CHAPTER 3

FREQUENCY RESPONSE ANALYSIS

3.1 SINUSOIDAL TRANSFER FUNCTION AND FREQUENCY RESPONSE

The response of a system for the sinusoidal input is called sinusoidal response. The ratio of sinusoidal response and sinusoidal input is called ***sinusoidal transfer function*** of the system and in general, it is denoted by $T(j\omega)$. The sinusoidal transfer function is the frequency domain representation of the system, and so it is also called ***frequency domain transfer function.***

The sinusoidal transfer, function $T(j\omega)$ can be obtained as shown below.

1. Contruct a physical model of a system using basic elements/parameters.
2. Determine the differential equations governing the system from the physical model of the system.
3. Take Laplace transform of differential equations in order to convert them to s-domain equation.
4. Determine s-domain transfer function, T(s), which is ratio of s-domain output and input.
5. Determine the frequency domain transfer function, $T(j\omega)$ by replacing s by $j\omega$ in the s-domain transfer function, T(s).

> *Note : If the s-domain transfer function, T(s) is known, then frequency domain transfer function, $T(j\omega)$ can be obtained directly from T(s) by replacing s by $j\omega$.*
>
> i.e., $T(s) \xrightarrow{s=j\omega} T(j\omega)$

Consider a linear time invariant system with frequency domain transfer function, $T(j\omega)$ shown in fig 3.1. Let the system be excited by a sinusoidal signal frequency ω, amplitude A, and phase θ. Now the response or output will also be a sinusoidal signal of same frequency ω, but the amplitude and phase of response will be modified by amplitude and phase of the transfer function respectively.

Now, the amplitude of the response is given by the product of the amplitude of the input and transfer function. The phase of the response is given by the sum of the phase of the input and transfer function.

Let, $T(j\omega) = |T(j\omega)| \angle T(j\omega)$

where, $|T(j\omega)|$ = Magnitude of $T(j\omega)$, and, $\angle T(j\omega)$ = Phase of $T(j\omega)$.

Let, Input, $r(t) = A \sin(\omega t + \theta) = A \angle\theta$

where, A = Amplitude of input, ω = Frequency of input, and θ = Phase of input.

Now, Response, $c(t) = r(t) \times T(j\omega) = A\angle\theta \times |T(j\omega)| \angle T(j\omega) = A \times |T(j\omega)| \angle(\theta + \angle T(j\omega)) = B\angle\phi$

where, $B = A \times |T(j\omega)|$ = Magnitude of response, and, $\phi = \theta + \angle T(j\omega)$ = Phase of response.

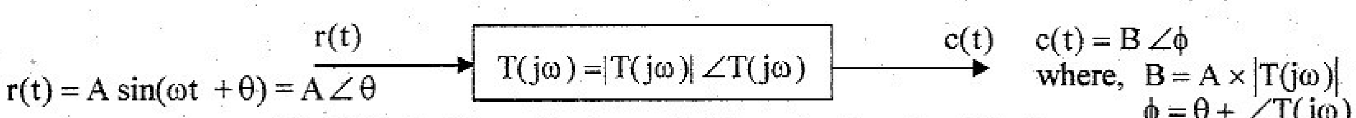

Fig 3.1 : System with sinusoidal transfer function $T(j\omega)$.

FREQUENCY RESPONSE

The frequency domain transfer function T(jω) is a complex function of ω. Hence it can be seperated into magnitude function and phase function. Now, the magnitude and phase functions will be real functions of ω, and they are called ***frequency response.***

The frequency response can be evaluated for open loop system and closed loop system. The frequency domain transfer function of open loop and closed loop systems can be obtained from the s-domain transfer function by replacing s by jω shown below.

Open loop transfer function : $G(s) \xrightarrow{s=j\omega} G(j\omega) = |G(j\omega)| \angle G(j\omega)$ .....(3.1)

Loop transfer function : $G(s)H(s) \xrightarrow{s=j\omega} G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$ .....(3.2)

Closed loop transfer function: $M(s) \xrightarrow{s=j\omega} M(j\omega) = |M(j\omega)| \angle M(j\omega)$ .....(3.3)

where, $|G(j\omega)|, |M(j\omega)|, |G(j\omega)H(j\omega)|$ are Magnitude functions
$\angle G(j\omega),\ \angle M(j\omega),\ \angle G(j\omega)H(j\omega)$ are Phase functions.

Note : For unity feedback system, H(s) = 1 and open loop and loop transfer functions are same.

The advantages of frequency response analysis are the following.

1. The absolute and relative stability of the closed loop system can be estimated from the knowledge of their open loop frequency response.
2. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments .
3. The transfer function of complicated systems can be determined experimentally by frequency response tests.
4. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in frequency domain.
5. When the system is designed by use of the frequency response analysis, the effects of noise disturbance and parameters variations are relatively easy to visualize and incorporate corrective measures.
6. The frequency response analysis and designs can be extended to certain nonlinear control systems.

3.2 FREQUENCY DOMAIN SPECIFICATIONS

The performance and characteristics of a system in frequency domain are measured in terms of frequency domain specifications. The requirements of a system to be designed are usually specified in terms of these specifications.

The frequency domain specifications are,

1. Resonant peak , M_r
2. Resonant Frequency , ω_r
3. Bandwidth, ω_b
4. Cut-off rate
5. Gain margin, K_g
6. Phase margin, γ

Resonant Peak (M_r)

The maximum value of the magnitude of closed loop transfer function is called the resonant peak, M_r. A large resonant peak corresponds to a large overshoot in transient response.

Resonant Frequency (ω_r)

The frequency at which the resonant peak occurs is called resonant frequency, ω_r. This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of transient response.

Bandwidth (ω_b)

The Bandwidth is the range of frequencies for which normalized gain of the system is more than –3 db. The frequency at which the gain is –3 db is called cut-off frequency. Bandwidth is usually defined for closed loop system and it transmits the signals whose frequencies are less than the cut-off frequency. The Bandwidth is a measure of the ability of a feedback system to reproduce the input signal, noise rejection characteristics and rise time. A large bandwidth corresponds to a small rise time or fast response.

Cut-off Rate

The slope of the log-magnitude curve near the cut off frequency is called cut-off rate. The cut -off rate indicates the ability of the system to distinguish the signal from noise.

Gain Margin ,K_g

The gain margin, K_g is defined as the value of gain, to be added to system, in order to bring the system to the verge of instability.

The gain margin, K_g is given by the reciprocal of the magnitude of open loop transfer function at phase cross over frequency. The frequency at which the phase of open loop transfer function is 180° is called the phase cross-over frequency, ω_{pc}.

$$\text{Gain Margin, } K_g = \frac{1}{|G(j\omega_{pc})|} \quad .....(3.4)$$

The gain margin in db can be expressed as,

$$K_g \text{ in db} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|} \quad .....(3.5)$$

Note : $|G(j\omega_{pc})|$ *is the magnitude of* $G(j\omega)$ at $\omega = \omega_{pc}$

The Gain margin in db is given by the negative of the db magnitude of $G(j\omega)$ at phase cross-over frequency. The gain margin indicates the additional gain that can be provided to system without affecting the stability of the system.

Phase Margin (γ)

The phase margin γ, is defined as the additional phase lag to be added at the gain cross over frequency in order to bring the system to the verge of instability. The gain cross over frequency ω_{gc} is the frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the db magnitude is zero).

The phase margin γ, is obtained by adding 180° to the phase angle ϕ of the open loop transfer function at the gain cross over frequency

$$\text{Phase margin, } \gamma = 180° + \phi_{gc}, \quad .....(3.6)$$

$$\text{where, } \phi_{gc} = \angle G(j\omega_{gc})$$

Note : $\angle G(j\omega_{gc})$ *is the phase angle of* $G(j\omega)$ at $\omega = \omega_{gc}$

The phase margin indicates the additional phase lag that can be provided to the system without affecting stability.

