CHAPTER 2

TIME RESPONSE ANALYSIS

2.1 TIME RESPONSE

The time response of the system is the output of the closed loop system as a function of time. It is denoted by c(t). The time response can be obtained by solving the differential equation governing the system. Alternatively, the response c(t) can be obtained from the transfer function of the system and the input to the system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = M(s)$ .....(2.1)

The Output or Response in s-domain, C(s) is given by the product of the transfer function and the input, R(s). On taking inverse Laplace transform of this product the time domain response, c(t) can be obtained.

Response in s-domain, $C(s) = R(s)\,M(s)$ .....(2.2)

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\{R(s) \times M(s)\}$ .....(2.3)

where, $M(s) = \frac{G(s)}{1+G(s)H(s)}$

The time response of a control system consists of two parts : *the transient and the steady state response.* The transient response is the response of the system when the input changes from one state to another. The steady state response is the response as time, **t** approaches infinity.

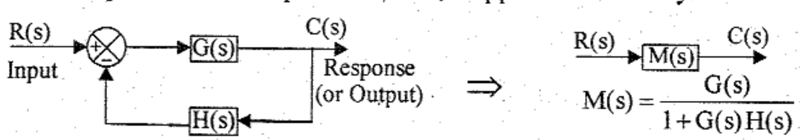


Fig 2.1 : Closed loop system.

2.2 TEST SIGNALS

The knowledge of input signal is required to predict the response of a system. In most of the systems the input signals are not known ahead of time and also it is difficult to express the input signals mathematically by simple equations. The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity and a constant acceleration. Hence test signals which resembles these characteristics are used as input signals to predict the performance of the system. The commonly used test input signals are impulse, step, ramp, acceleration and sinusoidal signals.

The standard test signals are,

1. a) Step signal
   b) Unit step signal
2. a) Ramp signal
   b) Unit ramp signal
3. a) Parabolic signal
   b) Unit parabolic signal
4. Impulse signal
5. Sinusoidal signal.

Since the test signals are simple functions for time, they can be easily generated in laboratories. The mathematical and experimental analysis of control systems using these signals can be carried out easily. The use of the test signals can be justified because of a correlation existing between the response characteristics of a system to a test input signal and capability of the system to cope with actual input signals.

STEP SIGNAL

The step signal is a signal whose value changes from zero to A at t = 0 and remains constant at A for t > 0. The step signal resembles an actual steady input to a system. A special case of step signal is unit step in which **A** is unity.

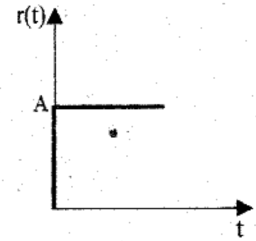


Fig 2.2 : Step signal.

The mathematical representation of the step signal is,

$$r(t) = 1 \;;\; t \geq 0$$
$$= 0 \;;\; t < 0 \qquad .....(2.4)$$

RAMP SIGNAL

The ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. The ramp signal resembles a constant velocity input to the system. A special case of ramp signal is unit ramp signal in which the value of **A** is unity.

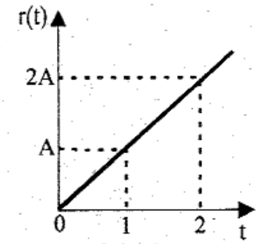


Fig 2.3 : Ramp signal.

The mathematical representation of the ramp signal is,

$$r(t) = A\,t \;;\; t \geq 0$$
$$= 0 \;;\; t < 0 \qquad .....(2.5)$$

PARABOLIC SIGNAL

In parabolic signal, the instantaneous value varies as square of the time from an initial value of zero at t = 0. The sketch of the signal with respect to time resembles a parabola. The parabolic signal resembles a constant acceleration input to the system. A special case of parabolic signal is unit parabolic signal in which A is unity.

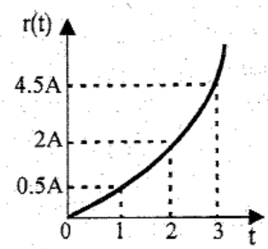


Fig 2.4 : Parabolic signal.

The mathematical representation of the parabolic signal is,

$$r(t) = \frac{At^2}{2} \;;\; t \geq 0$$
$$= 0 \;;\; t < 0 \qquad .....(2.6)$$

***Note :** Integral of step signal is ramp signal. Integral of ramp signal is parabolic signal.*

IMPULSE SIGNAL

A signal of very large magnitude which is available for very short duration is called ***impulse signal.*** Ideal impulse signal is a signal with infinite magnitude and zero duration but with an area of A. The unit impulse signal is a special case, in which A is unity.

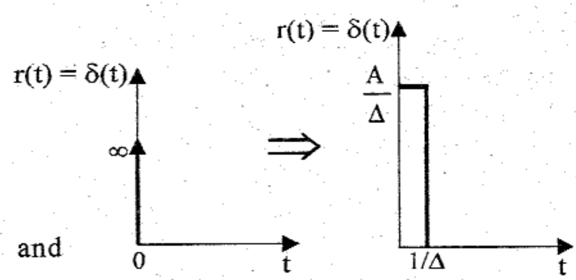


Fig 2.5 : Impulse signal.

The impulse signal is denoted by δ(t) and mathematically it is expressed as,

$$\delta(t) = \infty;\; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)dt = A$$
$$= 0;\; t \neq 0 \qquad .....(2.7)$$

Since a perfect impulse cannot be achieved in practice it is usually approximated by a pulse of small width but with area, A. Mathematically an impulse signal is the derivative of a step signal. Laplace transform of the impulse function is unity.

TABLE 2-1 : Standard Test Signals

Name of the signal	Time domain equation of signal, r(t)	Laplace transform of the signal, R(s)
Step	A	$\frac{A}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	At	$\frac{A}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{At^2}{2}$	$\frac{A}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Impulse	$\delta(t)$	1

2.3 IMPULSE RESPONSE

The response of the system, with input as impulse signal is called ***weighing function*** or ***impulse response*** of the system. It is also given by the inverse Laplace transform of the system transfer function, and denoted by m(t).

$$\text{Impulse response, } m(t) = \mathcal{L}^{-1}\{R(s)\,M(s)\} = \mathcal{L}^{-1}\{M(s)\} \quad .....(2.8)$$

$$\text{where, } M(s) = \frac{G(s)}{1+G(s)H(s)}$$

R(s) = 1, for impulse

Since impulse response (or weighing function) is obtained from the transfer function of the system, it shows the characteristics of the system. Also the response for any input can be obtained by convolution of input with impulse response.

2.4 ORDER OF A SYSTEM

The input and output relationship of a control system can be expressed by n^{th} order differential equation shown in equation (2.9).

$$a_0\frac{d^n}{dt^n}p(t) + a_1\frac{d^{n-1}}{dt^{n-1}}p(t) + a_2\frac{d^{n-2}}{dt^{n-2}}p(t) + ...+ a_{n-1}\frac{d}{dt}p(t) + a_n\,p(t) = b_0\frac{d^m}{dt^m}q(t)$$
$$+ b_1\frac{d^{m-1}}{dt^{m-1}}q(t) + b_2\frac{d^{m-2}}{dt^{m-2}}q(t) + ...+ b_{m-1}\frac{d}{dt}q(t) + b_m\,q(t) \quad .....(2.9)$$

where, p(t) = Output / Response ; q(t) = Input / Excitation.

The order of the system is given by the order of the differential equation governing the system. If the system is governed by n^{th} order differential equation, then the system is called ***n^{th} order system.***

Alternatively, the order can be determined from the transfer function of the system. The transfer function of the system can be obtained by taking Laplace transform of the differential equation governing the system and rearranging them as a ratio of two polynomials in **s**, as shown in equation (2.10).

$$\text{Transfer function, } T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + b_2 s^{m-2} + \ldots\ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_{n-1} s + a_n} \quad .....(2.10)$$

where, $P(s)$ = Numerator polynomial

$Q(s)$ = Denominator polynomial

The order of the system is given by the maximum power of **s** in the denominator polynomial, $Q(s)$.

Here, $Q(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots + a_{n-1}s + a_n$.

Now, **n** is the order of the system

When n = 0, the system is zero order system.

When n = 1, the system is first order system.

When n = 2, the system is second order system and so on.

Note : *The order can be specified for both open loop system and closed loop system.*

The numerator and denominator polynomial of equation (2.10) can be expressed in the factorized form as shown in equation (2.11).

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots\ldots(s+p_n)} \quad .....(2.11)$$

where, $z_1, z_2, \ldots\ldots z_m$ are zeros of the system.

$p_1, p_2, \ldots\ldots p_n$ are poles of the system.

Now, the value of **n** gives the number of poles in the transfer function. Hence the order is also given by the number of poles of the transfer function.

Note : *The zeros and poles are critical value, of s, at which the function T(s) attains extreme values 0 or ∞. When s takes the value of a zero, the function T(s) will be zero. When s takes the value of a pole, the function T(s) will be infinite.*

2.5 REVIEW OF PARTIAL FRACTION EXPANSION

The time response of the system is obtained by taking the inverse Laplace transform of the product of input signal and transfer function of the system. Taking inverse Laplace transform requires the knowledge of partial fraction expansion. In control systems three different types of transfer function are encountered. They are,

Case 1 : Functions with separate poles.

Case 2 : Functions with multiple poles.

Case 3 : Functions with complex conjugate poles.

The partial fraction of all the three cases are explained with an example.

Case 1 : When the transfer function has distinct poles

$$\text{Let, } T(s) = \frac{K}{s(s+p_1)(s+p_2)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+p_1)(s+p_2)}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{s+p_2}$$

The residues A, B and C are given by,

$$A=T(s)\times s\big|_{s=0}\quad B=T(s)\times(s+p_1)\big|_{s=-p_1}\quad C=T(s)\times(s+p_2)\big|_{s=-p_2}$$

Example

$$\text{Let, } T(s)=\frac{2}{s(s+1)(s+2)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{2}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A=T(s)\times s\big|_{s=0}=\frac{2}{s(s+1)(s+2)}\times s\Big|_{s=0}=\frac{2}{(s+1)(s+2)}\Big|_{s=0}=\frac{2}{1\times 2}=1$$

B is obtained by multiplying T(s) by (s +1) and letting s = −1.

$$B=T(s)\times(s+1)\big|_{s=-1}=\frac{2}{s(s+1)(s+2)}\times(s+1)\Big|_{s=-1}=\frac{2}{s(s+2)}\Big|_{s=-1}=\frac{2}{-1(-1+2)}=-2$$

C is obtained by multiplying T(s) by (s +2) and letting s = −2.

$$C=T(s)\times(s+2)\big|_{s=-2}=\frac{2}{s(s+1)(s+2)}\times(s+2)\Big|_{s=-2}=\frac{2}{s(s+1)}\Big|_{s=-2}=\frac{2}{-2(-2+1)}=+1$$

$$\therefore T(s)=\frac{2}{s(s+1)(s+2)}=\frac{1}{s}-\frac{2}{s+1}+\frac{1}{s+2}$$

Case 2 : When the transfer function has multiple poles

$$\text{Let, } T(s)=\frac{K}{s(s+p_1)(s+p_2)^2}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+p_1)(s+p_2)^2}=\frac{A}{s}+\frac{B}{s+p_1}+\frac{C}{(s+p_2)^2}+\frac{D}{(s+p_2)}$$

The residues A, B, C and D are given by,

$$A=T(s)\times s\big|_{s=0}\qquad B=T(s)\times(s+p_1)\big|_{s=-p_1}$$

$$C=T(s)\times(s+p_2)^2\big|_{s=-p_2}\qquad D=\frac{d}{ds}\left[T(s)\times(s+p_2)^2\right]\Big|_{s=-p_2}$$

Example

$$\text{Let, } T(s)=\frac{2}{s(s+1)(s+2)^2}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s)=\frac{K}{s(s+1)(s+2)^2}=\frac{A}{s}+\frac{B}{(s+1)}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)}$$

A is obtained by multiplying T(s) by s and letting s = 0.

$$A = T(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)^2} \times s\Big|_{s=0} = \frac{2}{(s+1)(s+2)^2}\Big|_{s=0} = \frac{2}{1 \times 2^2} = 0.5$$

B is obtained by multiplying T(s) by (s +1) and letting s = –1.

$$B = T(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)^2} \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+2)^2}\Big|_{s=-1} = \frac{2}{-1(-1+2)^2} = -2$$

C is obtained by multiplying T(s) by $(s+2)^2$ and letting s = –2.

$$C = T(s) \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)^2} \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)}\Big|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

D is obtained by differentiating the product T(s) $(s+2)^2$ with respect to s and then letting s = –2.

$$D = \frac{d}{ds}\left[T(s) \times (s+2)^2\right]\Big|_{s=-2} = \frac{d}{ds}\left[\frac{2}{s(s+1)}\right]\Big|_{s=-2} = \frac{-2(2s+1)}{s^2(s+1)^2}\Big|_{s=-2} = \frac{-2(2(-2)+1)}{(-2)^2(-2+1)^2} = +1.5$$

$$\therefore\ T(s) = \frac{2}{s(s+1)(s+2)^2} = \frac{0.5}{s} - \frac{2}{s+1} + \frac{1}{(s+2)^2} + \frac{1.5}{s+2}$$

Case 3 : When the transfer function has complex conjugate poles

$$\text{Let, } T(s) = \frac{K}{(s+p_1)(s^2+bs+c)}$$

By partial fraction expansion, T(s) can be expressed as,

$$T(s) = \frac{K}{(s+p_1)(s^2+bs+c)} = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c} \qquad .....(2.12)$$

The residue A is given by, $A = T(s) \times (s+p_1)\big|_{s=-p_1}$

The residues B and C are solved by cross multiplying the equation (2.12) and then equating the coefficient of like power of **s.**

Finally express T(s) as shown below,

$$T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2+bs+c}$$

$$\boxed{(x+y)^2 = x^2 + 2xy + y^2}$$

Let us express, $s^2 + bs$, in the form of $(x+y)^2$. This will require addition and subtraction of an extra term $(b/2)^2$.

$$\therefore\ T(s) = \frac{A}{s+p_1} + \frac{Bs+C}{s^2 + 2 \times \frac{b}{2}s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2} = \frac{A}{s+p_1} + \frac{Bs+C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

$$= \frac{A}{s+p_1} + \frac{Bs}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)} + \frac{C}{\left(s+\frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

Example

$$\text{Let, } T(s) = \frac{1}{(s+2)(s^2+s+1)}$$

By partial fraction expansion,

$$T(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

A is obtained by multiplying T(s) by (s +2) and letting s = –2.

$$\therefore A = T(s) \times (s+2)\Big|_{s=-2} = \frac{1}{(s+2)(s^2+s+1)} \times (s+2)\Big|_{s=-2} = \frac{1}{(-2)^2-2+1} = \frac{1}{3}$$

To solve B and C, cross multiply the following equation and substitute the value of A. Then equate the like power of s.

$$\frac{1}{(s+2)(s^2+s+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+s+1}$$

$$1 = A(s^2+s+1) + (Bs+C)(s+2)$$

$$1 = \frac{1}{3}(s^2+s+1) + Bs^2 + 2Bs + Cs + 2C$$

$$1 = \frac{s^2}{3} + \frac{s}{3} + \frac{1}{3} + Bs^2 + 2Bs + Cs + 2C$$

$$s^2+s+1 = s^2 + 2\times\frac{s}{2} + \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{2}\right)^2 = \left(s+\frac{1}{2}\right)^2 + \left(1-\frac{1}{4}\right) = (s+0.5)^2 + 0.75$$

On equating the coefficient of s^2 terms, $0 = \frac{1}{3} + B$; $\therefore B = -\frac{1}{3}$

On equating the coefficient of s terms, $0 = \frac{1}{3} + 2B + C$; $\therefore C = -\frac{1}{3} - 2B = -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$

$$T(s) = \frac{\frac{1}{3}}{s} + \frac{\frac{-1}{3}s + \frac{1}{3}}{s^2+s+1} = \frac{1}{3s} - \frac{1}{3}\frac{s}{(s^2+s+1)} + \frac{1}{3}\frac{1}{(s^2+s+1)}$$

$$= \frac{1}{3s} - \frac{1}{3}\frac{s}{(s+0.5)^2+0.75} + \frac{1}{3}\frac{1}{(s+0.5)^2+0.75}$$

2.6 RESPONSE OF FIRST ORDER SYSTEM FOR UNIT STEP INPUT

The closed loop order system with unity feedback is shown in fig 2.6.

