interpreted as follows.
    a. If there are sign changes in the first column of routh array then the system is unstable. The number of roots lying on right half of s-plane is equal to number of sign changes. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.
    b. If there are no sign changes in the first column of routh array then the all zeros row indicate the existence of purely imaginary roots and so the system is limitedly or marginally stable. The roots of auxiliary equation lies on imaginary axis and the remaining roots lies on left half of s-plane.

METHOD-2

1. Determine the auxiliary polynomial, A(s).
2. Divide the characteristic equation by auxiliary polynomial.

3. Construct Routh array using the coefficients of quotient polynomial.
4. The array is interpretted as follows.
   a. If there are sign changes in the first column of routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of s-plane is given by number of sign changes in first column of routh array.

      The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex.

      The total number of roots on right half of s-plane is given by the sum of number of sign changes and the number of roots of auxiliary polynomial with positive real part. The number of roots on imaginary axis can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

   b. If there is no sign change in the first column of routh array of quotient polynomial then the system is limitedly or marginally stable. Since there is no sign change all the roots of quotient polynomial are lying on the left half of s-plane.

      The roots of auxiliary polynomial are directly calculated to find whether they are purely imaginary or purely real or complex. The number of roots lying on imaginary axis and on the right half of s-plane can be estimated from the roots of auxiliary polynomial. The remaining roots are lying on the left half of s-plane.

Case-III : First element of a row is zero

While constructing routh array, if a zero is encountered as first element of a row then all the elements of the next row will be infinite. To overcome this problem let $0 \rightarrow \in$ and complete the construction of array in the usual way (as that of case-I)

Finally let $\in \rightarrow 0$ and determine the values of the elements of the array which are functions of $\in$. The resultant array is interpreted as follows.

Note : *If all the elements of a row are zeros then the solution is attempted by considering the polynomial as case-II polynomial. Even if there is a single element zero on s^1 row, it is considered as a row of all zeros.*

a. If there is no sign change in first column of routh array and if there is no row with all zeros, then all the roots are lying on left half of s-plane and the system is stable.

b. If there are sign changes in first column of routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the remaining roots are lying on the left half of s-plane.

c. If there is a row of all zeros after letting $\in \rightarrow 0$, then there is a possibility of roots on imaginary axis. Determine the auxiliary polynomial and divide the characteristic equation by auxiliary polynomial to eliminate the imaginary roots. The routh array is constructed using the coefficients of quotient polynomial and the characteristic equation is interpreted as explained in method-2 of case-II polynomial.

EXAMPLE 4.1

Using Routh criterion, determine the stability of the system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^4+8s^3+18s^2+16s+5=0$.

The given characteristic equation is 4th order equation and so it has 4 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

$$\begin{array}{llllll} s^4 & : & 1 & 18 & 5 & \text{.... Row-1} \\ s^3 & : & 8 & 16 & & \text{.... Row-2} \end{array}$$

The elements of s^3 row can be divided by 8 to simplify the computations.

$$\begin{array}{llllll} s^4 & : & 1 & 18 & 5 & \text{.... Row-1} \\ s^3 & : & 1 & 2 & & \text{.... Row-2} \\ s^2 & : & 16 & 5 & & \text{.... Row-3} \\ s^1 & : & 1.7 & & & \text{.... Row-4} \\ s^0 & : & 5 & & & \text{.... Row-5} \end{array}$$

Column-1

$$s^2 : \frac{1\times18-2\times1}{1} \quad \frac{1\times5-0\times1}{1}$$

$$s^2 : \; 16 \quad 5$$

$$s^1 : \frac{16\times2-5\times1}{16}$$

$$s^1 : \; 1.6875 \approx 1.7$$

$$s^0 : \frac{1.7\times5-0\times16}{17}$$

$$s^0 : \; 5$$

On examining the elements of first column of routh array it is observed that all the elements are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

RESULT

1. Stable system
2. All the four roots are lying on the left half of s-plane.

EXAMPLE 4.2

Construct Routh array and determine the stability of the system whose characterisitc equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

SOLUTION

The characteristic equation of the system is, $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$.