3.3 FREQUENCY DOMAIN SPECIFICATIONS OF SECOND ORDER SYSTEM

RESONANT PEAK (M_r)

Consider the closed loop transfer function of second order system,

$$\frac{C(s)}{R(s)} = M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad .....(3.7)$$

The sinusoidal transfer function M(jω) is obtained by letting s = jω.

$$\therefore M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \quad .....(3.8)$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(-\frac{\omega^2}{\omega_n^2} + j2\zeta\frac{\omega}{\omega_n} + 1\right)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\frac{\omega}{\omega_n}}$$

Let, Normalized frequency, $u = \left(\frac{\omega}{\omega_n}\right)$

$$\therefore M(j\omega) = \frac{1}{(1-u^2) + j2\zeta u}$$

Let, M = Magnitude of closed loop transfer function

α = Phase of closed loop transfer function.

$$M = |M(j\omega)| = \left[\frac{1}{(1-u^2)^2 + (2\zeta u)^2}\right]^{\frac{1}{2}} = \left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{1}{2}} \quad .....(3.9)$$

$$\alpha = \angle M(j\omega) = -\tan^{-1}\frac{2\zeta u}{1-u^2} \quad .....(3.10)$$

The resonant peak is the maximum value of M. The condition for maximum value of M can be obtained by differentiating the equation of M with respect to u and letting dM/du = 0 when $u = u_r$,

where, $u_r = \frac{\omega_r}{\omega}$ = Normalized resonant frequency.

On differentiating equation (3.9) with respect to u we get,

$$\frac{dM}{du} = \frac{d}{du}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{1}{2}} = -\frac{1}{2}\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{-\frac{3}{2}}\left[2(1-u^2)(-2u) + 8\zeta^2u\right]$$

$$= \frac{-\left[-4u(1-u^2) + 8\zeta^2u\right]}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}} = \frac{4u(1-u)^2 - 8\zeta^2u}{2\left[(1-u^2)^2 + 4\zeta^2u^2\right]^{\frac{3}{2}}} \quad .....(3.11)$$

Replace u by u_r in equation (3.11) and equate to zero.

$$\frac{4u_r(1-u_r^2) - 8\zeta^2u_r}{2\left[(1-u_r^2)^2 + 4\zeta^2u_r^2\right]^{\frac{3}{2}}} = 0 \quad .....(3.12)$$

The equation (3.12) will be zero if numerator is zero. Hence, on equating numerator to zero we get,

$$4u_r(1-u_r^2)-8\zeta^2u_r=0 \quad \Rightarrow \quad 4u_r-4u_r^3-8\zeta^2u_r=0$$

$$\therefore\ 4u_r^3=4u_r-8\zeta^2u_r \quad \Rightarrow \quad u_r^2=1-2\zeta^2 \quad \Rightarrow \quad u_r=\sqrt{1-2\zeta^2} \qquad \text{......(3.13)}$$

Therefore, the resonant peak occurs when $u_r=\sqrt{1-2\zeta^2}$

Put this condition in the equation for M and solve for M_r.

$$\therefore M_r=\left.\frac{1}{\left[(1-u^2)^2+4\zeta^2u^2\right]^{\frac{1}{2}}}\right|_{u=u_r}=\frac{1}{\left[(1-u_r^2)^2+4\zeta^2u_r^2\right]^{\frac{1}{2}}}=\frac{1}{\left[(1-(1-2\zeta^2))^2+4\zeta^2(1-2\zeta^2)\right]^{\frac{1}{2}}}$$

$$=\frac{1}{\left[4\zeta^4+4\zeta^2-8\zeta^4\right]^{\frac{1}{2}}}=\frac{1}{\left[4\zeta^2-4\zeta^4\right]^{\frac{1}{2}}}=\frac{1}{\left[4\zeta^2(1-\zeta^2)\right]^{\frac{1}{2}}}=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\therefore\ \boxed{\text{Resonant peak, } M_r=\frac{1}{2\zeta\sqrt{1-\zeta^2}}} \qquad \text{......(3.14)}$$

RESONANT FREQUENCY (ω_r)

Normalized resonant frequency, $u_r=\dfrac{\omega_r}{\omega_n}=\sqrt{1-2\zeta^2}$ .....(3.15)

The resonant frequency, $\omega_r=\omega_n\sqrt{1-2\zeta^2}$ .....(3.16)

BANDWIDTH (ω_b)

Let, Normalized bandwidth, $u_b=\dfrac{\omega_b}{\omega_n}$

When $u=u_b$, the magnitude M, of the closed loop system is $1/\sqrt{2}$ (or –3db).

Hence in the equation for M (equation 3.9), put $u=u_b$ and equate to $1/\sqrt{2}$.

$$\therefore\ M=\frac{1}{\left[(1-u_b^2)^2+4\zeta^2u_b^2\right]^{\frac{1}{2}}}=\frac{1}{\sqrt{2}} \qquad \text{.....(3.17)}$$

On squaring and cross multiplying we get,

$$(1-u_b^2)^2+4\zeta^2u_b^2=2 \quad \Rightarrow \quad 1+u_b^4-2u_b^2+4\zeta^2u_b^2=2 \quad \Rightarrow \quad u_b^4-2u_b^2(1-\zeta^2)-1=0$$

Let, $x=u_b^2$; $\quad \therefore x^2-2(1-2\zeta^2)x-1=0$

$$\therefore x=\frac{2(1-2\zeta^2)\pm\sqrt{4(1-2\zeta^2)^2+4}}{2}=\frac{2(1-2\zeta^2)\pm2\sqrt{(1+4\zeta^4-4\zeta^2)+1}}{2}$$

Let us take only the positive sign,

$$\therefore x=1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}$$

But, $u_b=\sqrt{x}$; $\quad \therefore u_b=\sqrt{x}=\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{\frac{1}{2}}$; $\quad$ Also, $u_b=\dfrac{\omega_b}{\omega_n}$

$$\therefore \boxed{\text{Bandwidth, } \omega_b = \omega_n u_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{\frac{1}{2}}} \quad .....(3.18)$$

PHASE MARGIN (γ)

The open loop transfer function of second order system,

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \quad .....(3.19)$$

The sinusoidal transfer function G(jω) is obtained by letting s = jω.

$$G(j\omega) = \frac{\omega_n^2}{j\omega\,(j\omega + 2\zeta\omega_n)} = \frac{\omega_n^2}{\omega_n\left(\frac{j\omega}{\omega_n}\right)\omega_n\left(2\zeta + j\frac{\omega}{\omega_n}\right)} = \frac{1}{j\frac{\omega}{\omega_n}\left(2\zeta + j\frac{\omega}{\omega_n}\right)} \quad .....(3.20)$$

Let normalized frequency, $u = \omega/\omega_n$

On substituting $u = \omega/\omega_n$ in equation (3.20) we get,

$$G(j\omega) = \frac{1}{ju(2\zeta + ju)} \quad .....(3.21)$$

$$\text{Magnitude of } G(j\omega) = |G(j\omega)| = \frac{1}{u\sqrt{4\zeta^2 + u^2}} = \frac{1}{\sqrt{u^4 + 4\zeta^2 u^2}} \quad .....(3.22)$$

$$\text{Phase of } G(j\omega) = -90^\circ - \tan^{-1}\frac{u}{2\zeta} \quad .....(3.23)$$

At the gain cross-over frequency ω_{gc}, the magnitude of G(jω) is unity.