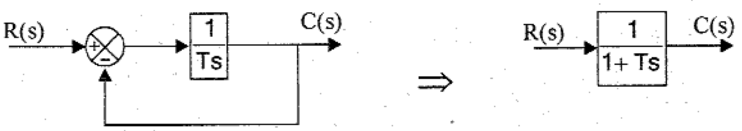


Fig 2.6 : *Closed loop for first order system.*

The closed loop transfer function of first order system, $\frac{C(s)}{R(s)} = \frac{1}{1+Ts}$

If the input is unit step then, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

$$\therefore \text{The response in s-domain, } C(s) = R(s)\frac{1}{(1+Ts)} = \frac{1}{s}\frac{1}{(1+Ts)} = \frac{1}{sT\left(\frac{1}{T}+s\right)} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}$$

By partial fraction expansion,

$$C(s) = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} = \frac{A}{s} + \frac{B}{\left(s+\frac{1}{T}\right)}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} \times s\Bigg|_{s=0} = \frac{\frac{1}{T}}{s+\frac{1}{T}}\Bigg|_{s=0} = \frac{\frac{1}{T}}{\frac{1}{T}} = 1$$

B is obtained by multiplying C(s) by (s+1/T) and letting s = –1/T.

$$B = C(s) \times \left(s+\frac{1}{T}\right)\Bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)} \times \left(s+\frac{1}{T}\right)\Bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{s}\Bigg|_{s=-\frac{1}{T}} = \frac{\frac{1}{T}}{\frac{-1}{T}} = -1$$

$$\therefore C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right\} = 1 - e^{-\frac{t}{T}} \qquad .....(2.13)$$

The equation (2.13) is the response of the closed loop first order system for unit step input. For step input of step value, A, the equation (2.13) is multiplied by A.

$\therefore$ For closed loop first order system, Unit step response $= 1 - e^{-\frac{t}{T}}$

Step response $= A\left(1 - e^{-\frac{t}{T}}\right)$

When, $t = 0$, $c(t) = 1 - e^{0} = 0$
When, $t = 1T$, $c(t) = 1 - e^{-1} = 0.632$
When, $t = 2T$, $c(t) = 1 - e^{-2} = 0.865$
When, $t = 3T$, $c(t) = 1 - e^{-3} = 0.95$
When, $t = 4T$, $c(t) = 1 - e^{-4} = 0.9817$
When, $t = 5T$, $c(t) = 1 - e^{-5} = 0.993$
When, $t = \infty$, $c(t) = 1 - e^{-\infty} = 1$

Here T is called Time constant of the system. In a time of 5T, the system is assumed to have attained steady state. The input and output signal of the first order system is shown in fig 2.7.

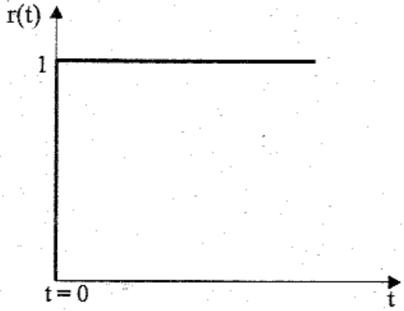


Fig 2.7a : Unit step input.

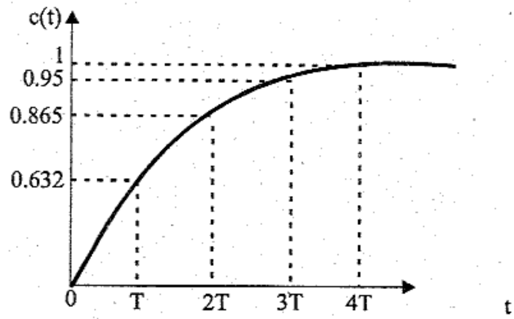


Fig 2.7b : Response for Unit step input.

Fig 2.7 : Response of first order system to Unit step input.

2.7 SECOND ORDER SYSTEM

The closed loop second order system is shown in fig 2.8

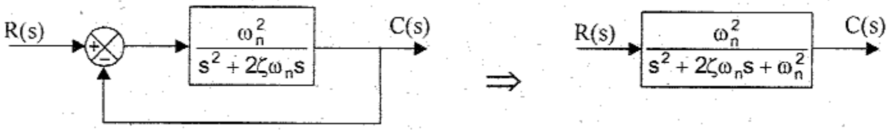


Fig 2.8 : Closed loop for second order system.

The standard form of closed loop transfer function of second order system is given by,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(2.14)$$

where, ω_n = Undamped natural frequency, rad/sec.

ζ = Damping ratio.

The ***damping ratio*** is defined as the ratio of the actual damping to the critical damping. The response c(t) of second order system depends on the value of damping ratio. Depending on the value of ζ, the system can be classified into the following four cases,

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Under damped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$

The characteristics equation of the second order system is,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad .....(2.15)$$

It is a quadratic equation and the roots of this equation is given by,

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \qquad .....(2.16)$$

When $\zeta = 0,\ s_1, s_2 = \pm j\omega_n$; $\begin{cases}\text{roots are purely imaginary} \\ \text{and the system is undamped}\end{cases}$ .....(2.17)

When $\zeta = 1,\ s_1, s_2 = -\omega_n$; $\begin{cases}\text{roots are real and equal and} \\ \text{the system is critically damped}\end{cases}$ .....(2.18)

When $\zeta > 1,\ s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$; $\begin{cases}\text{roots are real and unequal and} \\ \text{the system is overdamped}\end{cases}$ .....(2.19)

$$\text{When } 0 < \zeta < 1,\ s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{-1}\ \sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$= -\zeta\omega_n \pm j\omega_d$; $\begin{cases}\text{roots are complex conjugate} \\ \text{the system is underdamped}\end{cases}$ .....(2.20)

where, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ .....(2.21)

Here ω_d is called damped frequency of oscillation of the system and its unit is rad/sec.

2.7.1 RESPONSE OF UNDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For undamped system, z = 0.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(2.22)$$

When the input is unit step, $r(t) = 1$ and $R(s) = \frac{1}{s}$.

$$\therefore \text{The response in s-domain, } C(s) = R(s)\ \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{1}{s}\ \frac{\omega_n^2}{s^2 + \omega_n^2} \quad .....(2.23)$$

By partial fraction expansion,

$$C(s) = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times s\Bigg|_{s=0} = \frac{\omega_n^2}{s^2 + \omega_n^2}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

B is obtained by multiplying C(s) by $(s^2 + \omega_n^2)$ and letting $s^2 = -\omega_n^2$ or $s = j\omega_n$.

$$B = C(s) \times (s^2 + \omega_n^2)\Big|_{s=j\omega} = \frac{\omega_n^2}{s\,(s^2 + \omega_n^2)} \times (s^2 + \omega_n^2)\Bigg|_{s=j\omega} = \frac{\omega_n^2}{s}\Bigg|_{s=j\omega_n} = \frac{\omega_n^2}{j\omega_n} = -j\omega_n = -s$$

$$\therefore\ C(s) = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$	$\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$

$$\text{Time domain response, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2 + \omega_n^2}\right\} = 1 - \cos \omega_n t \quad .....(2.24)$$

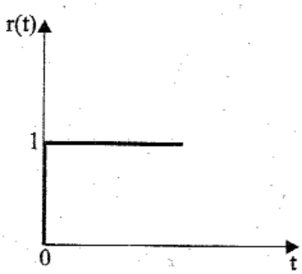


Fig 2.9.a : Input.

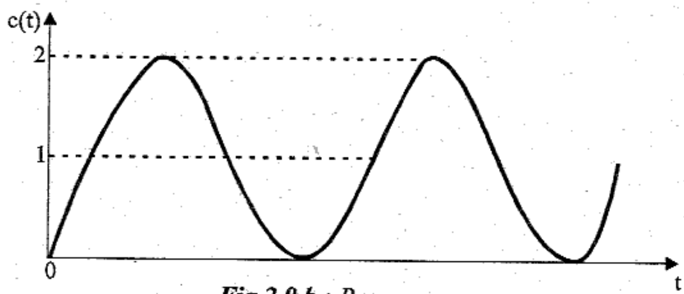


Fig 2.9.b : Response.

Fig 2.9 : Response of undamped second order system for unit step input.

Using equation (2.24), the response of undamped second order system for unit step input is sketched in fig 2.9, and observed that the response is completely oscillatory.

> ***Note :*** *Every practical system has some amount of damping. Hence undamped system does not exist in practice.*

The equation (2.24) is the response of undamped closed loop second order system for unit step input. For step input of step value A, the equation (2.24) should be multiplied by A.

> ∴ For closed loop undamped second order system,
>
> Unit step response $= 1 - \cos \omega_n t$
>
> Step response $= A(1 - \cos \omega_n t)$

2.7.2 RESPONSE OF UNDERDAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For underdamped system, $0 < \zeta < 1$ and roots of the denominator (characteristic equation) are complex conjugate.

The roots of the denominator are, $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

Since $\zeta < 1$, ζ^2 is also less then 1, and so $1 - \zeta^2$ is always positive.

$$\therefore\ s = -\zeta\omega_n \pm \omega_n\sqrt{(-1)(1-\zeta^2)} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore s = -\zeta\omega_n \pm j\omega_d$$

The response in s-domain, $C(s) = R(s)\,\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

For unit step input, r(t) = 1 and R(s) = 1/s.

$$\therefore C(s) = \frac{\omega_n^2}{s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

By partial fraction expansion, $C(s) = \dfrac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \dfrac{A}{s} + \dfrac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ .....(2.25)

A is obtained by multiplying C(s) by s and letting s = 0.

$$\therefore\ A = s \times C(s)\Big|_{s=0} = s \times \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

To solve for B and C, cross multiply equation (2.25) and equate like power of s.

On cross multiplication equation (2.25) after substituting A = 1, we get,

$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + (Bs + C)s$$
$$\omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2 + Bs^2 + Cs$$

Equating coefficients of s^2 we get, $0 = 1 + B$ $\quad\therefore B = -1$

Equating coefficient of s we get, $0 = 2\zeta\omega_n + C$ $\quad\therefore C = -2\zeta\omega_n$

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad .....(2.26)$$

Let us add and subtract $\zeta^2\omega_n^2$ to the denominator of second term in the equation (2.26).

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2 + \zeta^2\omega_n^2 - \zeta^2\omega_n^2} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2) + (\omega_n^2 - \zeta^2\omega_n^2)}$$

$$= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$\boxed{\omega_d = \omega_n\sqrt{1 - \zeta^2}}$$

$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \qquad .....(2.27)$$

Let us multiply and divide by ω_d in the third term of the equation (2.27).

$$\therefore\ C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\,\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s + a)^2 + \omega^2}$
$\mathcal{L}\{e^{-at}\cos\omega t\} = \frac{s + a}{(s + a)^2 + \omega^2}$

The response in time domain is given by,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\,\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right\}$$

$$= 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\,e^{-\zeta\omega_n t}\sin\omega_d t = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_n\sqrt{1 - \zeta^2}}\sin\omega_d t\right)$$

$$\boxed{\omega_d = \omega_n\sqrt{1 - \zeta^2}}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\left(\sqrt{1 - \zeta^2}\,\cos\omega_d t + \zeta\,\sin\omega_d t\right) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\left(\sin\omega_d t \times \zeta + \cos\omega_d t \times \sqrt{1 - \zeta^2}\right)$$

Let us express c(t) in a standard form as shown below.

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}(\sin\omega_d t \times \cos\theta + \cos\omega_d t \times \sin\theta)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\sin(\omega_d t + \theta) \qquad .....(2.28)$$

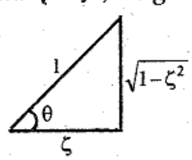
$$\text{where,}\ \left(\theta = \tan^{-1}\frac{\sqrt{1 - \zeta^2}}{\zeta}\right)$$

Note : On constructing right angle triangle with ζ and $\sqrt{1 - \zeta^2}$, we get

$\sin\theta = \sqrt{1 - \zeta^2}$

$\cos\theta = \zeta$

$\tan\theta = \frac{\sqrt{1 - \zeta^2}}{\zeta}$



The equation (2.28) is the response of under damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.28) should be multiplied by A.

∴ For closed loop under damped second order system,

$$\text{Unit step response} = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\text{Step response} = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)\ ;\ \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

Using equation (2.28) the response of underdamped second order system for unit step input is sketched and observed that the response oscillates before settling to a final value. The oscillations depends on the value of damping ratio.

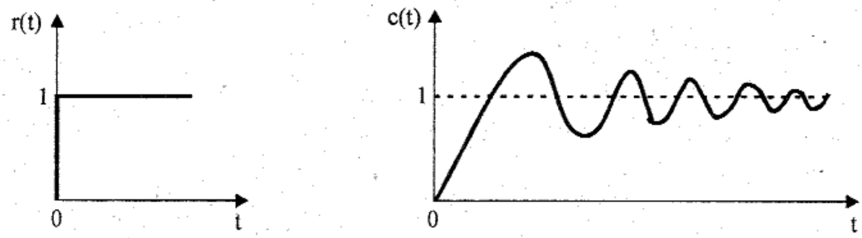


Fig 2.10.a : Input. *Fig 2.10.b : Response.*

Fig 2.10 : Response of under damped second order system for unit step input.

2.7.3 RESPONSE OF CRITICALLY DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For critical damping $\zeta = 1$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\omega_n)^2} \qquad .....(2.29)$$

When input is unit step, r(t) = 1 and R(s) = 1/s.

∴ The response in s-domain,

$$C(s) = R(s)\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{1}{s}\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{\omega_n^2}{s(s+\omega_n)^2} \qquad .....(2.30)$$

By partial fraction expansion, we can write,

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$A = s \times C(s)\Big|_{s=0} = \frac{\omega_n^2}{(s+\omega_n)^2}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+\omega_n)^2 \times C(s)\Big|_{s=-\omega_n} = \frac{\omega_n^2}{s}\Bigg|_{s=-\omega_n} = -\omega_n$$

$$C = \frac{d}{ds}\left[(s+\omega_n)^2 \times C(s)\right]\Bigg|_{s=-\omega_n} = \frac{d}{ds}\left(\frac{\omega_n^2}{s}\right)\Bigg|_{s=-\omega_n} = \frac{-\omega_n^2}{s^2}\Bigg|_{s=-\omega_n} = -1$$

$$\therefore C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n} = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}$$

$\mathcal{L}\{1\} = \frac{1}{s}$
$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$
$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right\}$$

$$c(t) = 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}$$

$$c(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \quad .....(2.31)$$

The equation (2.31) is the response of critically damped closed loop second order system for unit step input. For step input of step value, A, the equation (2.31) should be multiplied by A.

$\therefore$ For closed loop critically damped second order system,
Unit step response $= 1 - e^{-\omega_n t}(1+\omega_n t)$
Step response $= A\left[1 - e^{-\omega_n t}(1+\omega_n t)\right]$

Using equation (2.31), the response of critically damped second order system is sketched as shown in fig 2.11 and observed that the response has no oscillations.

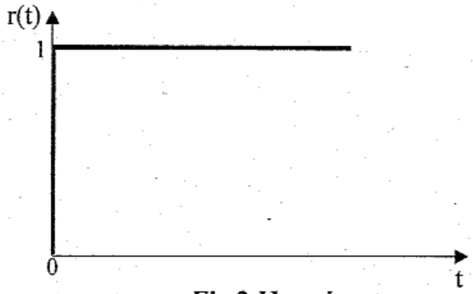


Fig 2.11.a : Input.

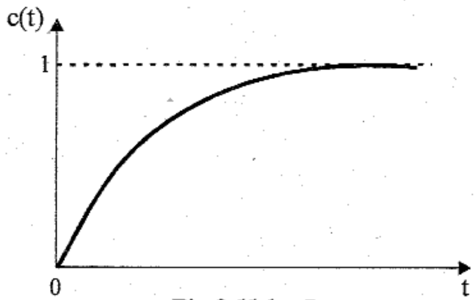


Fig 2.11.b : Response.

Fig 2.11 : Response of critically damped second order system for unit step input.

2.7.4 RESPONSE OF OVER DAMPED SECOND ORDER SYSTEM FOR UNIT STEP INPUT

The standard form of closed loop transfer function of second order system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For overdamped system $\zeta > 1$. The roots of the denominator of transfer function are real and distinct. Let the roots of the denominator be s_a, s_b.

$$s_a, s_b = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1} = -\left[\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}\right] \quad .....(2.32)$$

Let $s_1 = -s_a$ and $s_2 = -s_b$ $\qquad \therefore s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$ .....(2.33)

$$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \quad .....(2.34)$$

The closed loop transfer function can be written in terms of s_1 and s_2 as shown below.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)} \quad .....(2.35)$$

For unit step input r(t) = 1 and R(s) = 1/s.

$$\therefore C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)} = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$A = s \times C(s)\Big|_{s=0} = s \times \frac{\omega_n^2}{s(s+s_1)(s+s_2)}\Bigg|_{s=0} = \frac{\omega_n^2}{s_1 s_2}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$B = (s+s_1) \times C(s)\Big|_{s=-s_1} = \frac{\omega_n^2}{s(s+s_2)}\Bigg|_{s=-s_1} = \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{-\omega_n^2}{s_1\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]} = \frac{-\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_1} = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}$$

$$C = C(s) \times (s+s_2)\Big|_{s=-s_2} = \frac{\omega_n^2}{s(s+s_1)}\Bigg|_{s=-s_2} = \frac{\omega_n^2}{-s_2(-s_2+s_1)}$$

$$= \frac{\omega_n^2}{-s_2\left[-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]} = \frac{\omega_n^2}{\left[2\omega_n\sqrt{\zeta^2-1}\right]s_2} = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}$$

The response in time domain, c(t) is given by,

$$c(t) = \mathcal{L}^{-1}\left\{= \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,\frac{1}{(s+s_1)} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,\frac{1}{(s+s_2)}\right\}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\,e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_2}\,e^{-s_2 t}$$

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \quad .....(2.36$$

where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

The equation (2.36) is the response of overdamped closed loop system for unit step input. For step input of value, A, the equation (2.36) is multiplied by A.