The given characteristic polynomial is 6th order equation and so it has 6 roots. Since the highest power of s is even number, form the first row of routh array using the coefficients of even powers of s and form the second row using the coefficients of odd powers of s.

$$\begin{array}{lllllll} s^6 & : & 1 & 8 & 20 & 16 & \text{.... Row-1} \\ s^5 & : & 2 & 12 & 16 & & \text{.... Row-2} \end{array}$$

The elements of s^5 row can be divided by 2 to simplify the calculations.

$$
\begin{array}{lllllll}
s^6 & : & 1 & 8 & 20 & 16 & \text{.... Row-1} \\
s^5 & : & 1 & 6 & 8 & & \text{....Row-2} \\
s^4 & : & 1 & 6 & 8 & & \text{....Row-4} \\
s^3 & : & 0 & 0 & & & \text{....Row-4} \\
s^3 & : & 1 & 3 & & & \text{....Row-4} \\
s^2 & : & 3 & 8 & & & \text{....Row-5} \\
s^1 & : & 0.33 & & & & \text{....Row-6} \\
s^0 & : & 8 & & & & \text{....Row-7}
\end{array}
$$

Column-1

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots on imaginary axis. Hence the system is limitedly or marginally stable.

The auxiliary polynomial is,

$$s^4+6s^2+8=0$$

Let, $s^2=x$

$$\therefore x^2+6x+8=0$$

The roots of quadratic are, $x=\dfrac{-6\pm\sqrt{6^2-4\times 8}}{2}$

$$=-3\pm 1=-2 \text{ or } -4$$

The roots of auxiliary polynomial is,

$$s=\pm\sqrt{x}\ =\pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$=+j\sqrt{2},-j\sqrt{2},+j2 \text{ and } -j2$$

$$s^4 : \frac{1\times 8-6\times 1}{1} \quad \frac{1\times 20-8\times 1}{1} \quad \frac{1\times 16-0\times 1}{1}$$

$$s^4 : \quad 2 \quad 12 \quad 16$$

divide by 2

$$s^4 : \quad 1 \quad 6 \quad 8$$

$$s^3 : \frac{1\times 6-6\times 1}{1} \quad \frac{1\times 8-8\times 1}{1}$$

$$s^3 : \quad 0 \quad 0$$

The auxiliary equation is, A = s^4+6s^2+8. On differentiating A with respect to s we get,

$$\frac{dA}{ds}=4s^3+12s$$

The coefficients of $\frac{dA}{ds}$ are used to form s^3 row.

$$s^3 : \quad 4 \quad 12$$

divide by 4

$$s^3 : \quad 1 \quad 3$$

$$s^2 : \frac{1\times 6-3\times 1}{1} \quad \frac{1\times 8-0\times 1}{1}$$

$$s^2 : \quad 3 \quad 8$$

$$s^1 : \frac{3\times 3-8\times 1}{3}$$

$$s^1 : \ 0.33$$

$$s^0 : \frac{0.33\times 8-0\times 3}{0.33}$$

$$s^0 : \ 8$$

The roots of auxiliary polynomial are also roots of characteristic equation. Hence 4 roots are lying on imaginary axis and the remaining two roots are lying on the left half of s-plane.

RESULT

1. The system is limitedly or marginally stable.
2. Four roots are lying on imaginary axis and remaining two roots are lying on left half of s-plane.

EXAMPLE 4.3

Construct Routh array and determine the stability of the system represented by the characteristic equation, $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation.

SOLUTION

The characteristic equation of the system is, $s^5+s^4+2s^3+2s^2+3s+5=0$.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$
\begin{array}{llllll}
s^5 & : & 1 & 2 & 3 & \text{.... Row-1} \\
s^4 & : & 1 & 2 & 5 & \text{.... Row-2}
\end{array}
$$

$$\begin{array}{llll}
s^3 & : \quad \in & -2 & \text{.... Row-3} \\
s^2 & : \quad \dfrac{2\in+2}{\in} & 5 & \text{.... Row-4} \\
s^1 & : \quad \dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2} & & \text{.... Row-5} \\
s^0 & : \quad 5 & & \text{.... Row-6}
\end{array}$$

On letting $\in \to 0$, we get

$$\begin{array}{llllll}
s^5 & : & 1 & 2 & 3 & \text{.... Row-1} \\
s^4 & : & 1 & 2 & 5 & \text{.... Row-2} \\
s^3 & : & 0 & -2 & & \text{.... Row-3} \\
s^2 & : & \infty & 5 & & \text{.... Row-4} \\
s^1 & : & -2 & & & \text{.... Row-5} \\
s^0 & : & 5 & & & \text{.... Row-6}
\end{array}$$

+ to −

− to +

Column-1

On observing the elements of first column of routh array, it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and the system is unstable. The remaining three roots are lying on the left half of s-plane.