Let normalized gain cross over frequency, $u_{gc} = \omega_{gc}/\omega_n$

On substituting u by u_{gc} in the equation (3.22) and equating to unity, we get,

$$\therefore \text{At } u = u_{gc},\ |G(j\omega)| = \frac{1}{\sqrt{u_{gc}^4 + 4\zeta^2 u_{gc}^2}} = 1 \quad \Rightarrow \quad u_{gc}^4 + 4\zeta^2 u_{gc}^2 = 1 \quad \Rightarrow \quad u_{gc}^4 + 4\zeta^2 u_{gc}^2 - 1 = 0$$

Let, $\mathbf{x = u_{gc}^2}$; $\therefore x^2 + 4\zeta^2 x - 1 = 0$

$$\therefore x = \frac{-4\zeta^2 \pm \sqrt{16\zeta^4 + 4}}{2} = -2\zeta^2 \pm \sqrt{4\zeta^4 + 1}$$

Let us take only the positive sign,

$$\therefore x = -2\zeta^2 + \sqrt{4\zeta^4 + 1}$$

$$\text{But, } u_{gc} = \sqrt{x}\ ; \quad \therefore u_{gc} = \sqrt{x} = \left[-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right]^{\frac{1}{2}} \quad .....(3.24)$$

$$\text{The phase margin, } \gamma = 180 + \angle G(j\omega)\big|_{\omega = \omega_{gc},\ u = u_{gc}} \quad .....(3.25)$$

Substituting for ∠G(jω) from equation (3.23) in equation (3.25) we get,

$$\gamma = 180 + \left(-90^\circ - \tan^{-1}\frac{u_{gc}}{2\zeta}\right) = 90 - \tan^{-1}\left[\frac{\left[-2\zeta^2 + \sqrt{4\zeta^4 + 1}\right]^{\frac{1}{2}}}{2\zeta}\right] \quad .....(3.26)$$

***Note :** The gain margin of second order system is infinite.*

3.4 CORRELATION BETWEEN TIME AND FREQUENCY RESPONSE

The correlation between time and frequency response has an explicit form only for first and second order systems. The correlation for second-order system is discussed here.

Consider the magnitude and phase of a closed loop second order system as a function of normalized frequency, as given by equations (3.9) and (3.10).

Magnitude of closed loop system, $M = |M(j\omega)| = \dfrac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$

Phase of closed loop system, $\alpha = \angle M(j\omega) = -\tan^{-1}\dfrac{2\zeta u}{1-u^2}$

The magnitude and phase angle characteristics for normalized frequency u, for certain values of ζ are shown in fig 3.2 and 3.3. The frequency at which M has a peak value is known as the resonant frequency. The peak value of the magnitude is the resonant peak M_r. At this frequency the slope of the magnitude curve is zero. The frequency corresponding to M_r is u_r, which is the normalized resonant frequency.

From equations (3.14) and (3.15) we get,

Resonant peak, $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

Resonant frequency, $\omega_r = \omega_n\sqrt{1-2\zeta^2}$

When $\zeta = 0$, $\quad \omega_r = \omega_n\sqrt{1-2\zeta^2} = \omega_n$ .....(3.27)

When $\zeta = 0$, $\quad M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} = \infty$ .....(3.28)

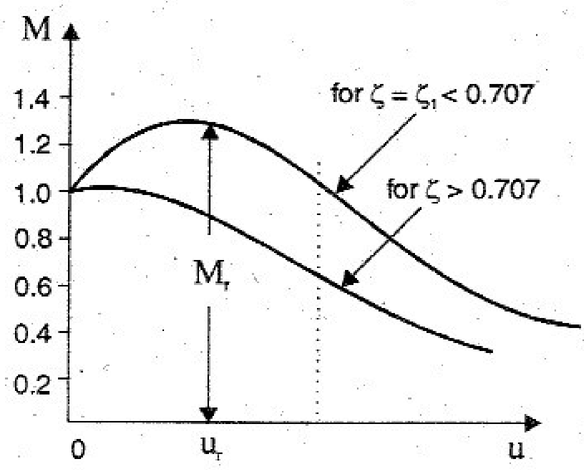

Fig 3.2 : Magnitude, M as a function of u.

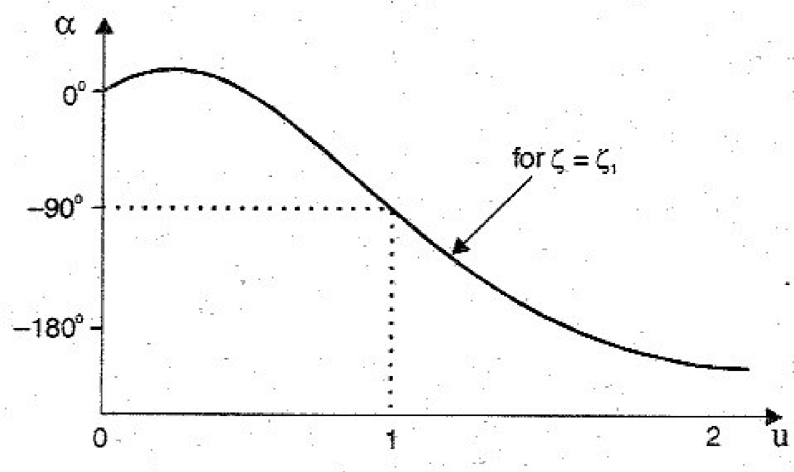

Fig 3.3 : Phase, α *as a function of u.*

From equations (3.27) and (3.28), it is clear that as ζ tends to zero, ω_r approaches ω_n, and M_r approaches infinity.

When $1-2\zeta^2 = 0$, $\omega_r = 0$, which means there is no resonant peak at this condition.

Let, $1-2\zeta^2 = 0$; $\therefore \zeta^2 = \dfrac{1}{2} \quad \Rightarrow \quad \zeta = \dfrac{1}{\sqrt{2}}$

For $0 < \zeta \le 1/\sqrt{2}$, the resonant frequency always has a value less than ω_n, and the resonant peak has a value greater than one.

For $\zeta > 1/\sqrt{2}$, the condition (dM/du) = 0, will not be satisfied for any real value of ω.

Hence when $\zeta > 1/\sqrt{2}$ the magnitude M decreases monotonically from M = 1 at u = 0 with increasing u. It follows that for $\zeta > 1/\sqrt{2}$ there is no resonant peak and the greatest value of M equals one.

The frequency at which M has a value of $1/\sqrt{2}$ is of special significance and is called the cut-off frequency ω_c. The signal frequencies above cut-off are greatly attenuated on passing through a system.

For feedback control system, the range of frequencies over which $M \ge 1/\sqrt{2}$ is defined as bandwidth ω_b. Control system being low-pass filters (at zero frequency M = 1), the bandwidth ω_b is equal to cut-off frequency ω_c.

In general the bandwidth of a control system indicates the noise-filtering characteristics of the system. Also, bandwidth gives a measure of the transient response.

$$\text{The normalized bandwidth, } u_b = \frac{\omega_b}{\omega_n} = \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + \zeta^4}\right]^{\frac{1}{2}}$$

From the equation of u_b it is clear that u_b is a function of ζ alone. The graph between u_b and ζ is shown in fig 3.4.

The expression for the damped frequency of oscillation ω_d and peak overshoot M_p of the step response, for $0 \le \zeta \le 1$ are,

$$\text{Damped frequency, } \omega_d = \omega_n\sqrt{1-\zeta^2} \quad \text{and} \quad \text{Peak overshoot, } M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Comparison of the equation of M_r and M_p reveals that both are functions of only ζ.

The sketch of M_r and M_p for various value of ζ are shown in fig 3.5. The sketches reveals that a system with a given value of M_r must exhibit a corresponding value of M_p if subjected to a step input. For $\zeta > 1/\sqrt{2}$, the resonant peak M_r does not exist and the correlation breaks down. This is not a serious problem as for this range of ζ, the step response oscillations are well damped and M_p is negligible.