∴ For closed loop over damped second order system,

Unit step response $= 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$ where, $s_1 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

Step response $= A\left[1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\,\frac{1}{s_1}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)\right]$ $s_2 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

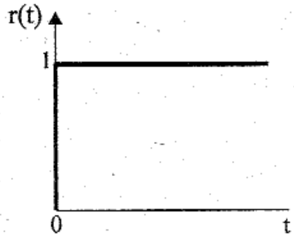


Fig 2.12.a : Input.

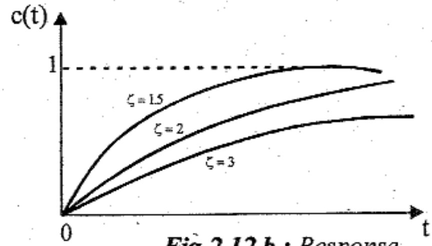


Fig 2.12.b : Response.

Fig 2.12 : Response of over damped second order system for unit step input.

Using equation (2.36), the response of overdamped second order system is sketched as shown in fig 2.12 and observed that the response has no oscillations but it takes longer time for the response to reach the final steady value.

2.8 TIME DOMAIN SPECIFICATIONS

The desired performance characteristics of control systems are specified in terms of time domain specifications. Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses, whenever they are subjected to inputs or disturbances.

The desired performance characteristics of a system of any order may be specified in terms of the transient response to a unit step input signal. The response of a second order system for unit-step input with various values of damping ratio is shown in fig 2.13.

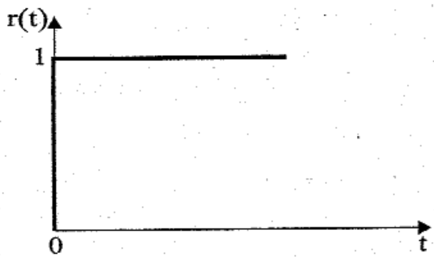


Fig 2.13.a : Input.

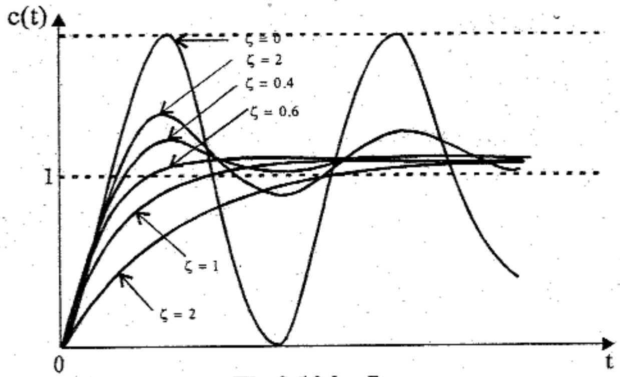


Fig 2.13.b : Response.

Fig 2.13 : Unit step response of second order system.

The transient response of a system to a unit step input depends on the initial conditions. Therefore to compare the time response of various systems it is necessary to start with standard initial conditions. The most practical standard is to start with the system at rest and so output and all time derivatives before $t = 0$ will be zero. The transient response of a practical control system often exhibits damped oscillation before reaching steady state. A typical damped oscillatory response of a system is shown in fig 2.14.

The transient response characteristics of a control system to a unit step input is specified in terms of the following time domain specifications.

1. Delay time, t_d
2. Rise time, t_r
3. Peak time, t_p
4. Maximum overshoot, M_p
5. Settling time, t_s

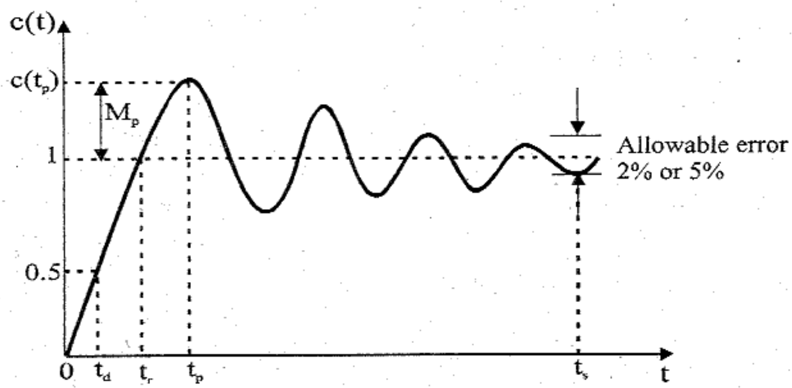


Fig 2.14 : Damped oscillatory response of second order system for unit step input.

The time domain specifications are defined as follows.

1. DELAY TIME (t_d) : It is the time taken for response to reach 50% of the final value, for the very first time.

2. RISE TIME (t_r) : It is the time taken for response to raise from 0 to 100% for the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

3. PEAK TIME (t_p) : It is the time taken for the response to reach the peak value the very first time. (or) It is the time taken for the response to reach the peak overshoot, M_p.

4. PEAK OVERSHOOT (M_p) : It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value.

Let, $c(\infty)$ = Final value of c(t).

$c(t_p)$ = Maximum value of c(t).

Now, Peak overshoot, $$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \quad .....(2.37)$$

% Peak overshoot, $$\%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 \quad .....(2.38)$$

5. SETTLING TIME (t_s) : It is defined as the time taken by the response to reach and stay within a specified error. It is usually expressed as % of final value. The usual tolerable error is 2 % or 5% of the final value.

EXPRESSIONS FOR TIME DOMAIN SPECIFICATIONS

Rise time (t_r)

The unit step response of second order system for underdamped case is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$ **(Refer fig 2.14).**

$$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 1$$

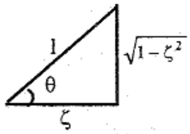
$$\therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0$$

Since $-e^{-\zeta\omega_n t_r} \neq 0$, the term, $\sin(\omega_d t_r + \theta) = 0$

When, $\phi = 0, \pi, 2\pi, 3\pi \ldots$, $\sin\phi = 0$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\omega_d t_r = \pi - \theta$$

> *Note : On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*
>
> $$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}$$
>
> 
>

$$\therefore \text{Rise Time, } t_r = \frac{\pi - \theta}{\omega_d} \qquad .....(2.39)$$

Here, $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$; Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ ***(refer note)***

$$\therefore \text{Rise time, } t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in sec} \qquad .....(2.40)$$

> ***Note :** θ or $\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ should be measured in radians.*

Peak time (t_p)

To find the expression for peak time, t_p, differentiate c(t) with respect to t and equate to 0.

$$\text{i.e., } \left.\frac{d}{dt}c(t)\right|_{t=t_p} = 0$$

The unit step response of under damped second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Differentiating c(t) with respect to **t**.

$$\frac{d}{dt}c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(-\zeta\omega_n)\sin(\omega_d t + \theta) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\cos(\omega_d t + \theta)\omega_d$$

Put, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\therefore \frac{d}{dt}c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\zeta\omega_n)\sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_d t + \theta) - \sqrt{1-\zeta^2}\cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\cos\theta\sin(\omega_d t + \theta) - \sin\theta\cos(\omega_d t + \theta)\right] \quad \textbf{\textit{(refer note)}}$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\sin(\omega_d t + \theta)\cos\theta - \cos(\omega_d t + \theta)\sin\theta\right]$$

$$=\frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\left[\sin((\omega_d t+\theta)-\theta)\right] = \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin(\omega_d t)$$

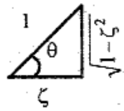
at $t=t_p$, $\frac{d}{dt}c(t)=0$

$$\therefore \frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t_p}\sin(\omega_d t_p)=0$$

Since, $e^{-\zeta\omega_n t_p}\neq 0$, the term, $\sin(\omega_d t_p)=0$

When $\phi=0, \pi, 2\pi, 3\pi$, $\sin\phi=0$

$\therefore\ \omega_d t_p=\pi$

> ***Note :** On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*
>
> $\sin\theta=\sqrt{1-\zeta^2}$
>
> $\cos\theta=\zeta$
>
> 
>

$$\therefore \text{Peak time, } t_p=\frac{\pi}{\omega_d} \qquad .....(2.41)$$

The damped frequency of oscillation, $\omega_d=\omega_n\sqrt{1-\zeta^2}$

$$\therefore \text{Peak time, } t_p=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad .....(2.42)$$

Peak overshoot (M_p)

$$\%\text{Peak overshoot, } \%M_p=\frac{c(t_p)-c(\infty)}{c(\infty)}\times 100 \qquad .....(2.43)$$

where, $c(t_p)$ = Peak response at $t=t_p$.

$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

At $t=\infty$, $\quad c(t) = c(\infty) = 1-\frac{e^{-\infty}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta) = 1-0 = 1$

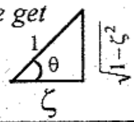
> $t_p=\frac{\pi}{\omega_d}$

At $t=t_p$, $\quad c(t)=c(t_p)=1-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d t_p+\theta)$

$$=1-\frac{e^{-\zeta\omega_n\frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d\frac{\pi}{\omega_d}+\theta\right)$$

> $\omega_d=\omega_n\sqrt{1-\zeta^2}$
>
> $\sin(\pi+\theta)=-\sin\theta$

$$=1-\frac{e^{-\zeta\omega_n\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}\sin(\pi+\theta)}{\sqrt{1-\zeta^2}}$$

> ***Note :** On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$, we get*
>
> $\sin\theta=\sqrt{1-\zeta^2}$
>
> 
>

$$=1+\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin\theta$$

$$=1+\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2} = 1+e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \qquad .....(2.44)$$

$$\text{Percentage Peak Overshoot, } \%M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100 = \frac{1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1}{1} \times 100$$

$$= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\therefore \text{ Percentage Peak Overshoot, } \%M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 \quad .....(2.45)$$

Settling time (t_s)

The response of second order system has two components. They are,

1. Decaying exponential component, $\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$.
2. Sinusoidal component, $\sin(\omega_d t + \theta)$.

In this the decaying exponential term dampens (or) reduces the oscillations produced by sinusoidal component. Hence the settling time is decided by the exponential component. The settling time can be found out by equating exponential component to percentage tolerance errors.

For 2 % tolerance error band, at $t = t_s$, $\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.02$

For least values of ζ, $e^{-\zeta\omega_n t_s} = 0.02$.

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = \ln(0.02) \quad \Rightarrow \quad -\zeta\omega_n t_s = -4 \quad \Rightarrow \quad t_s = \frac{4}{\zeta\omega_n}$$

For the second order system, the time constant, $T = \frac{1}{\zeta\omega_n}$

$$\therefore \text{ Settling time, } t_s = \frac{1}{\zeta\omega_n} = 4T \quad \text{(for 2\% error)} \quad .....(2.46)$$

For 5% error, $e^{-\zeta\omega_n t_s} = 0.05$

On taking natural logarithm we get,

$$-\zeta\omega_n t_s = \ln(0.05) \quad \Rightarrow \quad -\zeta\omega_n t_s = -3 \quad \Rightarrow \quad t_s = \frac{3}{\zeta\omega_n}$$

$$\therefore \text{ Settling time, } t_s = \frac{3}{\zeta\omega_n} = 3T \quad \text{(for 5\% error)} \quad .....(2.47)$$

In general for a specified percentage error, Settling time can be evaluted using equation (2.48).

$$\therefore \text{ Settling time, } t_s = \frac{\ln(\%\text{ error})}{\zeta\omega_n} = \frac{\ln(\%\text{ error})}{T} \quad .....(2.48)$$

EXAMPLE 2.1

Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.

SOLUTION

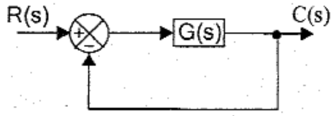


Fig 1 : Closed loop system.

The closed loop system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}} = \frac{\frac{4}{s(s+5)}}{\frac{s(s+5)+4}{s(s+5)}} = \frac{4}{s(s+5)+4} = \frac{4}{s^2+5s+4} = \frac{4}{(s+4)(s+1)}$$

The response in s-domain, $C(s) = R(s)\,\frac{4}{(s+1)(s+4)}$

Since the input is unit step, $R(s) = \frac{1}{s}$; $\quad \therefore C(s) = \frac{4}{s(s+1)(s+4)}$

By partial fraction expansion, we can write,

$$C(s) = \frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = C(s) \times s\big|_{s=0} = \left.\frac{4}{(s+1)(s+4)}\right|_{s=0} = \frac{4}{1\times 4} = 1$$

$$B = C(s) \times (s+1)\big|_{s=-1} = \left.\frac{4}{s(s+4)}\right|_{s=-1} = \frac{4}{-1(-1+4)} = \frac{-4}{3}$$

$$C = C(s) \times (s+4)\big|_{s=-4} = \left.\frac{4}{s(s+1)}\right|_{s=-4} = \frac{4}{-4(-4+1)} = \frac{1}{3}$$

The time domain response c(t) is obtained by taking inverse Laplace transform of C(s).

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s+4}\right\}$

$$= 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t} = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$$

RESULT

Response of unity feedback system, $c(t) = 1 - \frac{1}{3}\left[4e^{-t} - e^{-4t}\right]$

EXAMPLE 2.2

A positional control system with velocity feedback is shown in fig 1. What is the response of the system for unit step input.

SOLUTION

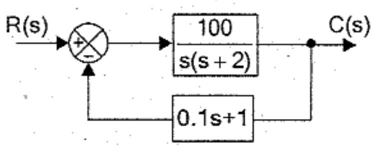


Fig 1 : Positional control system.

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

Given that, $G(s) = \frac{100}{s(s+2)}$ and $H(s) = 0.1s+1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{100}{s(s+2)}}{1+\left(\frac{100}{s(s+2)}\right)(0.1s+1)} = \frac{\frac{100}{s(s+2)}}{\frac{s(s+2)+100(0.1s+1)}{s(s+2)}} = \frac{100}{s^2+2s+10s+100} = \frac{100}{s^2+12s+100}$$

Here $(s^2 + 12s + 100)$ is characteristic polynomial. The roots of the characteristic polynomial are,

$$s_1, s_2 = \frac{-12 \pm \sqrt{144-400}}{2} = \frac{-12 \pm j16}{2} = -6 \pm j8$$

The roots are complex conjugate. The system is underdamped and so the response of the system will have damped oscillations.

$$\text{The response in s-domain, } C(s) = R(s)\,\frac{100}{s^2+12s+100}$$

$$\text{Since input is unit step, } R(s) = \frac{1}{s}$$

$$\therefore C(s) = \frac{1}{s}\,\frac{100}{s^2+12s+100} = \frac{100}{s(s^2+12s+100)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \left.\frac{100}{s^2+12s+100}\right|_{s=0} = \frac{100}{100} = 1$$

The residue B and C are evaluated by cross multiplying the following equation and equating the coefficients of like power of s.

$$\frac{100}{s\,(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

$$100 = A(s^2+12s+100) + (Bs+C)\,s$$

$$100 = As^2 + 12As + 100A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B$ $\quad\therefore B = -A = -1$

On equating coefficients of s we get, $0 = 12A + C$ $\quad\therefore C = -12A = -12$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-12}{s^2+12s+100} = \frac{1}{s} - \frac{s+12}{s^2+12s+36+64} = \frac{1}{s} - \frac{s+6+6}{(s+6)^2+8^2}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2} = \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

$$\text{Time response, } c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{8}\,\frac{8}{(s+6)^2+8^2}\right\}$$

$$= 1 - e^{-6t}\cos 8t - \frac{6}{8}\,e^{-6t}\sin 8t = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right]$$

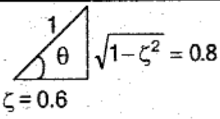
The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$. The damping ratio ζ is evaluated by comparing the closed loop transfer function of the system with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{100}{s^2+12s+100}$$

On comparing we get, $\omega_n^2 = 100$ $\quad\therefore \omega_n = 10$

$2\zeta\omega_n = 12$ $\quad\therefore \zeta = \frac{12}{2\omega_n} = \frac{12}{2 \times 10} = 0.6$