$$s^3 : \frac{1\times 2-2\times 1}{1} \quad \frac{1\times 3-5\times 1}{1}$$

$$s^3 : \quad 0 \qquad -2$$

Replace 0 by $\in$

$$s^3 : \quad \in \qquad -2$$

$$s^2 : \frac{\in\times 2-(-2\times 1)}{\in} \quad \frac{\in\times 5-0\times 1}{\in}$$

$$s^2 : \frac{2\in+2}{\in} \qquad 5$$

$$s^1 : \frac{\dfrac{2\in+2}{\in}\times(-2)-(5\times\in)}{\dfrac{2\in+2}{\in}}$$

$$s^1 : \frac{-(5\in^2+4\in+4)}{2\in+2}$$

$$s^0 : \frac{\dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2}\times 5-0\times\dfrac{2\in+2}{\in}}{\dfrac{-\left(5\in^2+4\in+4\right)}{2\in+2}}$$

$$s^0 : 5$$

RESULT

(a). The system is unstable.

(b). Two roots are lying on right half of s-plane and three roots are lying on left half of s-plane.

EXAMPLE 4.4

By routh stability criterion determine the stability of the system represented by the characteristic equation, $9s^5-20s^4+10s^3-s^2-9s-10=0$. Comment on the location of roots of characteristic equation.

SOLUTION

The characteristic polynomial of the system is, $9s^5-20s^4+10s^3-s^2-9s-10=0$

On examining the coefficients of the characteristic polynomial, it is found that some of the coefficients are negative and so some roots will lie on the right half of s-plane. Hence the system is unstable. The routh array can be constructed to find the number of roots lying on right half of s-plane.

The given characteristic polynomial is 5th order equation and so it has 5 roots. Since the highest power of s is odd number, form the first row of routh array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s.

$$\begin{array}{llllll}
s^5 & : & 9 & 10 & -9 & \text{.... Row-1} \\
s^4 & : & -20 & -1 & -10 & \text{.... Row-2} \\
s^3 & : & 9.55 & -13.5 & & \text{.... Row-3} \\
s^2 & : & -29.3 & -10 & & \text{.... Row-4} \\
s^1 & : & -16.8 & & & \text{.... Row-5} \\
s^0 & : & -10 & & & \text{.... Row-6}
\end{array}$$

+ to −

− to +

+ to −

Column-1

$$s^3 : \frac{-20\times 10-(-1)\times 9}{-20} \quad \frac{-20\times(-9)-(-10)\times 9}{-20}$$

$$s^3 : 9.55 \qquad -13.5$$

$$s^2 : \frac{9.55\times(-1)-(-13.5)\times(-20)}{9.55} \quad \frac{9.55\times(-10)}{9.55}$$

$$s^2 : -29.3 \qquad -10$$

By examining the elements of Ist column of routh array it is observed that there are three sign changes and so three roots are lying on the right half of s-plane and the remaining two roots are lying on the left half of s-plane.

$$s^1 : \frac{-29.3\times(-13.5)-(-10)\times 9.55}{-29.3}$$

s^1: − 16.8

$$s^0 : \frac{-16.8\times(-10)}{-16.8}$$

s^0: − 10

RESULT

(a). The system is unstable.

(b). Three roots are lying on right half of s-plane and two roots are lying on left half of s-plane.

EXAMPLE 4.5

The characteristic polynomial of a system is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$. Determine the location of roots on s-plane and hence the stability of the system.

SOLUTION

METHOD-I

The characteristic equation is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

s^7	:	1	24	24	23	.... Row-1
s^6	:	9	24	24	15	.... Row-2

Divide s^6 row by 3 to simplify the computations.

	s^7	:	1	24	24	23	.... Row-1
	s^6	:	3	8	8	5	.... Row-2
	s^5	:	1	1	1		.... Row-3
	s^4	:	1	1	1		.... Row-4
	s^3	:	0	0			.... Row-5
	s^3	:	2	1			.... Row-5
	s^2	:	0.5	1			.... Row-6
+ to −	s^1	:	-3				.... Row-7
− to +	s^0	:	1				.... Row-8