The comparison of the equation of ω_r and ω_d reveals that there exists a definite correlation between them. The sketch of ω_r/ω_d with respect to ζ is shown in fig 3.6.

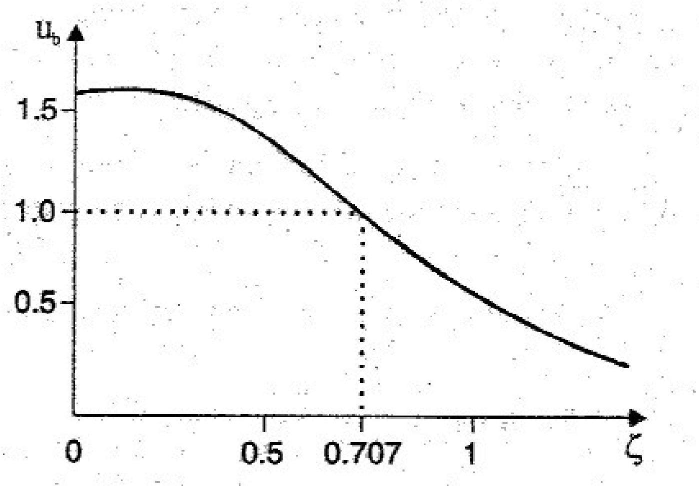

Fig 3.4 : Normalised bandwidth as a function of ζ.

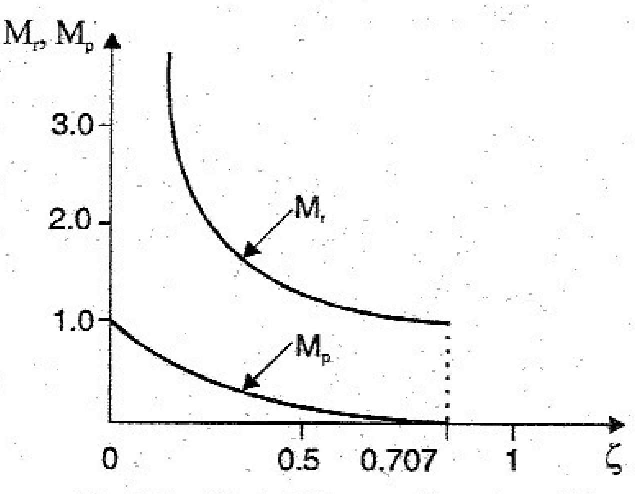

Fig 3.5 : M_r and M_p as a function of ζ.

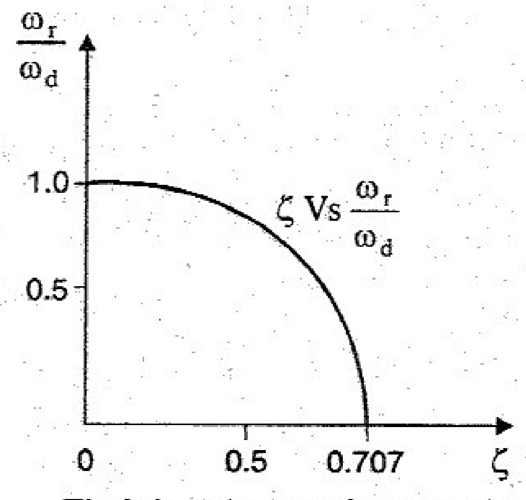

Fig 3.6 : ω_r/ω_d as a function of ζ.

3.5 FREQUENCY RESPONSE PLOTS

Frequency response analysis of control systems can be carried either analytically or graphically. The various graphical techniques available for frequency response analysis are,

1. Bode plot
2. Polar plot (or Nyquist plot)
3. Nichols plot
4. M and N circles
5. Nichols chart

The Bode plot, Polar plot and Nichols plot are usually drawn for open loop systems. From the open loop response plot, the performance and stability of closed loop system are estimated. The M and N circles and Nichols chart are used to graphically determine the frequency response of unity feedback closed loop system from the knowledge of open loop response.

The frequency response plots are used to determine the frequency domain specifications, to study the stability of the systems and to adjust the gain of the system to satisfy the desired specifications.

3.6 BODE PLOT

The Bode plot is a frequency response plot of the sinusoidal transfer function of a system. A Bode plot consists of two graphs. One is a plot of the magnitude of a sinusoidal transfer function versus log ω. The other is a plot of the phase angle of a sinusoidal transfer function versus log ω.

The Bode plot can be drawn for both open loop and closed loop system. Usually the bode plot is drawn for open loop system. The standard representation of the logarithmic magnitude of open loop transfer function of G(jω) is 20 log |G(jω)| where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel,usually abbreviated as db. The curves are drawn on semilog paper, using the log scale (abcissa) for frequency and the linear scale (ordinate) for either magnitude (in decibels) or phase angle (in degrees).

The main advantage of the bode plot is that multiplication of magnitudes can be converted into addition. Also a simple method for sketching an approximate log-magnitude curve is available.

Consider the open loop transfer function, $G(s) = \frac{K\,(1+sT_1)}{s\,(1+sT_2)\,(1+sT}$

$$G(j\omega) = \frac{K\,(1+j\omega T_1)}{j\omega\,(1+j\omega T_2)\,(1+j\omega T_3)}$$

$$= \frac{K\angle 0^\circ\ \sqrt{1+\omega^2 T_1^2}\ \angle\tan^{-1}\omega T_1}{\omega\angle 90^\circ\ \sqrt{1+\omega^2 T_2^2}\ \angle\tan^{-1}\omega T_2 \sqrt{1+\omega^2 T_3^2}\ \angle\tan^{-1}\omega T_3}$$

The magnitude of $G(j\omega) = |G(j\omega)| = \frac{K\sqrt{1+\omega^2 T_1^2}}{\omega\sqrt{1+\omega^2 T_2^2}\ \sqrt{1+\omega^2 T_3^2}}$

The phase angle of the $G(j\omega) = \angle G(j\omega) = \tan^{-1}\omega T_1 - 90^\circ - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$

The magnitude of G(jω) can be expressed in decibels as shown below.

$$|G(j\omega)| \text{ in db} = 20\log|G(j\omega)|$$

$$= 20\log\left[\frac{K\sqrt{1+\omega^2 T_1^2}}{\omega\sqrt{1+\omega^2 T_2^2}\ \sqrt{1+\omega^2 T_3^2}}\right.$$

$$= 20 \log \left[\frac{K}{\omega} \times \sqrt{1+\omega^2 T_1^2} \times \frac{1}{\sqrt{1+\omega^2 T_2^2}} \times \frac{1}{\sqrt{1+\omega^2 T_3^2}} \right]$$

$$= 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_2^2}} + 20 \log \frac{1}{\sqrt{1+\omega^2 T_3^2}}$$

$$= 20 \log \frac{K}{\omega} + 20 \log \sqrt{1+\omega^2 T_1^2} - 20 \log \sqrt{1+\omega^2 T_2^2} - 20 \log \sqrt{1+\omega^2 T_3^2} \quad .....(3.29)$$

From the equation (3.29) it is clear that, when the magnitude is expressed in db, the multiplication is converted to addition. Hence in magnitude plot, the db magnitudes of individual factors of $G(j\omega)$ can be added.

Therefore to sketch the magnitude plot, a knowledge of the magnitude variations of individual factor is essential. The magnitude plot and phase plot of various factors of $G(j\omega)$ are explained in the following section.