Constructing right angled triangle with ζ and $\sqrt{1-\zeta^2}$ we get,



$\cos\theta = 0.6$

$\sin\theta = 0.8$

$$\sin\theta = 0.8 \;;\; \cos\theta = 0.6 \;;\; \tan\theta = \frac{0.8}{0.6}$$

$$\therefore \theta = \tan^{-1}\frac{0.8}{0.6} = 53° = 53° \times \frac{\pi}{180°}\text{rad} = 0.925 \text{ rad.}$$

$$\therefore \text{ Time response, } c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right] = 1 - e^{-6t}\frac{10}{8}\left[\frac{6}{10}\sin 8t + \frac{8}{10}\cos 8t\right]$$

$$= 1 - \frac{10}{8}e^{-6t}[\sin 8t \times 0.6 + \cos 8t \times 0.8] = 1 - 1.25\, e^{-6t}[\sin 8t \cos\theta + \cos 8t \sin\theta]$$

$$= 1 - 1.25\, e^{-6t}[\sin\,(8t+\theta)] = 1 - 1.25\, e^{-6t}\sin(8t + 0.925)$$

Note : θ is expressed in radians

RESULT

The response in time domain,

$$c(t) = 1 - e^{-6t}\left[\frac{6}{8}\sin 8t + \cos 8t\right] \quad \text{or} \quad c(t) = 1 - 1.25\, e^{-6t}\sin(8t + 0.925)$$

EXAMPLE 2.3

The response of a servomechanism is, $c(t) = 1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}$ when subject to a unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

SOLUTION

Given that, $c(t) = 1 + 0.2\, e^{-60t} - 1.2\, e^{-10t}$

On taking Laplace transform of c(t) we get,

$$C(s) = \frac{1}{s} + 0.2\frac{1}{(s+60)} - 1.2\frac{1}{(s+10)} = \frac{(s+60)\,(s+10) + 0.2\,s\,(s+10) - 1.2\,s\,(s+60)}{s\,(s+60)\,(s+10)}$$

$$= \frac{s^2 + 70s + 600 + 0.2\,s^2 + 2s - 1.2s^2 - 72s}{s\,(s+60)\,(s+10)} = \frac{600}{s\,(s+60)\,(s+10)} = \frac{1}{s}\;\frac{600}{(s+60)\,(s+10)}$$

Since input is unit step, R(s) = 1/s.

$$\therefore C(s) = R(s)\,\frac{60}{(s+60)\,(s+10)} = R(s)\,\frac{600}{s^2 + 70s + 600}$$

$$\therefore \text{ The closed loop transfer function of the system, } \frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}$$

The damping ratio and natural frequency of oscillation can be estimated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{600}{s^2 + 70s + 600}$$

On comparing we get,

$$\omega_n^2 = 600$$

$$\therefore \omega_n = \sqrt{600} = 24.49 \text{ rad / sec}$$

$$2\zeta\omega_n = 70$$

$$\therefore \zeta = \frac{70}{2\omega_n} = \frac{70}{2 \times 24.49} = 1.43$$

RESULT

The closed loop transfer function of the system, $\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}$

Natural frequency of oscillation, $\omega_n = 24.49$ rad/sec

Damping ratio, $\zeta = 1.43$

EXAMPLE 2.4

The unity feedback system is characterized by an open loop transfer function G(s) =K/s (s +10). Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine peak overshoot and time at peak overshoot for a unit step input.

SOLUTION

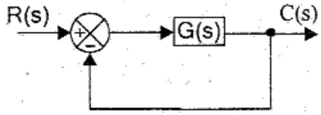


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (s +10)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s\,(s+10)+K} = \frac{K}{s^2+10s+K}$$

The value of K can be evaluated by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

On comparing we get,

$\omega_n^2 = K$	$2\zeta\omega_n = 10$	$K = 100$
$\therefore \omega_n = \sqrt{K}$	Put $\zeta = 0.5$ and $\omega_n = \sqrt{K}$	$\omega_n = 10$ rad / sec
	$\therefore 2 \times 0.5 \times \sqrt{K} = 10$	
	$\sqrt{K} = 10$	

The value of gain, K=100.

Percentage peak overshoot, $\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 = 0.163 \times 100 = 16.3\%$$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.363$ sec

RESULT

The value of gain,	K	=	100
Percentage peak overshoot,	$\%M_p$	=	16.3%
Peak time,	t_p	=	0.363 sec.

EXAMPLE 2.5

The open loop transfer function of a unity feedback system is given by G(s) = K/s (sT + 1), where K and T are positive constant. By what factor should the amplifier gain K be reduced, so that the peak overshoot of unit step response of the system is reduced from 75% to 25%.

SOLUTION

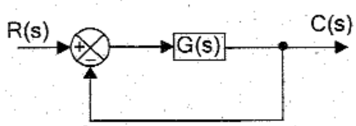


Fig 1 : Unity feedback system.

The unity feedback system is shown in fig 1.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

Given that, G(s) = K/s (sT +1)

$$\therefore \frac{C(s)}{R(s)} = \frac{K/s(sT+1)}{1+K/s(sT+1)} = \frac{K}{s(sT+1)+K} = \frac{K}{s^2T+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

Expression for ζ and ω_n can be obtained by comparing the transfer function with the standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{K/T}{s^2+\frac{1}{T}s+\frac{K}{T}}$$

On comparing we get,

$$\omega_n^2 = K/T \qquad \therefore \omega_n = \sqrt{K/T}$$

$$2\zeta\omega_n = 1/T \qquad \zeta = \frac{1}{2\omega_n T} = \frac{1}{2\sqrt{\frac{K}{T}}\,T} = \frac{1}{2\sqrt{KT}}$$

The peak overshoot, M_p is reduced by increasing the damping ratio ζ. The damping ratio ζ is increased by reducing the gain K.

When $M_p = 0.75$, Let $\zeta = \zeta_1$ and $K = K_1$

When $M_p = 0.25$, Let $\zeta = \zeta_2$ and $K = K_2$

Peak overshoot, $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

Taking natural logarithm on both sides, $\ln M_p = \frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$

On squaring we get, $(\ln M_p)^2 = \frac{\zeta^2\pi^2}{1-\zeta^2}$

On crossing multiplication we get,

$$(1-\zeta^2)(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 - \zeta^2(\ln M_p)^2 = \zeta^2\pi^2$$

$$(\ln M_p)^2 = \zeta^2\pi^2 + \zeta^2(\ln M_p)^2$$

$$(\ln M_p)^2 = \zeta^2\left[\pi^2 + (\ln M_p)^2\right]$$

$$\therefore \zeta^2 = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2} \quad ....(1)$$

$$\text{But } \zeta = \frac{1}{2\sqrt{KT}}, \quad \therefore \zeta^2 = \frac{1}{4KT} \quad ....(2)$$

On equating, equation (1) & (2) we get,

$$\frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$\frac{1}{K} = \frac{4T(\ln M_p)^2}{\pi^2+(\ln M_p)^2}$$

$$K = \frac{\pi^2+(\ln M_p)^2}{4T(\ln M_p)^2}$$

When, $K = K_1$, $M_p = 0.75$, $\therefore K_1 = \frac{\pi^2+(\ln 0.75)^2}{4T(\ln 0.75)^2} = \frac{9.952}{0.331T} = \frac{30.06}{T}$

When, $K = K_2$, $M_p = 0.25$, $\therefore K_2 = \frac{\pi^2+(\ln 0.25)^2}{4T(\ln 0.25)^2} = \frac{11.79}{7.68T} = \frac{1.53}{T}$

$$\therefore \frac{K_1}{K_2} = \frac{(1/T)\,30.06}{(1/T)\,1.53} = 19.6$$

$$K_1 = 19.6\,K_2 \quad \text{(or)} \quad K_2 = \frac{1}{19.6}K_1$$

To reduce peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times (approximately 20 times).

RESULT

The value of gain, K should be reduced approximately 20 times to reduce peak overshoot from 0.75 to 0.25.

EXAMPLE 2.6

A positional control system with velocity feedback is shown in fig 1. What is the response c(t) to the unit step input. Given that $\zeta = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

SOLUTION

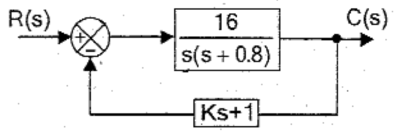


Fig 1

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$

Given that G(s) = 16/s(s + 0.8) and H(s) = Ks +1

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{16}{s\,(s+0.8)}}{1 + \frac{16}{s\,(s+0.8)}\,(Ks+1)} = \frac{16}{s\,(s+0.8) + 16(Ks+1)}$$

$$= \frac{16}{s^2 + 0.8s + 16Ks + 16} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

The values of K and ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{16}{s^2 + (0.8 + 16K)\,s + 16}$$

On comparing we get.

$$\omega_n^2 = 16 \qquad\qquad 0.8 + 16K = 2\zeta\omega_n$$

$$\therefore \omega_n = 4 \text{ rad/sec} \qquad \therefore K = \frac{2\zeta\omega_n - 0.8}{16} = \frac{2 \times 0.5 \times 4 - 0.8}{16} = 0.2$$

$$\therefore \frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16 \times 0.2)\,s + 16} = \frac{16}{s^2 + 4s + 16}$$

Given that the damping ratio, $\zeta = 0.5$. Hence the system is underdamped and so the response of the system will have damped oscillations. The roots of characteristic polynomial will be complex conjugate.

The response in s - domain, $C(s) = R(s)\,\frac{16}{s^2 + 4s + 16}$

For unit step input, R(s) = 1/s.

$$\therefore C(s) = \frac{1}{s}\,\frac{16}{s^2 + 4s + 16} = \frac{16}{s\,(s^2 + 4s + 16)}$$

By partial fraction expansion we can write,

$$C(s) = \frac{16}{s\,(s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

The residue A is obtained by multiplying C(s) by s and letting s = 0.

$$A = C(s) \times s\Big|_{s=0} = \frac{16}{s^2 + 4s + 16}\Big|_{s=0} = \frac{16}{16} = 1$$

The residues B and C are evaluated by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{16}{s\,(s^2 + 4s + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 16}$$

On cross multiplication we get, $16 = A\,(s^2 + 4s + 16) + (Bs + C)\,s$

$$16 = As^2 + 4As + 16A + Bs^2 + Cs$$

On equating the coefficients of s^2 we get, $0 = A + B \quad \therefore\ B = -A = -1$

On equating the coefficients of s we get, $0 = 4A + C \ \therefore\ C = -4A = -4$

$$\therefore\ C(s) = \frac{1}{s} + \frac{-s-4}{s^2+4s+16} = \frac{1}{s} - \frac{s+4}{s^2+4s+4+12}$$

$$= \frac{1}{s} - \frac{s+2+2}{(s+2)^2+12} = \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

The response in time domain,

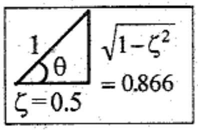
$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{\sqrt{12}}\ \frac{\sqrt{12}}{(s+2)^2+12}\right\}$$

$$= 1 - e^{-2t}\cos\sqrt{12}\,t - \frac{2}{2\sqrt{3}}\ e^{-2t}\sin\sqrt{12}\,t$$

$$= 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$$

The result can be converted to another standard form by constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$.

On constructing right angle triangle with ζ and $\sqrt{1-\zeta^2}$ we get,



$$\sin\theta = 0.866 = \sqrt{3}/2\ ;\quad \cos\theta = 0.5 \ = 1/2\ ;\ \tan\theta = 1.732$$

$$\therefore\ \theta = \tan^{-1}1.732 = 60° = 1.047 \text{ rad}$$

$\therefore$ The response in time domain,

$$c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}} \times 2 \times \sin\sqrt{12}\,t \times \frac{1}{2} + \frac{2}{\sqrt{3}} \times \cos\sqrt{12}\,t \times \frac{\sqrt{3}}{2}\right]$$

$$= 1 - e^{-2t}\frac{2}{\sqrt{3}}\left[\sin\sqrt{12}\,t\cos\theta + \cos\sqrt{12}\,t\sin\theta\right]$$

Note : θ is expressed in radians.

$$= 1 - \frac{2}{\sqrt{3}}\ e^{-2t}\left[\sin(\sqrt{12}\,t + \theta)\right] = 1 - \frac{2}{\sqrt{3}}e^{-2t}\left[\sin(\sqrt{12}\,t + 1.047)\right]$$

Damped frequency of oscillation $\left.\right\}\ \omega_d = \omega_n\sqrt{1-\zeta^2} = 4\sqrt{1-0.5^2} = 3.464$ rad / sec

$$\therefore\ \text{Rise time, } t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1.047}{3.464} = 0.6046 \text{ sec}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.464} = 0.907 \text{ sec}$$

% Maximum overshoot $\left.\right\}\ \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.5\times\pi}{\sqrt{1-0.5^2}}} \times 100 = 0.163 \times 100 = 16.3\%$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.5 \times 4} = 0.5 \text{ sec}$$

For 5% error, Settling time, $t_s = 3T = 3 \times 0.5 = 1.5$ sec

For 2% error, Settling time, $t_s = 4T = 4 \times 0.5 = 2$ sec

RESULT

The time domain response, $c(t) = 1 - e^{-2t}\left[\frac{1}{\sqrt{3}}\sin(\sqrt{12}\,t) + \cos(\sqrt{12}\,t)\right]$

(or) $c(t) = 1 - \frac{2}{\sqrt{3}}e^{-2t}\left[\sin(\sqrt{12}\,t + 1.047)\right]$

Rise time, $t_r = 0.6046$ sec
Peak time, $t_p = 0.907$ sec
% Maximum overshoot, $\%M_p = 16.3\%$
Settling time, $t_s = 1.5$ sec, for 5% error
$= 2$ sec, for 2% error

EXAMPLE 2.7

A unity feedback control system is characterized by the following open loop transfer function G(s) = (0.4 s +1)/s(s +0.6). Determine its transient response for unit step input and sketch the response. Evaluate the maximum overshoot and the corresponding peak time.

SOLUTION

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Given that, G(s) = (0.4 s +1)/s(s +0.6)

For unity feedback system, H(s) = 1.

$$\therefore \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{0.4\,s+1}{s(s+0.6)}}{1+\frac{0.4\,s+1}{s(s+0.6)}} = \frac{0.4\,s+1}{s\,(s+0.6)+0.4\,s+1}$$

$$= \frac{0.4\,s+1}{s^2+0.6s+0.4s+1} = \frac{0.4\,s+1}{s^2+s+1}$$

The s - domain response, $C(s) = R(s) \times \frac{0.4\,s+1}{s^2+s+1}$

For step input, R(s) = 1/s.

$$\therefore C(s) = \frac{1}{s}\,\frac{0.4\,s+1}{s^2+s+1} = \frac{0.4\,s+1}{s(s^2+s+1)}$$

By partial fraction expansion C(s) can be expressed as,

$$C(s) = \frac{0.4\,s+1}{s\,(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

The residue A is solved by multiplying C(s) by s and letting s = 0.

$$\therefore A = C(s) \times s\Big|_{s=0} = \frac{0.4\,s+1}{s^2+s+1}\Bigg|_{s=0} = 1$$

The residues B and C are solved by cross multiplying the following equation and equating the coefficients of like powers of s.

$$\frac{0.4\,s+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

On cross multiplication we get,

$$0.4\,s+1 = A(s^2+s+1) + (Bs+C)\,s$$

$$0.4\,s+1 = As^2 + As + A + Bs^2 + Cs$$

On equating coefficients of s^2 we get, $0 = A + B$ $\quad\therefore B = -A = -1$

On equating coefficients of s we get, $0.4 = A + C$ $\quad\therefore C = 0.4 - A = -0.6$

$$\therefore C(s) = \frac{1}{s} + \frac{-s-0.6}{s^2+s+1} = \frac{1}{s} - \frac{s+0.6}{s^2+s+0.25+0.75} = \frac{1}{s} - \frac{s+0.6}{\left(s^2+2\times 0.5s+0.5^2\right)+0.75}$$

$$= \frac{1}{s} - \frac{s+0.5+0.1}{(s+0.5)^2+0.75} = \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\;\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}$$

The time domain response is obtained by taking inverse Laplace transform of C(s).

∴ The response in time domain,

$$c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\frac{\sqrt{0.75}}{(s+0.5)^2+0.75}\right\}$$

$$= 1 - e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}}\,e^{-0.5t}\sin\sqrt{0.75}\,t$$

$$= 1 - e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$$

The transient response is the part of the output which vanishes as t tends to infinity. Here as t tends to infinity the exponential component $e^{-0.5t}$ tends to zero. Hence the transient response is given by the damped sinusoidal component.

The transient response of $c(t) = e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$

The value of ζ and ω_n can be estimated by comparing the characteristic equation of the system with standard form of second order characteristic equation.

$$\therefore s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + s + 1$$

On comparing we get,

$$\omega_n^2 = 1 \qquad \therefore \omega_n = 1\ \text{rad/sec}$$

$$2\zeta\omega_n = 1 \qquad \therefore \zeta = \frac{1}{2\omega_n} = \frac{1}{2} = 0.5$$

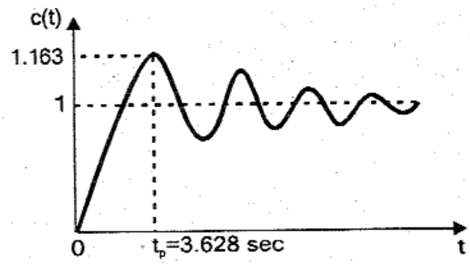


Fig 1 : Response of under damped system.