Column-1

$$s^5 : \frac{3\times24-8\times1}{3} \qquad \frac{3\times24-8\times1}{3} \qquad \frac{3\times23-5\times1}{3}$$

s^5 : 21.33 21.33 21.33

Divide by 21.33

s^5 : 1 1 1

$$s^4 : \frac{1\times8-1\times3}{1} \qquad \frac{1\times8-1\times3}{1} \qquad \frac{1\times5-0\times3}{1}$$

s^4 : 5 5 5

Divide by 5

s^4 : 1 1 1

$$s^3 : \frac{1\times1-1\times1}{1} \qquad \frac{1\times1-1\times1}{1}$$

s^3 : 0 0

The auxiliary polynomial is,

$$A = s^4 + s^2 + 1$$

Differentiate A with respect to s.

$$\frac{dA}{ds} = 4s^3 + 2s$$

s^3 : 4 2

Divide by 2

s^3 : 2 1

On examining the first column elements of routh array it is found that there are two sign changes. Hence two roots are lying on the right half of s-plane and so the system is unstable.

The row of all zeros indicates the possibility of roots on imaginary axis. This can be tested by evaluating the roots of auxiliary polynomial.

The auxiliary equation is, $s^4 + s^2 + 1 = 0$

Put, $s^2 = x$ in the auxiliary equation,

$$s^4 + s^2 + 1 = x^2 + x + 1 = 0$$

The roots of quadratic are, $x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

$= 1\angle120^\circ$ or $1\angle-120^\circ$

But $s^2 = x$, $\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle120^\circ}$ or $\pm\sqrt{1\angle-120^\circ}$

$= \pm\sqrt{1}\angle120^\circ/2$ or $\pm\sqrt{1}\angle-120^\circ/2$

$= \pm1\angle60^\circ$ or $\pm1\angle-60^\circ$

$= \pm(0.5 + j0.866)$ or $\pm(0.5 - j0.866)$

$s^2 : \frac{2\times1-1\times1}{2} \quad \frac{2\times1-0\times1}{2}$

$s^2 : \quad 0.5 \quad\quad 1$

$s^1 : \frac{0.5\times1-1\times2}{0.5}$

$s^1 : -3$

$s^0 : \frac{-3\times1}{-3}$

$s^0 : 1$

Two roots of auxiliary polynomial are lying on the right half of s-plane and the remaining two on the left half of s-plane. The roots of auxiliary equation are also the roots of characteristic polynomial. The two roots lying on the right half of s-plane are indicated by two sign changes in the first column of routh array. The remaining five roots are lying on the left half of s-plane. No roots are lying on imaginary axis.

RESULT

1. The system is unstable.
2. Two roots are lying on right half of s-plane and five roots are lying on left half of s-plane.

METHOD-II

The characteristic equation is, $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$

The given characteristic polynomial is 7th order equation and so it has 7 roots. Since the highest power of s is odd number, form the first row of array using the coefficients of odd powers of s and form the second row using the coefficients of even powers of s as shown below.

$s^7 \; : \; 1 \quad 24 \quad 24 \quad 23$ Row-1

$s^6 \; : \; 9 \quad 24 \quad 24 \quad 15$ Row-2

Divide s^6 row by 3 to simplify the computations.

$s^7 \; : \; 1 \quad 24 \quad 24 \quad 23$ Row-1

$s^6 \; : \; 3 \quad 8 \quad 8 \quad 5$ Row-2

$s^5 \; : \; 1 \quad 1 \quad 1$ Row-3

$s^4 \; : \; 1 \quad 1 \quad 1$ Row-4

$s^3 \; : \; 0 \quad 0$ Row-5

$s^5 : \frac{3\times24-8\times1}{3} \quad \frac{3\times24-8\times1}{3} \quad \frac{3\times23-5\times1}{3}$

$s^5 : 21.33 \quad 21.33 \quad 21.33$

Divide by 21.33

$s^5 : \quad 1 \quad 1 \quad 1$

$s^4 : \frac{1\times8-1\times3}{1} \quad \frac{1\times8-1\times3}{1} \quad \frac{1\times5-0\times3}{1}$

$s^4 : \quad 5 \quad 5 \quad 5$

Divide by 5

$s^4 : \quad 1 \quad 1 \quad 1$

$s^3 : \frac{1\times1-1\times1}{1} \quad \frac{1\times1-1\times1}{1}$

$s^3 : \quad 0 \quad 0$

Since we get a row of zeros, there exists an even polynomial, the even polynomial is nothing but, the auxiliary polynomial. The auxiliary polynomial is,

$s^4 + s^2 + 1 = 0$

Divide the characteristic equation by auxiliary polynomial to get the quotient polynomial.

The characteristic polynomial can be expressed as a product of quotient polynomial and auxiliary polynomial.