BASIC FACTORS OF G(jω)

The basic factors that very frequently occur in a typical transfer function $G(j\omega)$ are,

1. Constant gain, K
2. Integral factor, $\frac{K}{j\omega}$ or $\frac{K}{(j\omega)^n}$
3. Derivative factor, $K \times j\omega$ or $K \times (j\omega)^n$
4. First order factor in denominator, $\frac{1}{1+j\omega T}$ or $\frac{1}{(1+j\omega T)^m}$
5. First order factor in numerator, $(1+j\omega T)$ or $(1+j\omega T)^m$
6. Quadratic factor in denominator, $\left[\frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}\right]$
7. Quadratic factor in numerator, $\left[1+2\zeta\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]$

Constant Gain, K

Let, $G(s) = K$

$\therefore G(j\omega) = K = K\angle 0°$

$A = |G(j\omega)|$ in db $= 20 \log K$

$\phi = \angle G(j\omega) = 0°$

The magnitude plot for a constant gain K is a horizontal straight line at the magnitude of 20 log K db. The phase plot is straight line at 0°.

When $K > 1$, 20 log K is positive.

When $0 < K < 1$, 20 log K is negative.

When $K = 1$, 20 log K is zero.

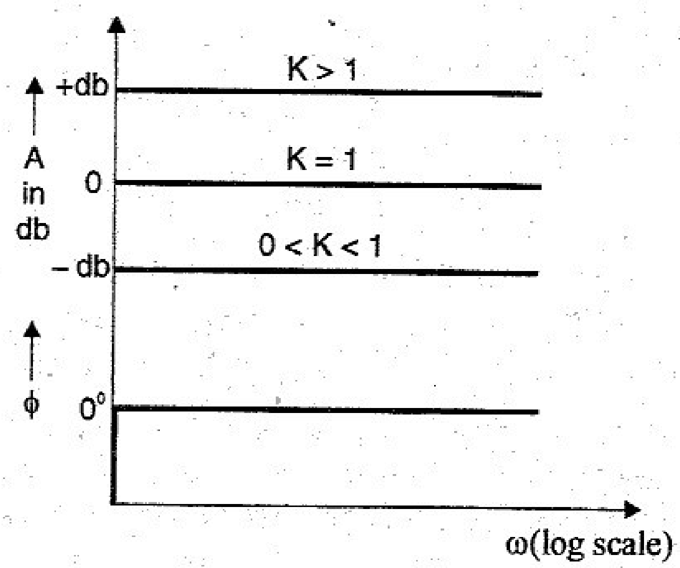

Fig 3.7 : Bode plot of constant gain, K.

Integral Factor

Let, $G(s) = \frac{K}{s}$

$$\therefore G(j\omega) = \frac{K}{j\omega} = \frac{K}{\omega} \angle -90^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (K/\omega)$$

$$\phi = \angle G(j\omega) = -90^\circ$$

When $\omega = 0.1\,K$, $A = 20 \log (1/0.1) = 20$ db

When $\omega = K$, $A = 20 \log 1 = 0$ db

When $\omega = 10\,K$, $A = 20 \log (1/10) = -20$ db

From the above analysis it is evident that the magnitude plot of the integral factor is a straight line with a slope of –20 db/dec and passing through zero db, when $\omega = K$. Since the $\angle G(j\omega)$ is a constant and independent of ω the phase plot is a straight line at –90°.

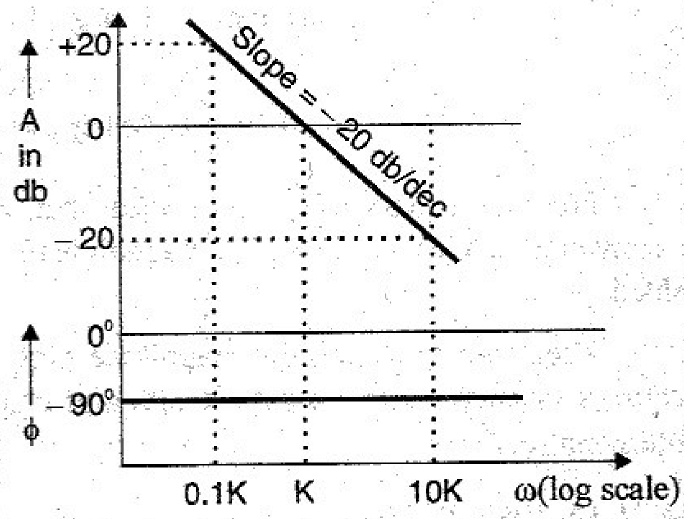

Fig 3.8 : Bode plot of integral factor, $\frac{K}{j\omega}$.

When an integral factor has multiplicity of n, then,

$$G(s) = K/s^n$$

$$G(j\omega) = K/(j\omega)^n = K/\omega^n \angle -90n^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log \frac{K}{\omega^n}$$

$$= 20 \log \left(\frac{K^{\frac{1}{n}}}{\omega}\right)^n = 20\,n \log \left(\frac{K^{\frac{1}{n}}}{\omega}\right)$$

$$\phi = \angle G(j\omega) = -90\,n^\circ$$

Now the magnitude plot of the integral factor is a straight line with a slope of –20n db/dec and passing through zero db when $\omega = K^{1/n}$. The phase plot is a straight line at –90n°.

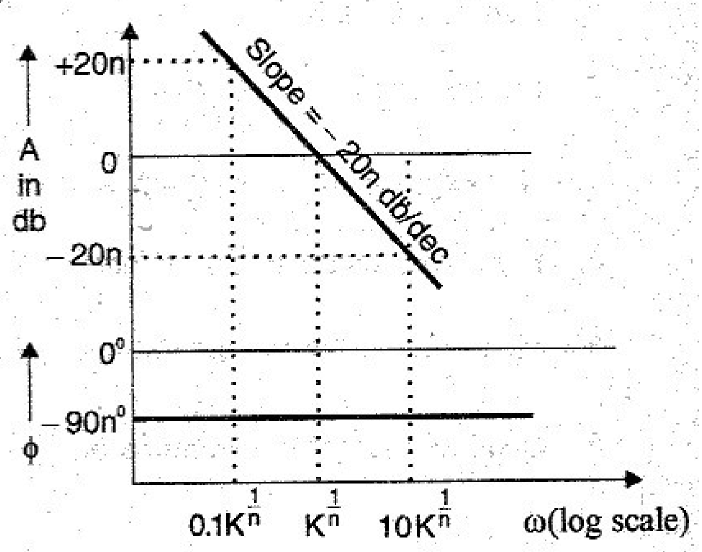

Fig 3.9 : Bode plot of integral factor, $K/(j\omega)^n$.

Derivative Factor

Let, $G(s) = Ks$

$$\therefore G(j\omega) = K\,j\omega = K\omega \angle 90^\circ$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega)$$

$$\phi = \angle G(j\omega) = +90^\circ$$

When $\omega = 0.1/K$, $A = 20 \log (0.1) = -20$ db

When $\omega = 1/K$, $A = 20 \log 1 = 0$ db

When $\omega = 10/K$, $A = 20 \log 10 = +20$ db

From the above analysis it is evident that the magnitude plot of the derivative factor is a straight line with a slope of +20 db/dec and passing through zero db when $\omega = 1/K$. Since the $\angle G(j\omega)$ is a constant and independent of ω, the phase plot is a straight line at +90°.

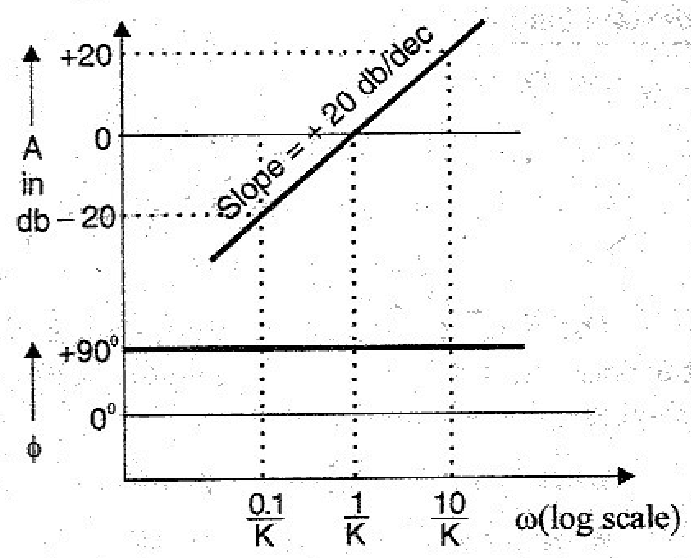

Fig 3.10 : Bode plot of derivative factor, $K \times j\omega$.