Maximum overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 0.163$

% Maximum overshoot, $\%M_p = M_p \times 100 = 0.163 \times 100 = 16.3\%$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{1\times\sqrt{1-0.5^2}} = 3.628$ sec

The response of the system is underdamped and it is shown in fig 1.

RESULT

Transient response of the system, $c(t) = e^{-0.5t}\left[0.1155\sin(\sqrt{0.75}\,t) + \cos(\sqrt{0.75}\,t)\right]$

% Maximum peak overshoot, $\%M_p$ = 16.3 %

Peak time, t_p = 3.628 sec

EXAMPLE 2.8

A unity feedback control system has an amplifier with gain K_A = 10 and gain ratio, G(s) = 1/s(s +2) in the feed forward path. A derivative feedback, H(s) = sK_o is introduced as a minor loop around G(s). Determine the derivative feedback constant, K_o so that the system damping factor is 0.6.

SOLUTION

The given system can be represented by the block diagram shown in fig 1.

Here, $K_A = 10$; $G(s) = \frac{1}{s(s+2)}$ and $H(s) = sK_o$

The closed loop transfer function of the system can be obtained by block diagram reduction techniques.

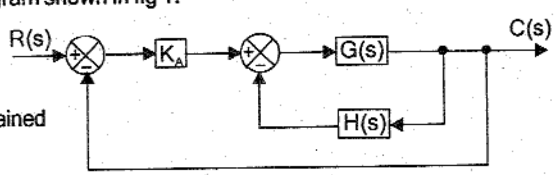
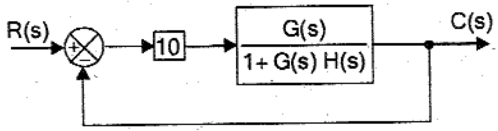
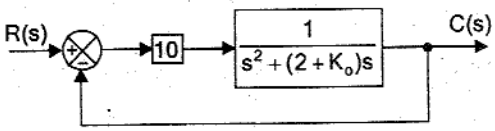


Fig 1.

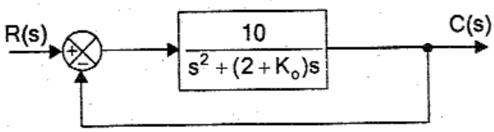
Step 1: Reducing the inner feedback loop.



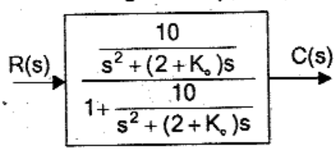
$$\frac{G(s)}{1+G(s)\,H(s)} = \frac{\frac{1}{s\,(s+2)}}{1+\frac{1}{s\,(s+2)}\,sK_o} = \frac{1}{s\,(s+2)+sK_o} = \frac{1}{s^2+2s+sK_o} = \frac{1}{s^2+(2+K_o)\,s}$$

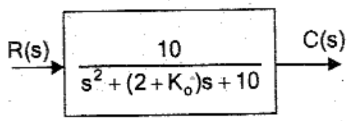


Step 2 : Combining blocks in cascade



Step 3 : Reducing the unity feedback path





The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{10}{s^2+(2+K_o)\,s+10}$ .....(1)

The given system is a second order system. The value of K_o can be determined by comparing the system transfer function with standard form of second order transfer function given below.

Standard form of Second order transfer function $\Big\}\ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ .....(2)

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10$$

$$\therefore\ \omega_n = \sqrt{10} = 3.162 \text{ rad / sec}$$

$$2+K_o = 2\zeta\omega_n$$

$$\therefore\ K_o = 2\zeta\omega_n - 2$$

$$= 2\times 0.6\times 3.162 - 2 = 1.7944$$

RESULT

The value of constant, $K_o = 1.7944$

EXAMPLE 2.9

A unity feedback control system has an open loop transfer function, G(s) = 10/s(s+2). Find the rise time, percentage overshoot, peak time and settling time for a step input of 12 units.

SOLUTION

Note : The formulae for rise time, percentage overshoot and peak time remains same for unit step and step input.

The unity feedback system is shown in fig 1.

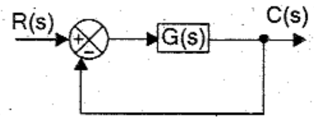


Fig 1 : Unity feedback system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

The closed loop transfer function,

Given that, G(s) = 10/s (s+2)

$$\therefore \frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s(s+2)+10} = \frac{10}{s^2+2s+10} \quad .....(1)$$

The values of damping ratio ζ and natural frequency of oscillation ω_n are obtained by comparing the system transfer function with standard form of second order transfer function.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\} \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad .....(2)$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 10 \qquad\qquad 2\zeta\omega_n = 2$$

$$\therefore \omega_n = \sqrt{10} = 3.162 \text{ rad / sec} \qquad \therefore \zeta = \frac{2}{2\omega_n} = \frac{1}{3.162} = 0.316$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-0.316^2}}{0.316} = 1.249 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.162\sqrt{1-0.316^2} = 3 \text{ rad / sec}$$

$$\text{Rise time, } t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.249}{3} = 0.63 \text{ sec}$$

$$\text{Percentage overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = e^{\frac{-0.316\pi}{\sqrt{1-0.316^2}}} \times 100$$

$$= 0.3512 \times 100 = 35.12\%$$

$$\text{Peak overshoot} = \frac{35.12}{100} \times 12 \text{ units} = 4.2144 \text{ units}$$

$$\text{Peak time, } t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.047 \text{ sec}$$

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.316 \times 3.162} = 1 \text{ sec}$$

$\therefore$ For 5% error, Settling time, $t_s = 3T = 3$ sec

For 2% error, Settling time, $t_s = 4T = 4$ sec

RESULT

Rise time, t_r	=	0.63 sec
Percentage overshoot, $\%M_p$	=	35.12%
Peak overshoot	=	4.2144 units, (for a input of 12 units)
Peak time, t_p	=	1.047 sec
Settling time, t_s	=	3 sec for 5% error
	=	4 sec for 2% error

EXAMPLE 2.10

A closed loop servo is represented by the differential equation $\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\,e$

Where c is the displacement of the output shaft, r is the displacement of the input shaft and e = r – c. Determine undamped natural frequency, damping ratio and percentage maximum overshoot for unit step input.

SOLUTION

The mathematical equations governing the system are,

$$\frac{d^2c}{dt^2}+8\frac{dc}{dt}=64\,e \qquad .....(1)$$

$$e = r - c \qquad .....(2)$$

Put $e = r - c$ in equation (1),

$$\therefore \frac{d^2c}{dt^2}+8\frac{dc}{dt}=64(r-c) \qquad .....(3)$$

Let $\mathcal{L}\{c\} = C(s)$ and $\mathcal{L}\{r\} = R(s)$

On taking Laplace transform of equation (3) we get,

$$s^2 C(s) + 8s\,C(s) = 64\,[R(s) - C(s)]$$

$$\therefore s^2 C(s) + 8s\,C(s) + 64\,C(s) = 64\,R(s)$$

$$(s^2 + 8s + 64)\,C(s) = 64\,R(s)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{64}{s^2+8s+64} \qquad .....(4)$$

The ratio C(s)/R(s) is the closed loop transfer function of the system. On comparing the system transfer function with standard form of second order transfer function, we can estimate the values of ζ and ω_n.

$$\left.\begin{array}{l}\text{Standard form of}\\ \text{Second order transfer function}\end{array}\right\}\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad .....(5)$$

On comparing equation (1) & (2) we get,

$$\omega_n^2 = 64 \qquad\qquad 2\zeta\omega_n = 8$$

$$\therefore \omega_n = 8\text{ rad / sec} \qquad\qquad \zeta = \frac{8}{2\omega_n} = \frac{8}{2\times 8} = 0.5$$

$$\text{Percentage peak overshoot, } \%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\times 100 = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}}\times 100 = 16.3\%$$

RESULT

Undamped natural frequency of oscillation, ω_n = 8 rad/sec

Damping ratio, ζ = 0.5

Percentage peak overshoot, $\%M_p$ = 16.3%

2.9 TYPE NUMBER OF CONTROL SYSTEMS

The type number is specified for loop transfer function G(s) H(s). The number of poles of the loop transfer function lying at the origin decides the type number of the system. In general, if **N** is the number of poles at the origin then the type number is **N.**

The loop transfer function can be expressed as a ratio of two polynomials in **s.**

$$G(s)\,H(s) = K\frac{P(s)}{Q(s)} = K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)\,\ldots\ldots\ldots\ldots}{s^N\,(s+p_1)\,(s+p_2)\,(s+p_3)\,\ldots\ldots} \quad .....(2.49)$$

where, $z_1, z_2, z_3, \ldots\ldots\ldots\ldots$ are zeros of transfer function

$p_1, p_2, p_3, \ldots\ldots\ldots\ldots$ are poles of transfer function

K = Constant

N = Number of poles at the origin

The value of **N** in the denominator polynomial of loop transfer function shown in equation (2.49) decides the type number of the system.

If N = 0, then the system is type – 0 system

If N = 1, then the system is type – 1 system

If N = 2, then the system is type – 2 system

If N = 3, then the system is type – 3 system and so on.

2.10 STEADY STATE ERROR

The steady state error is the value of error signal e(t), when **t** tends to infinity. The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non linearity of system components. The steady state performance of a stable control system is generally judged by its steady state error to step, ramp and parabolic inputs.

Consider a closed loop system shown in fig 2.15.

Let, R(s) = Input signal

E(s) = Error signal

C(s) H(s) = Feedback signal

C(s) = Output signal or response

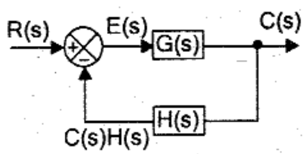


Fig 2.15.

The error signal, E(s) = R(s) – C(s) H(s) .....(2.50)

The output signal, C(s) = E(s) G(s) .....(2.51)

On substituting for C(s) from equation (2.51) in equation (2.50) we get,

$$E(s) = R(s) - [E(s)\,G(s)]\,H(s)$$

$$E(s) + E(s)\,G(s)\,H(s) = R(s)$$

$$E(s)\,[1 + G(s)\,H(s)] = R(s)$$

$$\therefore E(s) = \frac{R(s)}{1+G(s)\,H(s)} \quad .....(2.52)$$

Let, e(t) = error signal in time domain.

$$\therefore e(t) = \mathcal{L}^{-1}\{E(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)}{1+G(s)\,H(s)}\right\} \qquad .....(2.53)$$

Let, e_{ss} = steady state error.

The steady state error is defined as the value of e(t) when **t** tends to infinity.

$$\therefore e_{ss} = \underset{t\to\infty}{Lt}\; e(t) \qquad .....(2.54)$$

The final value theorem of Laplace transform states that,

$$\textbf{If,}\quad \mathbf{F(s) = \mathcal{L}\{f(t)\}}\ \textbf{then,}\quad \underset{t\to\infty}{\mathbf{Lt}}\ \mathbf{f(t)} = \underset{s\to 0}{\mathbf{Lt}}\ \mathbf{s\,F(s)} \qquad .....(2.55)$$

Using final value theorem,

$$\text{The steady state error, } e_{ss} = \underset{t\to\infty}{Lt}\; e(t) = \underset{s\to 0}{Lt}\; sE(s) = \underset{s\to 0}{Lt}\;\frac{sR(s)}{1+G(s)\,H(s)} \qquad .....(2.56)$$

2.11 STATIC ERROR CONSTANTS

When a control system is excited with standard input signal, the steady state error may be zero, constant or infinity. The value of steady state error depends on the type number and the input signal. Type-0 system will have a constant steady state error when the input is step signal. Type-1 system will have a constant steady state error when the input is ramp signal or velocity signal. Type-2 system will have a constant steady state error when the input is parabolic signal or acceleration signal. For the three cases mentioned above the steady state error is associated with one of the constants defined as follows,

$$\text{Positional error constant,}\quad K_p = \underset{s\to 0}{Lt}\; G(s)\,H(s) \qquad .....(2.57)$$

$$\text{Velocity error constant,}\quad K_v = \underset{s\to 0}{Lt}\; s\,G(s)\,H(s) \qquad .....(2.58)$$

$$\text{Acceleration error constant,}\quad K_a = \underset{s\to 0}{Lt}\; s^2 G(s)\,H(s) \qquad .....(2.59)$$

The K_p, K_v and K_a are in general called static error constants.

2.12 STEADY STATE ERROR WHEN THE INPUT IS UNIT STEP SIGNAL

$$\text{Steady state error, } e_{ss} = \underset{s\to 0}{Lt}\;\frac{sR(s)}{1+G(s)\,H(s)}$$

When the input is unit step, $R(s) = 1/s$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\;\frac{s\frac{1}{s}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\;\frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\underset{s\to 0}{Lt}\;G(s)\,H(s)} = \frac{1}{1+K_p} \qquad .....(2.60)$$

$$\text{where, } K_p = \underset{s\to 0}{Lt}\; G(s)\,H(s)$$

The constant K_p is called ***positional error constant.***

Type-0 system

$$K_p = \underset{s\to 0}{Lt}\; G(s)\,H(s) = \underset{s\to 0}{Lt}\; K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \text{constant}$$

Hence in type-0 systems when the input is unit step there will be a constant steady state error.

Type-1 system

$$K_p = \underset{s\to 0}{Lt}\, G(s)\, H(s) = \underset{s\to 0}{Lt}\; K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

In systems with type number 1 and above, for unit step input the value of K_p is infinity and so the steady state error is zero.

2.13 STEADY STATE ERROR WHEN THE INPUT IS UNIT RAMP SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{Lt}\; \frac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit ramp, $R(s) = \frac{1}{s^2}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\; \frac{s\frac{1}{s^2}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt}\; \frac{1}{s+sG(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\; sG(s)\,H(s)} = \frac{1}{K_v} \qquad .....(2.61)$$

where, $K_v = \underset{s\to 0}{Lt}\; s\,G(s)\,H(s)$

The constant K_v is called ***velocity error constant.***

Type-0 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\, H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = 1/K_v = 1/0 = \infty$$

Hence in type-0 systems when the input is unit ramp, the steady state error is infinity.

Type-1 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\, H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3......} = \text{constant}$$

$$\therefore e_{ss} = 1/K_v = \text{constant}$$

Hence in type-1 systems when the input is unit ramp there will be a constant steady state error.

Type-2 system

$$K_v = \underset{s\to 0}{Lt}\, sG(s)\, H(s) = \underset{s\to 0}{Lt}\; sK\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = 1/K_v = 1/\infty = 0$$

In systems with type number 2 and above, for unit ramp input, the value of K_v is infinity so the steady state error is zero.

2.14 STEADY STATE ERROR WHEN THE INPUT IS UNIT PARABOLIC SIGNAL

Steady state error, $e_{ss} = \underset{s\to 0}{Lt} \dfrac{sR(s)}{1+G(s)\,H(s)}$

When the input is unit parabola, $R(s) = \dfrac{1}{s^3}$

$$\therefore e_{ss} = \underset{s\to 0}{Lt} \frac{s\frac{1}{s^3}}{1+G(s)\,H(s)} = \underset{s\to 0}{Lt} \frac{1}{s^2+s^2\,G(s)\,H(s)} = \frac{1}{\underset{s\to 0}{Lt}\; s^2G(s)\,H(s)} = \frac{1}{K_a} \qquad .....(2.62)$$

where, $K_a = \underset{s\to 0}{Lt}\; s^2G(s)\,H(s)$

The constant K_a is called ***acceleration error constant.***

Type-0 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-0 systems for unit parabolic input, the steady state error is infinity.

Type-1 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = 0$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Hence in type-1 systems for unit parabolic input, the steady state error is infinity.

Type-2 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^2(s+p_1)\,(s+p_2)\,(s+p_3)......} = K\frac{z_1.z_2.z_3......}{p_1.p_2.p_3.....} = \text{constant}$$

$$\therefore e_{ss} = \frac{1}{K_a} = \text{constant}$$

Hence in type-2 system when the input is unit parabolic signal there will be a constant steady state error.

Type-3 system

$$K_a = \underset{s\to 0}{Lt}\, s^2G(s)\,H(s) = \underset{s\to 0}{Lt}\; s^2K\frac{(s+z_1)\,(s+z_2)\,(s+z_3)......}{s^3\,(s+p_1)\,(s+p_2)\,(s+p_3)......} = \infty$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0$$

In systems with type number 3 and above for unit parabolic input the value of K_a is infinity and so the steady state error is zero.

TABLE-2.2 : Static Error Constant for Various Type Number of Systems

Error Constant	Type number of system			
	0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

TABLE-2.3 : Steady State Error for Various Types of Inputs

Input Signal	Type number of system			
	0	1	2	3
Unit Step	$\frac{1}{1+K_p}$	0	0	0
Unit Ramp	∞	$\frac{1}{K_v}$	0	0
Unit Parabolic	∞	∞	$\frac{1}{K_a}$	0

2.15 GENERALIZED ERROR COEFFICIENT

The drawback in static error coefficients is that it does not show the variation of error with time and input should be a standard input. The generalized error coefficients gives the steady state error as a function of time. Also using the generalized error coefficients, the steady state error can be found for any type of input.