When derivative factor has multiplicity of n then,

$$G(s) = K s^n$$

$$\therefore G(j\omega) = K(j\omega)^n = K\omega^n \angle 90n°$$

$$A = |G(j\omega)| \text{ in db} = 20 \log (K\omega^n)$$

$$= 20 \log (K^{1/n}\omega)^n = 20\, n \log (K^{1/n}\omega)$$

$$\phi = \angle G(j\omega) = 90n°$$

Now the magnitude plot of the derivative factor is a straight line with a slope of +20n db/dec and passing through zero db when $\omega = 1/K^{1/n}$. The phase plot is a straight line at +90n°.

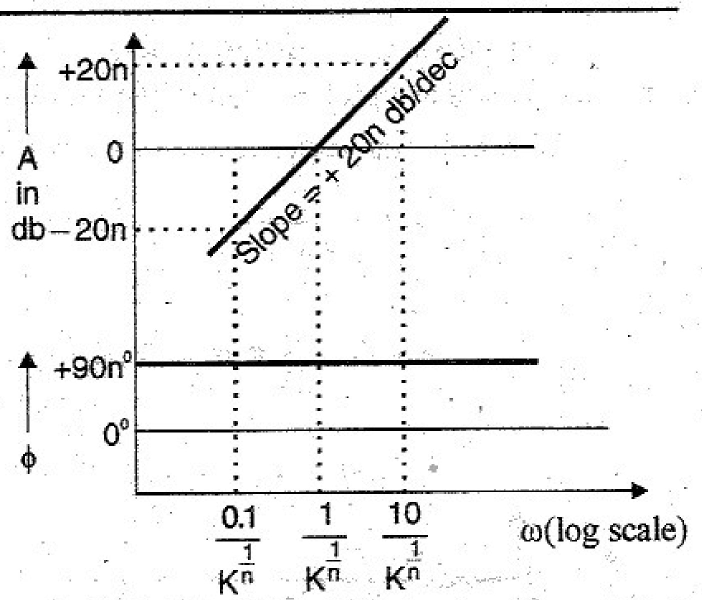

Fig 3.11 : Bode plot of derivative factor, $K(j\omega)^n$.

First order factor in denominator

$$G(s) = \frac{1}{1+sT}$$

$$\therefore G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$

Let, $A = |G(j\omega)|$ in db.

$$\therefore A = |G(j\omega)|_{\text{in db}} = 20 \log \frac{1}{\sqrt{1+\omega^2T^2}} = -20 \log \sqrt{1+\omega^2T^2}$$

At very low frequencies, $\omega T << 1$; $\therefore A = -20 \log \sqrt{1+\omega^2T^2} \approx -20 \log 1 = 0$

At very high frequencies, $\omega T >> 1$; $\therefore A = -20 \log \sqrt{1+\omega^2T^2} \approx -20 \log \sqrt{\omega^2T^2} = -20 \log \omega T$

At $\omega = \frac{1}{T}$, $A = -20 \log 1 = 0$

At $\omega = \frac{10}{T}$, $A = -20 \log 10 = -20$ db

The above analysis shows that the magnitude plot of the factor 1/(1+jωT) can be approximated by two straight lines, one is a straight line at 0 db for the frequency range, **$0 < \omega < 1/T$**, and the other is a straight line with slope –20 db/dec for the frequency range, **$1/T < \omega < \infty$**. The two straight lines are asymptotes of the exact curve.

The frequency at which the two asymptotes meet is called ***corner frequency*** or ***break frequency***. For the factor 1/(1+jωT) the frequency, ω = 1/T is the corner frequency, ω_c. It divides the frequency response curve into two regions, a curve for low frequency region and a curve for high frequency region.

The actual magnitude at the corner frequency, $\omega_c = \frac{1}{T}$ is,

$$A = -20 \log \sqrt{1+1} = -3 \text{ db.}$$

Hence by this approximation the loss in db at the corner frequency is –3 db.

The phase plot is obtained by calculating the phase angle of G(jω) for various values of ω

Phase angle, $\phi = \angle G(j\omega) = -\tan^{-1} \omega T$

At the corner frequency, $\omega = \omega_c = \frac{1}{T}$, $\phi = -\tan^{-1}\omega T = -\tan^{-1} 1 = -45°$

As $\omega \to 0$, $\phi \to 0°$

As $\omega \to \infty$, $\phi \to -90°$

The phase angle of the factor, $1/(1+j\omega T)$, varies from 0° to −90° as ω is varied from zero to infinity. The phase plot is a curve passing through −45° at ω_c.

When the first order factor in the denominator has a multiplicity of m, then,

$$G(s)=\frac{1}{(1+sT)^m}\ ; \qquad \therefore G(j\omega)=\frac{1}{(1+j\omega T)^m}=\frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T}$$

$$A=|G(j\omega)| \text{ in db}=20\log\frac{1}{\left(\sqrt{1+\omega^2T^2}\right)^m}=-20\,m\log\sqrt{1+\omega^2T^2}$$

$$\phi=\angle G(j\omega)=-m\tan^{-1}\omega T$$

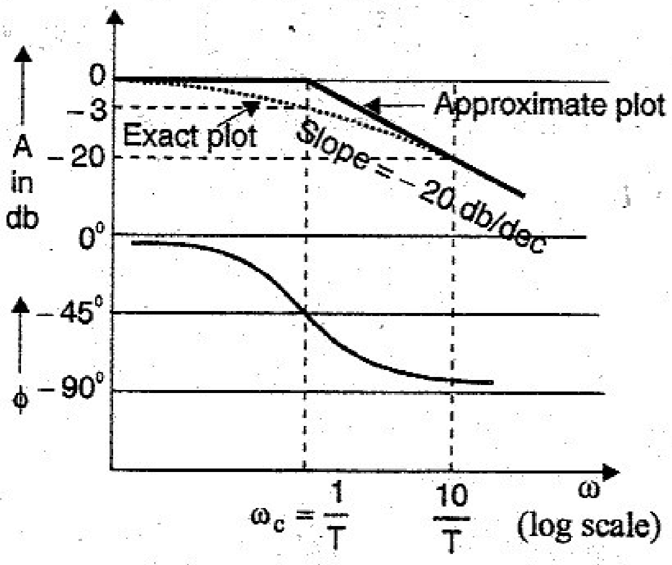

Fig 3.12 : Bode plot of the factor $\frac{1}{1+j\omega T}$.

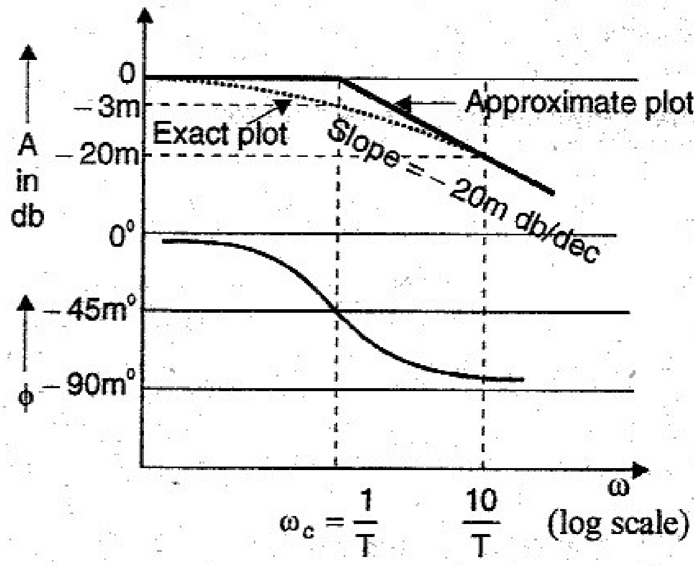

Fig 3.13 : Bode plot of the factor $1/(1+j\omega T)^m$.

Now the magnitude plot of the factor $1/(1+j\omega T)^m$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range, $0<\omega<1/T$, and the other is a straight line with slope −20 m db/dec for the frequency range, $1/T<\omega<\infty$. The corner frequency, $\omega_c = 1/T$ and the loss in db at the corner frequency is −3m db.

The phase angle of the factor $1/(1+j\omega T)^m$ varies from 0° to −90m° as ω is varied from zero to infinity. The phase plot is a curve passing through −45m° at ω_c.

FIRST ORDER FACTOR IN THE NUMERATOR

$$G(s)=1+sT$$

$$G(j\omega)=1+j\omega T=\sqrt{1+\omega^2T^2}\ \angle\tan^{-1}\omega T$$

$$A=|G(j\omega)| \text{ in db}=20\log\sqrt{1+\omega^2T^2}$$

$$\phi=\angle G(j\omega)=\tan^{-1}\omega T$$

By an analysis similar to that of previous section it can be shown that the magnitude plot of the factor $(1+j\omega T)$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range $0<\omega<1/T$ and the other is a straight line with slope +20 db/dec for the frequency range $1/T<\omega<\infty$. The two straight lines are asymptotes of the exact curve.

The frequency at which the two asymptotes meet is called the corner frequency or break frequency. For the factor $(1+j\omega T)$, the frequency, $(\omega = 1/T)$ is the corner frequency, ω_c. By this approximation the loss in db at the corner frequency is +3 db. The phase angle of the factor $(1+j\omega T)$ varies from zero to +90° as ω is varied from 0 to ∞. The phase plot is a curve passing through +45° at ω_c.

When the first order factor in the numerator has a multiplicity of m, then,

$$G(s) = (1+sT)^m$$

$$G(j\omega) = (1+j\omega T)^m = \left(\sqrt{1+\omega^2T^2}\right)^m \angle m\tan^{-1}\omega T$$

$$A = |G(j\omega)| \text{ in db} = 20\log\left(\sqrt{1+\omega^2T^2}\right)^m = 20m\log\sqrt{1+\omega^2T^2}$$

$$\phi = \angle G(j\omega) = m\tan^{-1}\omega T$$

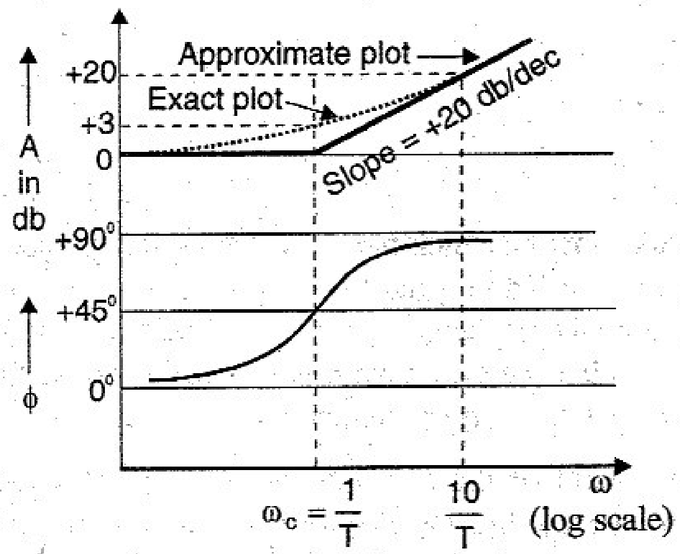

Fig 3.14 : Bode plot of the factor (1+jωT).

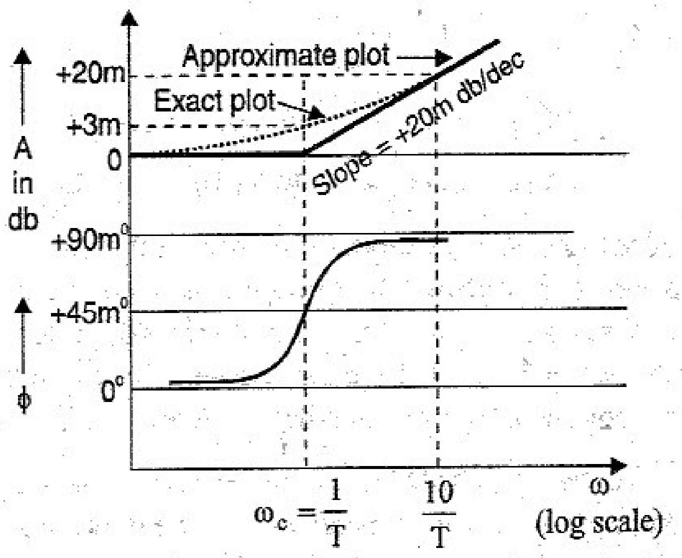

Fig 3.15 : Bode plot of the factor $(1+j\omega T)^m$.

Now the magnitude plot of the factor $(1+j\omega T)^m$ can be approximated by two straight lines, one is a straight line at zero db for the frequency range $0 < \omega < 1/T$ and the other is a straight line with a slope of +20m db/dec for the frequency range $1/T < \omega < \infty$. The corner frequency, $\omega_c = 1/T$ and the loss in db at this corner frequency is +3m db.

The phase angle of the factor $(1+j\omega T)^m$ varies from zero to +90m° as ω is varied from zero to infinity. The phase plot is a curve passing through +45 m° at ω_c.

QUADRATIC FACTOR IN THE DENOMINATOR

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{1+2\zeta\frac{s}{\omega_n}+\left(\frac{s}{\omega_n}\right)^2}$$

$$\therefore G(j\omega) = \frac{1}{1+j\frac{2\zeta\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2} = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+j2\zeta\frac{\omega}{\omega_n}} = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}}\angle -\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}$$

Let, $A = |G(j\omega)|$ in db.

$$A = 20\log\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}} = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega^2}{\omega_n^2}}$$

$$= -20\log\sqrt{1+\frac{\omega^4}{\omega_n^4}-2\frac{\omega^2}{\omega_n^2}+4\zeta^2\frac{\omega^2}{\omega_n^2}} = -20\log\sqrt{1-\frac{\omega^2}{\omega_n^2}(2-4\zeta^2)+\frac{\omega^4}{\omega_n^4}}$$

At very low frequencies when $\omega << \omega_n$, the magnitude is,

$$A = -20 \log \sqrt{1 - \frac{\omega^2}{\omega_n^2}(2-4\zeta^2) + \frac{\omega^4}{\omega_n^4}} \approx -20 \log 1 = 0$$

At very high frequencies when $\omega >> \omega_n$, the magnitude is,

$$A = -20 \log \sqrt{1 - \frac{\omega^2}{\omega_n^2}(2-4\zeta^2) + \frac{\omega^4}{\omega_n^4}} \approx -20 \log \sqrt{\frac{\omega^4}{\omega_n^4}} = -20 \log \frac{\omega^2}{\omega_n^2} = -20 \log \left(\frac{\omega}{\omega_n}\right)^2$$

$$\therefore A = -40 \log \frac{\omega}{\omega_n}$$

At $\omega = \omega_n$, $A = -40 \log 1 = 0$ db

At $\omega = 10\omega_n$, $A = -40 \log 10 = -40$ db

From the above analysis it is evident that the magnitude plot of the quadratic factor in the denominator can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega_n$ and the other is a straight line with slope –40 db/dec for the frequency range $\omega_n < \omega < \infty$. The two straight lines are asymptotes of the exact curve. The frequency at which the two asymptotes meet is called the corner frequency. For the quadratic factor, the frequency ω_n is the corner frequency, ω_c.