The error signal in s-domain, E(s) can be expressed as a product of two s-domain functions.

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)} = \frac{1}{1+G(s)\,H(s)}R(s) = F(s)\,R(s) \quad .....(2.63)$$

$$\text{where, } F(s) = \frac{1}{1+G(s)\,H(s)}$$

Let, $e(t) = \mathcal{L}^{-1}\{E(s)\}$ (error signal in time domain)

$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$r(t) = \mathcal{L}^{-1}\{R(s)\}$ (input signal in time domain)

The convolution theorem of Laplace transform states that the Laplace transform of the convolution of two time domain signals is equal to the product of their individual Laplace transform.

i.e., $\mathcal{L}\{f(t) * r(t)\} = F(s)\,R(s)$

where $*$ is the symbol for convolution operation

$$\therefore \mathcal{L}^{-1}\{F(s)\,R(s)\} = f(t) * r(t) \quad .....(2.64)$$

From equation (2.63) & (2.64) we can write,

$e(t) = f(t) * r(t)$

Mathematically the convolution of f(t) and r(t) is defined as,

$$f(t) * r(t) = \int_{-\infty}^{+\infty} f(T)\,r(t-T)\,dT \quad ; \quad \text{where T is a dummy variable}$$

$$\therefore \; e(t) = \int_{-\infty}^{+\infty} f(T)\,r(t-T)\,dT$$

It is assumed that the input signal starts only at t = 0 and does not exist before t = 0. Also we are interested in finding error signal at any time t after t = 0 (i.e., for t > 0). Hence in the above equation the limit of integral can be changed as 0 to t.

$$\therefore e(t) = \int_{0}^{t} f(T)\,r(t-T)\,dT$$

Using Taylor's series expansion the signal r(t–T) can be expressed as,

$$r(t-T) = r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) + ...+(-1)^n \frac{T^n}{n!}\overset{n.}{r}(t)...$$

where, $\dot{r}(t) = 1^{st}$ derivative of r(t)

$\ddot{r}(t) = 2^{nd}$ derivative of r(t)

⋮

$\overset{n.}{r}(t) = n^{th}$ derivative of r(t)

On substituting the Taylor's series expansion of r(t – T), the error e(t) can be written as,

$$e(t) = \int_0^t f(T)\left[r(t) - T\,\dot{r}(t) + \frac{T^2}{2!}\ddot{r}(t) - \frac{T^3}{3!}\dddot{r}(t) + ...+(-1)^n \frac{T^n}{n!}\overset{n.}{r}(t)...\right] dT$$

$$e(t) = \int_0^t f(T)\,r(t)\,dT - \int_0^t f(T)\,T\,\dot{r}(t)\,dT + \int_0^t f(T)\frac{T^2}{2!}\ddot{r}(t)\,dT$$

$$-\int_0^t f(T)\frac{T^3}{3!}\dddot{r}(t) + ...+\int_0^t f(T)\,(-1)^n\frac{T^n}{n!}\overset{n.}{r}(t)\,dT....\infty$$

Since $r(t), \dot{r}(t), \ddot{r}(t), ... \overset{n.}{r}(t)$ are constants when the integration is done with respect to T, the error signal can be written as,

$$e(t) = r(t)\int_0^t f(T)\,dT - \dot{r}(t)\int_0^t Tf(T)\,dt + \frac{\ddot{r}(t)}{2!}\int_0^t T^2 f(T)\,dt$$

$$-\frac{\dddot{r}(t)}{3!}\int_0^t T^3 f(T)\,dt + ...+(-1)^n\frac{\overset{n.}{r}(t)}{n!}\int_0^t T^n f(T)\,dt...$$

Let, $C_0 = +\int_0^t f(T)\,dT$ $\qquad C_3 = -\int_0^t T^3 f(T)\,dT$

$C_1 = -\int_0^t Tf(T)\,dT$ $\qquad$ ⋮

$C_2 = +\int_0^t T^2 f(T)\,dT$ $\qquad C_n = (-1)^n\int_0^t T^n f(T)\,dT$

$$e(t) = r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\,\frac{C_2}{2!} + \dddot{r}(t)\,\frac{C_3}{3!} + ...+\overset{n.}{r}(t)\,\frac{C_n}{n!} + ...$$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + ...+\frac{C_n}{n!}\overset{n.}{r}(t)..... \qquad .....(2.65)$$

The equation (2.65) is the general equation for error signal, e(t).

The coefficients C_0, C_1, C_2 ,.......C_n are called the generalized error coefficients or dynamic error coefficients.

The steady state error e_{ss} is obtained by taking limit $t \to \infty$ on e(t).

$\therefore$ Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\left[r(t)\,C_0 + \dot{r}(t)\,C_1 + \ddot{r}(t)\,\frac{C_2}{2!} + \dddot{r}(t)\,\frac{C_3}{3!} + ...+\overset{n.}{r}(t)\,\frac{C_n}{n!} + ...\right]$

$$= C_0\,r(t) + C_1\,\dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + ...+\frac{C_n}{n!}\overset{n.}{r}(t)..... \qquad .....(2.66)$$

2.16 EVALUATION OF GENERALIZED ERROR COEFFICIENTS

The generalized error coefficient is given by,

$$C_n = (-1)^n \int_0^t T^n f(T)\, dT \; ; \qquad \text{where } F(s) = \frac{1}{1 + G(s)\,H(s)}$$

We know that $\mathcal{L}\{f(T)\} = F(s)$, hence by the definition of Laplace transform,

$$F(s) = \int_0^t f(T)\, e^{-sT} dT \qquad .....(2.67)$$

On taking $\underset{s \to 0}{Lt}$ on both sides of equation (2.67) we get,

$$\underset{s \to 0}{Lt}\, F(s) = \underset{s \to 0}{Lt} \int_0^t f(T)\, e^{-sT} dT$$

$$= \int_0^t f(T)\, \underset{s \to 0}{Lt}\, e^{-sT} dT = \int_0^t f(T)\, dT = C_0$$

$$\therefore \boxed{\mathbf{C_0 = \underset{s \to 0}{Lt}\, F(s)}} \qquad .....(2.68)$$

On differentiating equation (2.68) with respect to s we get,

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^t f(T)\, e^{-sT}\, dT$$

$$= \int_0^t f(T)\, \frac{d}{ds}(e^{-sT}) dT = \int_0^t f(T)\, (-T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T)\, e^{-sT} dT \qquad .....(2.69)$$

On taking $\underset{s \to 0}{Lt}$ on both sides of equation (2.69) we get,

$$\underset{s \to 0}{Lt}\, \frac{d}{ds} F(s) = \underset{s \to 0}{Lt} \; -\int_0^t Tf(T)\, e^{-sT} dT$$

$$= -\int_0^t Tf(T)\, \underset{s \to 0}{Lt}\, e^{-sT} dT = -\int_0^t Tf(T)\, dT = C_1$$

$$\therefore \boxed{\mathbf{C_1 = \underset{s \to 0}{Lt}\, \frac{d}{ds} F(s)}} \qquad .....(2.70)$$

On differentiating equation (2.68) on both sides with respect to s we get,

$$\frac{d}{ds}\left[\frac{d}{ds}(F(s))\right] = \frac{d}{ds}\left[-\int_0^t Tf(T)\, e^{-sT}\, dT\right]$$

$$\frac{d^2}{ds^2} F(s) = \left[-\int_0^t Tf(T)\, \frac{d}{ds}(e^{-sT}) dT\right] = -\int_0^t Tf(T)\, (-T)\, e^{-sT} dT$$

$$\frac{d^2 (F(s))}{ds^2} = \int_0^t T^2 f(T)\, e^{-sT} dT \qquad .....(2.71)$$

Applying the limit $s \to 0$ on both sides of the equation (2.71) we get,

$$\underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\int_0^t T^2\, f(T)\, e^{-st}\, dT$$

$$= \int_0^t T^2\, f(T)\, \underset{s\to 0}{Lt}\, e^{-st}\, dT = \int_0^t T^2\, f(T)\, dT = C_2$$

$$\therefore \boxed{C_2 = \underset{s\to 0}{Lt}\frac{d^2}{ds^2}F(s)} \quad .....(2.72)$$

Similarly it can be shown that,

$$\boxed{C_n = \underset{s\to 0}{Lt}\frac{d^n}{ds^n}F(s)} \quad .....(2.73)$$

2.17 CORRELATION BETWEEN STATIC AND DYNAMIC ERROR COEFFICIENTS

The values of dynamic error coefficients can be used to calculate static error coefficients. The following expressions shows the relationship between them.

$$C_0 = \frac{1}{1+K_p} \quad .....(2.74)$$

$$C_1 = \frac{1}{K_v} \quad .....(2.75)$$

$$C_2 = \frac{1}{K_a} \quad .....(2.76)$$

Proof

$$C_0 = \underset{s\to 0}{Lt}\, F(s) = \underset{s\to 0}{Lt}\frac{1}{1+G(s)\,H(s)} = \frac{1}{1+\underset{s\to 0}{Lt}\, G(s)\,H(s)} = \frac{1}{1+K_p}$$

2.18 ALTERNATE METHOD FOR GENERALIZED ERROR COEFFICIENTS

The error signal in s-domain, $E(s) = \frac{R(s)}{1+G(s)\,H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} \quad .....(2.77)$$

The equation (2.77) can be expressed as a power series of s as shown in equation (2.78).

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)\,H(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + ... \quad .....(2.78)$$

$$\therefore E(s) = C_0\, R(s) + C_1 s R(s) + \frac{C_2}{2!}s^2\, R(s) + \frac{C_3}{3!}s^3\, R(s) + ... \quad .....(2.79)$$

On taking inverse Laplace transform of equation (2.79) we get,

$$e(t) = C_0\, r(t) + C_1 s r(t) + \frac{C_2}{2!}s^2\, r(t) + \frac{C_3}{3!}s^3\, r(t) + ... \quad .....(2.80)$$

The equation (2.80) is same as that of equation (2.65) in section 2.14.This method will be useful to find the generalized error coefficients without using differentiation, but using laplace transform.

EXAMPLE 2.11

For a unity feedback control system the open loop transfer function, $G(s)=\frac{10(s+2)}{s^2(s+1)}$. Find

a) the position, velocity and acceleration error constants,

b) the steady state error when the input is R(s), where $R(s)=\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}$

SOLUTION

a) To find static error constants

For a unity feedback system, H(s)=1

Position error constant, $K_p=\underset{s\to0}{Lt}\,G(s)H(s)=\underset{s\to0}{Lt}\,G(s)=\underset{s\to0}{Lt}\,\frac{10(s+2)}{s^2(s+1)}=\infty$

Velocity error constant, $K_v=\underset{s\to0}{Lt}\,s\,G(s)H(s)=\underset{s\to0}{Lt}\,s\,G(s)=\underset{s\to0}{Lt}\,s\frac{10(s+2)}{s^2(s+1)}=\infty$

Acceleration error constant, $K_a=\underset{s\to0}{Lt}\,s^2G(s)H(s)=\underset{s\to0}{Lt}\,s^2G(s)$

$$=\underset{s\to0}{Lt}\,s^2\frac{10(s+2)}{s^2(s+1)}=\frac{10\times2}{1}=20$$

b) To find steady state error

Method-I

Steady state error for non-standard input is obtained using generalized error series, given below.

The error signal, $e(t)=r(t)C_0+\dot{r}(t)C_1+\ddot{r}(t)\frac{C_2}{2!}+......+\overset{n}{r}(t)\frac{C_n}{n!}+....$

Given that, $R(s)=\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}$

Input signal in time domain, $r(t)=\mathcal{L}^{-1}\{R(s)\}=\mathcal{L}^{-1}\left\{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}\right\}$

$$=3-2t+\frac{1}{3}\frac{t^2}{2!}=3-2t+\frac{t^2}{6}$$

$$\therefore\dot{r}(t)=\frac{d}{dt}r(t)=-2+\frac{1}{6}2t=-2+\frac{t}{3}$$

$$\ddot{r}(t)=\frac{d^2}{dt^2}r(t)=\frac{d}{dt}\dot{r}(t)=\frac{1}{3}$$

$$\dddot{r}(t)=\frac{d^3}{dt^3}r(t)=\frac{d}{dt}\ddot{r}(t)=0$$

The derivatives of r(t) is zero after second derivative. Hence we have to evaluate only three constants C_0, C_1 and C_2. The generalized error constants are given by,

$$C_0=\underset{s\to0}{Lt}\,F(s)\,;\qquad C_1=\underset{s\to0}{Lt}\,\frac{d}{ds}F(s)\,;\qquad C_2=\underset{s\to0}{Lt}\,\frac{d^2}{ds^2}F(s)$$

$$F(s)=\frac{1}{1+G(s)H(s)}=\frac{1}{1+G(s)}=\frac{1}{1+\frac{10(s+2)}{s^2(s+1)}}=\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}=\frac{s^3+s^2}{s^3+s^2+10s+20}$$

$$C_0=\underset{s\to0}{Lt}\,F(s)=\underset{s\to0}{Lt}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right]=0$$

$$C_1 = \underset{s\to0}{Lt}\frac{d}{ds}F(s) = \underset{s\to0}{Lt}\frac{d}{ds}\left[\frac{s^3+s^2}{s^3+s^2+10s+20}\right]$$

$$= \underset{s\to0}{Lt}\left[\frac{(s^3+s^2+10s+20)(3s^2+2s)-(s^3+s^2)(3s^2+2s+10)}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to0}{Lt}\left[\frac{3s^5+2s^4+3s^4+2s^3+30s^3+20s^2+60s^2+40s-3s^5-2s^4-10s^3-3s^4-2s^3-10s^2}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to0}{Lt}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right] = 0$$

$$C_2 = \underset{s\to0}{Lt}\frac{d^2}{ds^2}F(s) = \underset{s\to0}{Lt}\frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to0}{Lt}\frac{d}{ds}\left[\frac{20s^3+70s^2+40s}{(s^3+s^2+10s+20)^2}\right]$$

$$= \underset{s\to0}{Lt}\left[\frac{(s^3+s^2+10s+20)^2(60s^2+140s+40) - (20s^3+70s^2+40s)\,2\times(s^3+s^2+10s+20)\,(3s^2+2s+10)}{(s^3+s^2+10s+20)^4}\right] = \frac{20^2\times 40}{20^4} = \frac{1}{10}$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} = \left(3-2t+\frac{t^2}{6}\right)\times 0 + \left(-2+\frac{t}{3}\right)\times 0 + \frac{1}{3}\times\frac{1}{10}\times\frac{1}{2!} = \frac{1}{60}$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\frac{1}{60} = \frac{1}{60}$

Method - II

The error signal in s-domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Given that, $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$; $G(s) = \frac{10(s+2)}{s^2(s+1)}$; $H(s) = 1$

$$\therefore E(s) = \frac{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}}{1+\frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}}{\frac{s^2(s+1)+10(s+2)}{s^2(s+1)}}$$

$$= \frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{s\to0}{Lt}\, s\,E(s)$

$$\therefore e_{ss} = \underset{s\to0}{Lt}\; s\left\{\frac{3}{s}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] - \frac{2}{s^2}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right] + \frac{1}{3s^3}\left[\frac{s^2(s+1)}{s^2(s+1)+10(s+2)}\right]\right\}$$

$$= \underset{s\to0}{Lt}\left\{\frac{3s^2(s+1)}{s^2(s+1)+10(s+2)} - \frac{2s(s+1)}{s^2(s+1)+10(s+2)} + \frac{(s+1)}{3s^2(s+1)+30\,(s+2)}\right\} = 0 - 0 + \frac{1}{60}$$

$$= \frac{1}{60}$$

Method - III

Error signal in s - domain, $E(s) = \dfrac{R(s)}{1+G(s)H(s)}$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

Given that, $G(s) = \dfrac{10(s+2)}{s^2(s+1)}$; $H(s) = 1$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\dfrac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^2(s+1)+10(s+2)}$$

$$= \frac{s^3+s^2}{s^3+s^2+10s+20} = \frac{s^2+s^3}{20+10s+s^2+s^3} = \frac{s^2}{20}+\frac{s^3}{40}+....$$

$$E(s) = R(s)\left[\frac{s^2}{20}+\frac{s^3}{40}+...\right] = \frac{1}{20}s^2R(s)+\frac{1}{40}s^3R(s)+...$$

$$\begin{array}{r|l} & \dfrac{s^2}{20}+\dfrac{s^3}{40}\;..... \\ 20+10s+s^2+s^3 & s^2+s^3 \\ & \underset{(-)}{s^2}\underset{(-)}{+}\dfrac{s^3}{2}\underset{(-)}{+}\dfrac{s^4}{20}\underset{(-)}{+}\dfrac{s^5}{20} \\ & \dfrac{s^3}{2}-\dfrac{s^4}{20}-\dfrac{s^5}{20} \\ & \underset{(-)}{\dfrac{s^3}{2}}\underset{(-)}{+}\dfrac{s^4}{4}\underset{(-)}{+}\dfrac{s^5}{40}\underset{(-)}{+}\dfrac{s^6}{40} \\ & -\dfrac{3s^4}{10}-\dfrac{3s^5}{40}-\dfrac{s^6}{40} \\ & \vdots \end{array}$$

Dividing numerator polynomial by denominator polynomial.

On taking inverse Laplace transform of the above equation we get,

$$e(t) = \frac{1}{20}\ddot{r}(t)+\frac{1}{40}\dddot{r}(t)+...$$

Given that, $R(s) = \dfrac{3}{s}-\dfrac{2}{s^2}+\dfrac{1}{3s^3}$

$$\therefore r(t) = \mathcal{L}^{-1}\{R(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s}-\frac{2}{s^2}+\frac{1}{3s^3}\right\} = 3-2t+\frac{1}{3}\frac{t^2}{2!} = 3-2t+\frac{t^2}{6}$$

$$\dot{r}(t) = \frac{d}{dt}r(t) = -2+\frac{1}{6}2t = -2+\frac{t}{3}$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{1}{3}$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\ddot{r}(t) = 0$$

$\therefore$ Error signal in time domain, $e(t) = \dfrac{1}{20}\ddot{r}(t) = \dfrac{1}{20}\left(\dfrac{1}{3}\right) = \dfrac{1}{60}$

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\, e(t) = \underset{t\to\infty}{Lt}\dfrac{1}{60} = \dfrac{1}{60}$

RESULT

(a) Position error constant, $K_p = \infty$

Velocity error constant, $K_v = \infty$

Acceleration error constant, $K_a = 20$

(b) When, $R(s) = \dfrac{3}{s}-\dfrac{2}{s^2}+\dfrac{1}{3s^3}$, Steady state error, $e_{ss} = \dfrac{1}{60}$

EXAMPLE 2.12

For servomechanisms with open loop transfer function given below explain what type of input signal give rise to a constant steady state error and calculate their values.

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$; b) $G(s) = \dfrac{10}{(s+2)(s+3)}$; c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

SOLUTION

a) $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$

The open loop system has a pole at origin. Hence it is a type-1 system. In systems with type number-1, the velocity (ramp) input will give a constant steady state error.

The steady state error with unit velocity input, $e_{ss} = \dfrac{1}{K_v}$

Velocity error constant, $K_v = \underset{s\to 0}{Lt}\ s\,G(s)\,H(s) = \underset{s\to 0}{Lt}\ s\,G(s)$

$$= \underset{s\to 0}{Lt}\ s\ \frac{20(s+2)}{s(s+1)(s+3)} = \frac{20\times 2}{1\times 3} = \frac{40}{3}$$

Steady state error, $e_{ss} = \dfrac{1}{K_v} = \dfrac{3}{40} = 0.075$

---

b) $G(s) = \dfrac{10}{(s+2)(s+3)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has no pole at origin. Hence it is a type-0 system. In systems with type number-0, the step input will give a constant steady state error.

The steady state error with unit step input, $e_{ss} = \dfrac{1}{1+K_p}$

Position error constant, $K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ G(s) = \underset{s\to 0}{Lt}\ \dfrac{10}{(s+2)(s+3)} = \dfrac{10}{2\times 3} = \dfrac{5}{3}$

Steady state error, $e_{ss} = \dfrac{1}{1+K_p} = \dfrac{1}{1+\dfrac{5}{3}} = \dfrac{3}{3+5} = \dfrac{3}{8} = 0.375$

---

c) $G(s) = \dfrac{10}{s^2(s+1)(s+2)}$

Let us assume unity feedback system, $\therefore H(s)=1$.

The open loop system has two poles at origin. Hence it is a type-2 system. In systems with type number-2, the acceleration (parabolic) input will give a constant steady state error.

The steady state error with unit acceleration input, $e_{ss} = \dfrac{1}{K_a}$

Acceleration error constant, $K_a = \underset{s\to 0}{Lt}\ s^2\,G(s)H(s) = \underset{s\to 0}{Lt}\ s^2\,G(s) = \underset{s\to 0}{Lt}\ s^2\ \dfrac{10}{s^2(s+1)(s+2)} = \dfrac{10}{1\times 2} = 5$

Steady state error, $e_{ss} = \dfrac{1}{K_a} = \dfrac{1}{5} = 0.2$

RESULT

1. In system (a) with unit velocity input, Steady state error = 0.075
2. In system (b) with unit step input, Steady state error = 0.375
3. In system (c) with unit acceleration input, Steady state error = 0.2

EXAMPLE 2.13

The open loop transfer function of a servo system with unity feedback is G(s) = 10/s(0.1s+1). Evaluate the static error constants of the system. Obtain the steady state error of the system, when subjected to an input given by the polynominal, $r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$.

SOLUTION

To find static error constant

For unity feedback system, H(s) = 1.

$\therefore$ Loop transfer function, G(s) H(s) = G(s)

The static error constants are K_p, K_v and K_a.

$$\text{Position error constant, } K_p = \underset{s\to 0}{Lt}\; G(s) = \underset{s\to 0}{Lt} \frac{10}{s(0.1s+1)} = \infty$$

$$\text{Velocity error constant, } K_v = \underset{s\to 0}{Lt}\; sG(s) = \underset{s\to 0}{Lt}\; s\frac{10}{s(0.1s+1)} = 10$$

$$\text{Acceleration error constant, } K_a = \underset{s\to 0}{Lt}\; s^2G(s) = \underset{s\to 0}{Lt}\; s^2\frac{10}{s(0.1s+1)} = 0$$

To find steady state error

Method - I

Steady state error for non-standard input is obtained using generalized error series, given below.

$$\text{The error signal, } e(t) = r(t)C_o + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} + ... + \overset{n}{r}(t)\frac{C_n}{n!} + ...$$

$$\text{Given that, } r(t) = a_o + a_1 t + \frac{a_2}{2}t^2$$

$$\therefore \dot{r}(t) = \frac{d}{dt}r(t) = \frac{d}{dt}\left(a_o + a_1 t + \frac{a_2}{2}t^2\right) = a_1 + a_2 t$$

$$\ddot{r}(t) = \frac{d^2}{dt^2}r(t) = \frac{d}{dt}\left(\frac{d}{dt}r(t)\right) = \frac{d}{dt}(a_1 + a_2 t) = a_2$$

$$\dddot{r}(t) = \frac{d^3}{dt^3}r(t) = \frac{d}{dt}\left(\frac{d^2}{dt^2}r(t)\right) = \frac{d}{dt}(a_2) = 0$$

Derivatives of r(t) is zero after 2nd derivative. Hence, let us evaluate three constants C_0, C_1 & C_2.

The generalized error constants are given by,

$$C_o = \underset{s\to 0}{Lt}\; F(s) \quad ; \qquad C_1 = \underset{s\to 0}{Lt} \frac{d}{ds} F(s) \; ; \qquad C_2 = \underset{s\to 0}{Lt} \frac{d^2}{ds^2} F(s)$$

$$F(s) = \frac{1}{1+G(s)H(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\frac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1\,s^2+s}{0.1s^2+s+10}$$

$$C_o = \underset{s\to 0}{Lt}\; F(s) = \underset{s\to 0}{Lt} \frac{0.1s^2+s}{0.1s^2+s+10} = 0$$

$$C_1 = \underset{s\to 0}{Lt} \frac{d}{ds}F(s) = \underset{s\to 0}{Lt} \frac{d}{ds}\left[\frac{0.1s^2+s}{0.1s^2+s+10}\right]$$

$$= \underset{s\to 0}{Lt}\left[\frac{(0.1s^2+s+10)(0.2s+1) - (0.1s^2+s)(0.2s+1)}{(0.1s^2+s+10)^2}\right] = \underset{s\to 0}{Lt} \frac{2s+10}{(0.1s^2+s+10)^2} = \frac{10}{10^2} = 0.1$$

$$C_2 = \underset{s\to 0}{Lt}\ \frac{d^2}{ds^2}F(s) = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{d}{ds}F(s)\right] = \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{2s+10}{(0.1s^2+s+10)^2}\right]$$

$$= \underset{s\to 0}{Lt}\ \frac{d}{ds}\left[\frac{(0.1s^2+s+10)^2\times 2-(2s+10)\times 2\,(0.1s^2+s+10)\,(0.2s+1)}{(0.1s^2+s+10)^4}\right]$$

$$\therefore C_2 = \frac{10^2\times 2-10\times 2\times 10\times 1}{10^4} = 0$$

Error signal, $e(t) = r(t)C_0 + \dot{r}(t)C_1 + \ddot{r}(t)\frac{C_2}{2!} = \dot{r}(t)C_1 + 0 + 0 = (a_1 + a_2t)\ 0.1$

$\therefore$ Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{t\to\infty}{Lt}\ \left[\ (a_1 + a_2t)\ 0.1\ \right] = \infty$

Method - II

The error signal in s - domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$

Given that, $r(t) = a_o + a_1t + \frac{a_2}{2}t^2$; $\quad G(s) = \frac{10}{s(0.1s+1)}$; $\quad H(s) = 1$

On taking Laplace transform of r(t) we get R(s),

$$\therefore\ R(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{2}\frac{2!}{s^3} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

$$\therefore\ E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{a_o}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{1+\frac{10}{s\,(0.1s+1)}} = \frac{\frac{a_o}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}}{\frac{s\,(0.1s+1)+10}{s\,(0.1s+1)}}$$

$$= \frac{a_0}{s}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s\,(0.1s+1)}{s\,(0.1s+1)+10}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

Steady state error, $e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$

$$\therefore e_{ss} = \underset{s\to 0}{Lt}\ s\left\{\frac{a_o}{s}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_1}{s^2}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right] + \frac{a_2}{s^3}\left[\frac{s(0.1s+1)}{s(0.1s+1)+10}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{a_os(0.1s+1)}{s(0.1s+1)+10} + \frac{a_1(0.1s+1)}{s(0.1s+1)+10} + \frac{a_2(0.1s+1)}{s[s(0.1s+1)+10]}\right\} = 0 + \frac{a_1}{10} + \infty\ = \infty$$

Method - III

Error signal in s - domain, $E(s) = \frac{R(s)}{1+G(s)H(s)}$; $\qquad \therefore \frac{E(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$

Given that, $G(s) = \frac{10}{s(0.1s+1)}$ and $H(s) = 1$.

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1+\frac{10}{s(0.1s+1)}} = \frac{s(0.1s+1)}{s(0.1s+1)+10} = \frac{0.1s^2+s}{0.1s^2+s+10} = \frac{s+0.1s^2}{10+s+0.1s^2} = \frac{s}{10} - \frac{s^3}{1000} + \ldots$$

Dividing numerator polynomial by denominator polynomial.

$$\therefore E(s) = \frac{s}{10}R(s) - \frac{s^3}{1000}R(s) + \ldots$$

On taking inverse Laplace transform,

$$e(t)=\frac{1}{10}\dot{r}-\frac{1}{1000}\dddot{r}(t)+.....$$

Given that, $r(t)=a_0+a_1t+\frac{a_2}{2}t^2$

$$\therefore\ \dot{r}=\frac{d}{dt}r(t)=a_1+a_2t$$

$$\ddot{r}(t)=\frac{d}{dt}\dot{r}(t)=a_2$$

$$\dddot{r}(t)=\frac{d}{dt}\ddot{r}(t)=0$$

$$\begin{array}{r|l} & \frac{s}{10}-\frac{s^3}{1000} \\ \hline 10+s+0.1s^2 & s+0.1s^2 \\ & \underset{(-)}{s}+\underset{(-)}{\frac{s^2}{10}}+\underset{(-)}{\frac{s^3}{100}} \\ \hline & -\frac{s^3}{100} \\ & \underset{(+)}{-\frac{s^3}{100}}\underset{(+)}{-\frac{s^4}{1000}}\underset{(+)}{-\frac{s^5}{10000}} \\ \hline & \frac{s^4}{1000}+\frac{s^5}{10000} \end{array}$$

$\therefore$ Error signal in time domain, $e(t)=\frac{1}{10}\dot{r}(t)=\frac{1}{10}(a_1+a_2t)$

Steady state error, $$e_{ss}=\underset{t\to\infty}{\mathrm{Lt}}\ e(t)=\underset{t\to\infty}{\mathrm{Lt}}\ \frac{1}{10}(a_1+a_2t)=\infty$$

RESULT

(a) Position error constant, $K_p=\infty$

(b) Velocity error constant, $K_v=10$

(c) Acceleration error constant, $K_a=0$

(d) When input, $r(t)=a_o+a_1t+\frac{a_2t^2}{2}$, Steady state error, $e_{ss}=\infty$

EXAMPLE 2.14

Consider a unity feedback system with a closed loop transfer function $\frac{C(s)}{R(s)}=\frac{Ks+b}{s^2+as+b}$. Determine open loop transfer function G(s). Show that steady state error with unit ramp input is given by $\frac{(a-K)}{b}$.

SOLUTION

For unity feedback system, H(s)=1

The closed loop transfer function, $M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}=\frac{G(s)}{1+G(s)}$

$$\therefore\frac{G(s)}{1+G(s)}=M(s)$$

On cross multiplication of the above equation we get,

$$G(s)=M(s)[1+G(s)]=M(s)+M(s)\,G(s)$$

$$\therefore\ G(s)-M(s)\,G(s)=M(s)\quad\Rightarrow\quad G(s)[1-M(s)]=M(s)\quad\Rightarrow\quad M(s)=\frac{Ks+b}{s^2+as+b}$$

$\therefore$ Open loop transfer function,

$$G(s)=\frac{M(s)}{1-M(s)}=\frac{\frac{Ks+b}{s^2+as+b}}{1-\frac{Ks+b}{s^2+as+b}}=\frac{Ks+b}{\left(s^2+as+b\right)-(Ks+b)}$$

$$=\frac{Ks+b}{s^2+as+b-Ks-b}=\frac{Ks+b}{s^2+(a-k)s}=\frac{Ks+b}{s[s+(a.K)]}$$

Velocity error constant, $K_V = \underset{s\to 0}{Lt}\ s\,G(s)H(s) = \underset{s\to 0}{Lt}\ s\,G(s) = \underset{s\to 0}{Lt}\ s\dfrac{Ks+b}{s[s+(a-K)]} = \dfrac{b}{a-K}$

With velocity input, Steady state error, $e_{ss} = \dfrac{1}{K_V} = \dfrac{a-K}{b}$

RESULT

Open loop transfer function, $G(s) = \dfrac{Ks+b}{s\,[s+(a-K)]}$

With velocity input, Steady state error, $e_{ss} = \dfrac{a-K}{b}$

EXAMPLE 2.15

A unity feedback system has the forward transfer function $G(s) = \dfrac{K_1(2s+1)}{s\,(5s+1)(1+s)^2}$. When the input r(t) = 1+6t, determine the minimum value of K_1 so that the steady error is less than 0.1.

SOLUTION

Given that, input r(t) = 1 + 6t

On taking laplace transform of r(t) we get R(s).

$$\therefore R(s) = \mathcal{L}\{r(t)\} = \mathcal{L}\{1+6t\} = \frac{1}{s}+\frac{6}{s^2}$$

The error signal in s-domain E(s) is given by,

$$\therefore E(s) = \frac{R(s)}{1+G(s)H(s)} = \frac{\frac{1}{s}+\frac{6}{s^2}}{1+\frac{K_1(2s+1)}{s(5s+1)(1+s)^2}} = \frac{\frac{1}{s}+\frac{6}{s^2}}{\frac{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}{s\,(5s+1)\,(1+s)^2}}$$

Here H(s) = 1

$$= \frac{1}{s}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s\,(5s+1)\,(1+s)^2}{s\,(5s+1)\,(1+s)^2+K_1(2s+1)}\right]$$

The steady state error e_{ss} can be obtained from final value theorem.

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s)$$

$$= \underset{s\to 0}{Lt}\ s\left\{\frac{1}{s}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right] + \frac{6}{s^2}\left[\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right]\right\}$$

$$= \underset{s\to 0}{Lt}\left\{\frac{s(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)} + \frac{6(5s+1)(1+s)^2}{s(5s+1)(1+s)^2+K_1(2s+1)}\right\} = 0+\frac{6}{K_1} = \frac{6}{K_1}$$

Given that, $e_{ss} < 0.1$, $\quad \therefore\ 0.1 = \dfrac{6}{K_1}$ $\quad$ or $\quad K_1 = \dfrac{6}{0.1} = 60$

RESULT

For steady state error, $e_{ss} < 0.1$, the value of K_1 should be greater than 60.

2.19 COMPONENTS OF AUTOMATIC CONTROL SYSTEM

The basic components of an automatic control system are Error detector, Amplifier and Controller, Actuator (Power actuator), Plant and Sensor or Feedback system. The block diagram of an automatic control system is shown in fig 2.16.