The two asymptotes of the exact curve are independent of the damping ratio, ζ. In the exact magnitude plot, resonant peak occurs near the corner frequency and the magnitude of resonant peak depends on ζ. Lower the value of ζ, larger will be the resonant peak. Hence by this approximation the error at the corner frequency depends on damping ratio ζ. The phase plot is obtained by calculating the phase angle of $G(j\omega)$ for various values of ω.

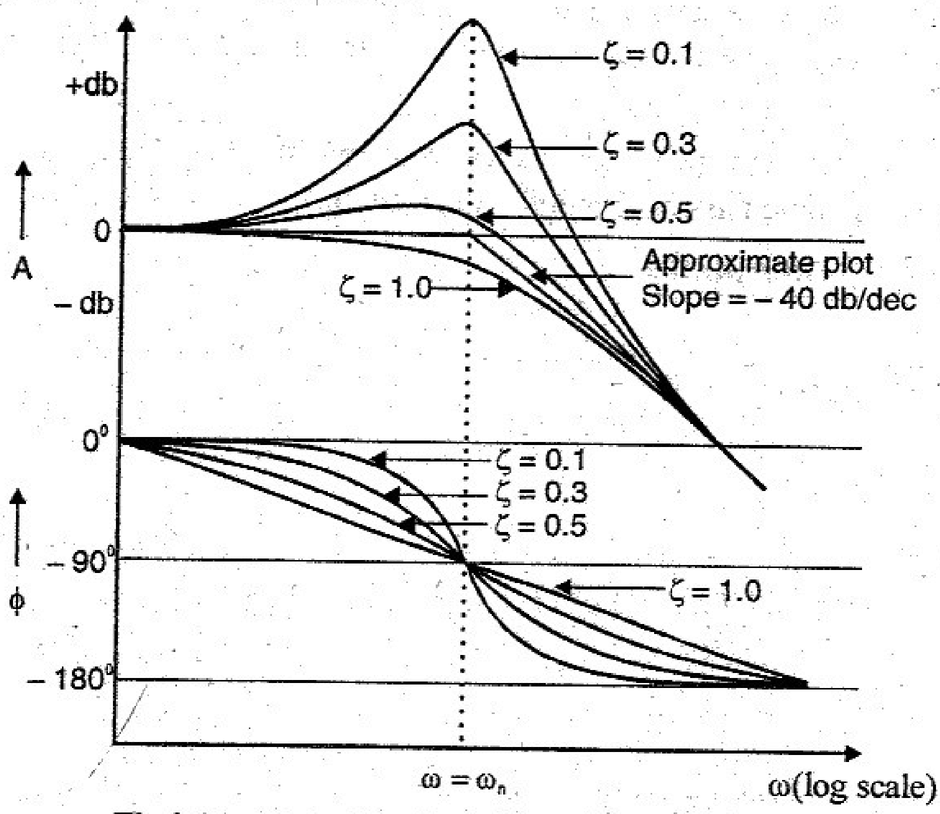

Fig 3.16 : Bode plot of quadratic factor in denominator.

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{\omega_n^2}}\right)$$

As $\omega = \omega_n$, $\phi = -\tan^{-1}\frac{2\zeta}{0} = -\tan^{-1}\infty = -90^\circ$

As $\omega \to 0$, $\phi \to 0$

As $\omega \to \infty$, $\phi \to -180^\circ$

The phase angle of the quadratic factor varies from 0 to -180^0 as ω is varied from 0 to ∞. The phase plot is a curve passing through -90^0 at ω_c. At the corner frequency phase angle is -90^0 and independent of ζ, but at all other frequency it depends on ζ.

QUADRATIC FACTOR IN THE NUMERATOR

$$G(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2} = 1 + 2\zeta\left(\frac{s}{\omega_n}\right) + \left(\frac{s}{\omega_n}\right)^2$$

$$G(j\omega) = 1 + j2\zeta\frac{\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2 = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2\frac{\omega^2}{\omega_n^2}}\ \angle\tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

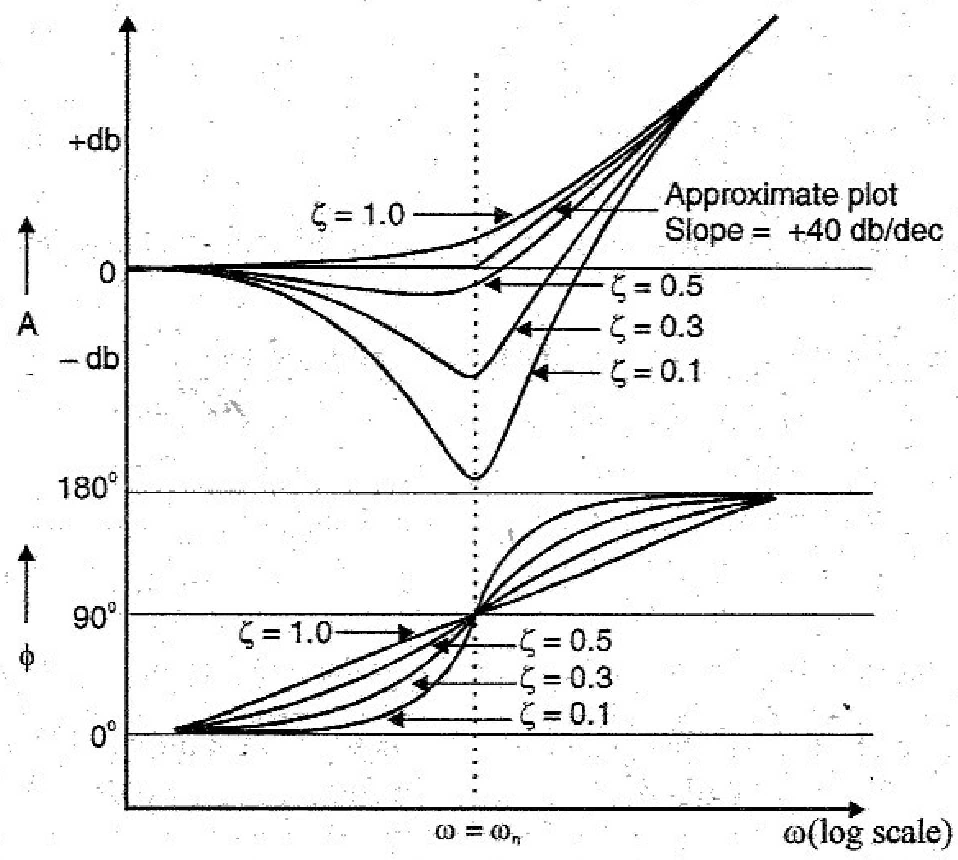

Fig 3.17 : Bode plot of quadratic factor in numerator.

Based on an analysis similar to that of denominator quadratic factor, the magnitude plot of the quadratic factor in the numerator can be approximated by two straight lines, one is a straight line at 0 db for the frequency range $0 < \omega < \omega_n$ and the other is a straight line with slope +40 db/dec for the frequency range $\omega_n < \omega < \infty$. The corner frequency is ω_n. Due to this approximation the error at the corner frequency depends on ζ.

The phase angle varies from 0 to $+180^\circ$, as ω is varied from 0 to ∞. At the corner frequency the phase angle is $+90^\circ$ and independent of ζ, but at all other frequency it depends on ζ.