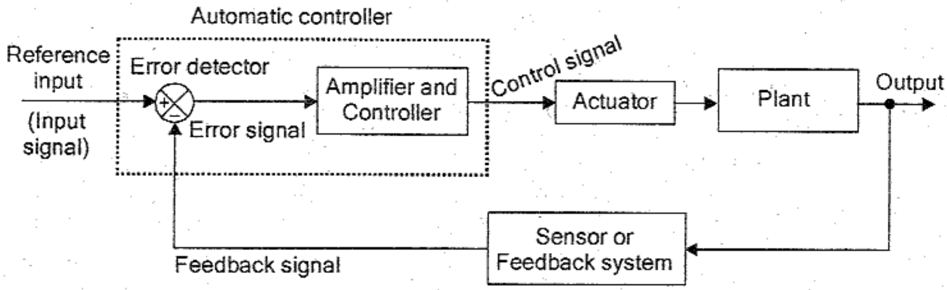


Fig 2.16: Block diagram of automatic control system.

The plant is the open loop system whose output is automatically controlled by closed loop system. The combined unit of error detector, amplifier and controller is called ***automatic controller***, because without this unit the system becomes open loop system.

In automatic control systems the reference input will be an input signal proportional to desired output. The feedback signal is a signal proportional to current output of the system. The error detector compares the reference input and feedback signal and if there is a difference it produces an error signal. An amplifier can be used to amplify the error signal and the controller modifies the error signal for better control action.

The actuator amplifies the controller output and converts to the required form of energy that is acceptable for the plant. Depending on the input to the plant, the output will change. This process continues as long as there is a difference between reference input and feedback signal. If the difference is zero, then there is no error signal and the output settles at the desired value.

Generally, the error signal will be a weak signal and so it has to be amplified and then modified for better control action. In most of the system the controller itself amplifies the error signal and integrates or differentiates to produce a control signal (i.e., modified error signal). The different types of controllers are P, PI, PD and PID controllers.

2.20 CONTROLLERS

A controller is a device introduced in the system to modify the error signal and to produce a control signal. The manner in which the controller produces the control signal is called the ***control action***. The controller modifies the transient response of the system. The electronic controllers using operational amplifiers are presented in this section.

The following six basic control actions are very common among industrial analog controllers.

1. Two-position or ON-OFF control action.
2. Proportional control action.
3. Integral control action.
4. Proportional- plus- integral control action.
5. Proportional-plus-derivative control action.
6. Proportional-plus-integral-plus-derivative control action.

Depending on the control actions provided the controllers can be classified as follows.

1. Two position or ON-OFF controllers.
2. Proportional controllers.
3. Integral controllers.
4. Proportional-plus-integral controllers.
5. Proportional-plus-derivative controllers.
6. Proportional-plus-integral-plus-derivative controllers.

ON-OFF (OR) TWO POSITION CONTROLLER

The ON-OFF or two position controller has only two fixed positions. They are either on or off. The on-off control system is very simple in construction and hence less expensive. For this reason, it is very widely used in both industrial and domestic control systems.

The ON-OFF control action may be provided by a relay. There are different types of relay. The most popular one is electromagnetic relay. It is a device which has NO (Normally Open) and NC (Normally Closed) contacts, whose opening and closing are controlled by the relay coil. When the relay coil is excited, the relay operates and the contacts change their positions (i.e., NO → NC and NC → NO).

Let the output signal from the controller be u(t) and the actuating error signal be e(t). In this controller, u(t) remains at either a maximum or minimum value.

$$u(t) = u_1 \,; \quad \text{for } e(t) < 0$$
$$\quad\;\; = u_2 \,; \quad \text{for } e(t) > 0$$

$$\boxed{E(s) = \mathcal{L}\{e(t)\} \,; \quad U(s) = \mathcal{L}\{u(t)\}}$$

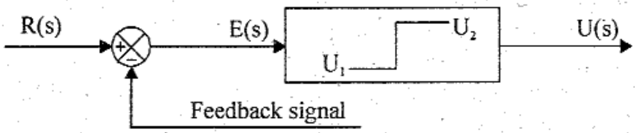


***Fig 2.17 :** Block diagram of on-off controller.*

PROPORTIONAL CONTROLLER (P - CONTROLLER)

The proportional controller is a device that produces a control signal, u(t) proportional to the input error signal, e(t).

In P-controller, $u(t) \propto e(t)$

$$\therefore \quad u(t) = K_p\, e(t) \qquad .....(2.81)$$

where, K_p = Proportional gain or constant

On taking Laplace transform of equation (2.81) we get,

$$U(s) = K_p E(s) \qquad .....(2.82)$$

$$\boxed{\therefore \text{ Transfer function of P - controller, } \frac{U(s)}{E(s)} = K_p} \qquad .....(2.83)$$

The equation (2.82) gives the output of the P-controller for the input E(s) and equation (2.83) is the transfer function of the P-controller. The block diagram of the P-controller is shown in fig 2.18.

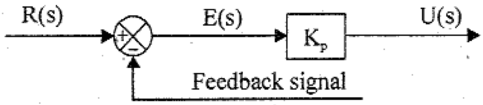


***Fig 2.18 :** Block diagram of proportional controller.*

From the equation (2.82), we can conclude that the proportional controller amplifies the error signal by an amount K_p. Also the introduction of the controller on the system increases the loop gain by an amount K_p. The increase in loop gain improves the steady state tracking accuracy, disturbance signal rejection and the relative stability and also makes the system less sensitive to parameter variations. But increasing the gain to very large values may lead to instability of the system. The drawback in P-controller is that it leads to a constant steady state error.

EXAMPLE OF ELECTRONIC P-CONTROLLER

The proportional controller can be realized by an amplifier with adjustable gain. Either the non-inverting operational amplifier or the inverting operational amplifier followed by sign changer will work as a proportional controller. The op-amp proportional controller is shown in fig 2.19 and 2.20.

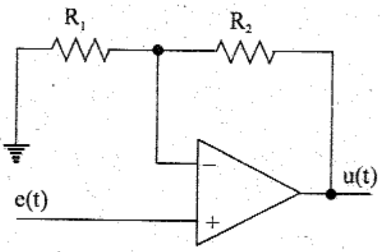


Fig 2.19 : Op-amp P-controller using non-inverting amplifier.

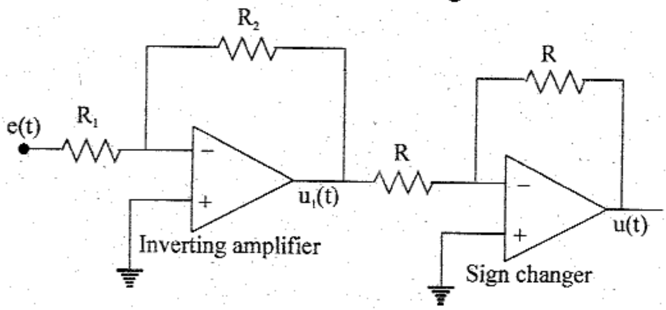


Fig 2.20 : Op-amp P-controller using inverting amplifier.

By deriving the transfer function of the controllers shown in fig 2.11 and 2.12 and comparing with the transfer function of P-controller defined by equation (2.83), it can be shown that they work as P-controllers.

ANALYSIS OF P-CONTROLLER SHOWN IN FIG 2.19

In fig 2.19, the input e(t) is applied to positive input. By symmetry of op-amp the voltage of negative input is also e(t). Also we assume an ideal op-amp so that input current is zero. Based on the above assumptions the equivalent circuit of the controller is shown in fig 2.21.

By voltage division rule,

$$e(t) = \frac{R_1}{R_1 + R_2}\, u(t) \;; \qquad \therefore u(t) = \frac{R_1 + R_2}{R_1}\, e(t) \qquad .....(2.84)$$

On taking Laplace transform of equation (2.84) we get,

$$U(s) = \frac{R_1 + R_2}{R_1}\, E(s) \qquad .....(2.85)$$

$$\therefore \frac{U(s)}{E(s)} = \frac{R_1 + R_2}{R_1} \qquad .....(2.86)$$

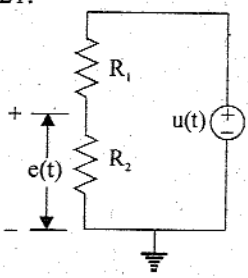


Fig 2.21 : Equivalent circuit of P-controller shown in fig 2.19.

The equation (2.86) is the transfer function of op-amp P-controller. On comparing equation (2.86) with equation (2.83) we get,

$$\text{Proportional gain, } K_P = \frac{R_1 + R_2}{R_1} \qquad .....(2.87)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain, K_p can be varied.

ANALYSIS OF P-CONTROLLER SHOWN IN FIG 2.20

The assumption made in op-amp circuit analysis are,

1. The voltages at both inputs are equal
2. The input current is zero.

Based on the above assumptions, the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.22 and 2.23.

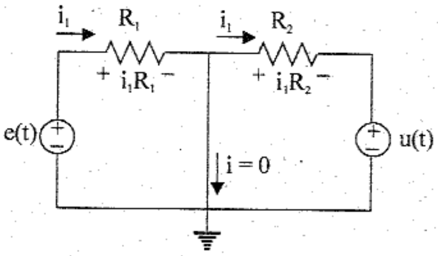


Fig 2.22 : Equivalent circuit of amplifier.

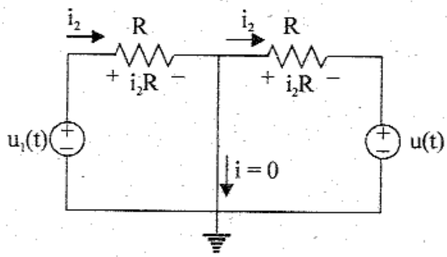


Fig 2.23 : Equivalent circuit of sign changer.

From fig 2.22, $e(t) = i_1 R_1 \;;\; \therefore\; i_1 = \frac{e(t)}{R_1}$ .....(2.88)

$u_1(t) = -i_1 R_2$ .....(2.89)

Substitute for i_1 from equation (2.88) in equation (2.89).

$$\therefore\; u_1(t) = -\frac{e(t)}{R_1} R_2 \quad .....(2.90)$$

From fig 2.23, $u(t) = -i_2 R \;;\; \therefore\; i_2 = -\frac{u(t)}{R}$ .....(2.91)

$u_1(t) = i_2 R$ .....(2.92)

Substitute for i_2 from equation (2.91) in equation (2.92).

$$\therefore\; u_1(t) = -\frac{u(t)}{R} R = -u(t) \quad .....(2.93)$$

On equating the equations (2.90) and (2.93) we get,

$$-u(t) = -\frac{e(t)}{R_1} R_2 \;; \qquad u(t) = \frac{R_2}{R_1} e(t) \quad .....(2.94)$$

On taking Laplace transform of equation (2.94) we get,

$$U(s) = \frac{R_2}{R_1} E(s) \quad .....(2.95)$$

$$\therefore\; \frac{U(s)}{E(s)} = \frac{R_2}{R_1} \quad .....(2.96)$$

The equation (2.96) is the transfer function of op-amp P-controller. On comparing equation (2.96) with equation (2.83) we get,

$$\text{Proportional gain, } K_p = \frac{R_2}{R_1} \quad .....(2.97)$$

Therefore by adjusting the values of R_1 and R_2 the value of gain K_p can be varied.

INTEGRAL CONTROLLER (I-CONTROLLER)

The integral controller is a device that produces a control signal u(t) which is proportional to integral of the input error signal, e(t).

In I-controller, $u(t) \ \alpha \int e(t)\,dt$; $\therefore u(t) = K_i \int e(t)\,dt$ .....(2.98)

where, K_i = Integral gain or constant.

On taking Laplace transform of equation (2.98) with zero initial conditions we get,

$$U(s) = K_i \frac{E(s)}{s} \quad .....(2.99)$$

$$\therefore \text{ Transfer function of I-controller, } \frac{U(s)}{E(s)} = \frac{K_i}{s} \quad .....(2.100)$$

The equation (2.99) gives the output of the I-controller for the input E(s) and equation (2.101) is the transfer function of the I-controller. The block diagram of I-controller is shown in fig 2.24.

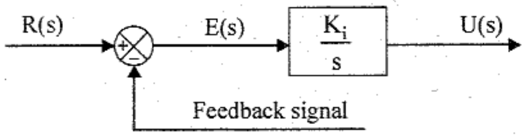


***Fig 2.24 :** Block diagram of an integral controller.*

The integral controller removes or reduces the steady error without the need for manual reset. Hence the I-controller is sometimes called ***automatic reset***. The drawback in integral controller is that it may lead to oscillatory response of increasing or decreasing amplitude which is undesirable and the system may become unstable.

EXAMPLE OF ELECTRONIC I-CONTROLLER

The integral controller can be realized by an integrator using op-amp followed by a sign changer as shown in fig 2.25.

By deriving the transfer function of the controller shown in fig 2.25 and comparing with the transfer function of I-controller defined by equation(2.101), it can be shown that it work as I-controller.

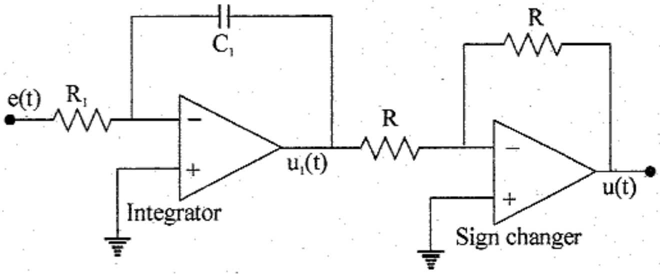


***Fig 2.25 :** I-controller using op-amp.*

ANALYSIS OF I-CONTROLLER SHOWN IN FIG 2.25

The assumptions made in op-amp circuit analysis are,

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.26 and 2.27.

From fig 2.26, $e(t) = i_1 R_1$; $\therefore i_1 = \dfrac{e(t)}{R_1}$ .....(2.101)

$$u_1(t) = -\frac{1}{C_1}\int i_1 dt \quad .....(2.102)$$

Substitute for i_1 from equation (2.101) in equation (2.102).

$$\therefore\ u_1(t) = -\frac{1}{C_1}\int \frac{e(t)}{R_1}dt = -\frac{1}{R_1C_1}\int e(t)dt \qquad .....(2.103)$$

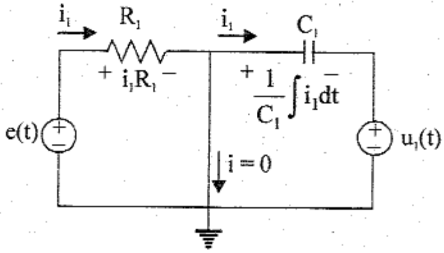


Fig 2.26 : Equivalent circuit of integrator.

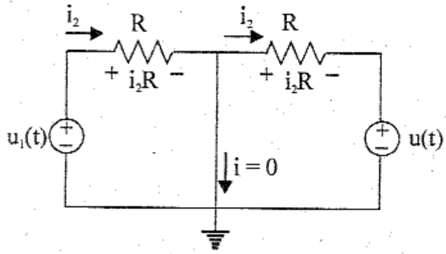


Fig 2.27 : Equivalent circuit of sign changer.

From fig 2.27, $u(t) = -i_2R$, $\therefore\ i_2 = \frac{-u(t)}{R}$ .....(2.104)

$$u_1(t) = i_2R \qquad .....(2.105)$$

Substitute for i_2 from equation (2.106) in equation (2.107),

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.106)$$

On equating the equations (2.103) and (2.106) we get,

$$-u(t) = -\frac{1}{R_1C_1}\int e(t)\,dt$$

$$\therefore\ u(t) = \frac{1}{R_1C_1}\int e(t)\,dt \qquad .....(2.107)$$

On taking Laplace transform of equation (2.107) with zero initial conditions we get,

$$U(s) = \frac{1}{R_1C_1}\frac{E(s)}{s} \qquad .....(2.108)$$

$$\therefore\ \frac{U(s)}{E(s)} = \frac{1}{R_1C_1}\frac{1}{s} \qquad .....(2.109)$$

The equation (2.109) is the transfer function of op-amp I-controller. On comparing equation (2.109) with equation (2.100) we get,

$$\text{Integral gain, } K_i = \frac{1}{R_1C_1} \qquad .....(2.110)$$

Therefore by adjusting the values of R_1 and C_1 the value of gain K_i can be varied.

PROPORTIONAL PLUS INTEGRAL CONTROLLER (PI-CONTROLLER)

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

In PI - controller, $u(t)\ \alpha\ \left[e(t) + \int e(t)\,dt\right]$; $\quad \therefore u(t) = K_p\,e(t) + \frac{K_p}{T_i}\int e(t)dt$ ....(2.111)

where, K_p = Proportional gain

T_i = Integral time.

On taking Laplace transform of equation (2.111) with zero initial conditions we get,

$$U(s) = K_p\, E(s) + \frac{K_p}{T_i}\frac{E(s)}{s} \quad .....(2.112)$$

$$\therefore \text{Transfer function of PI - controller, } \frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right) \quad .....(2.113)$$

The equation (2.112) gives the output of the PI-controller for the input E(s) and equation (2.113) is the transfer function of the PI controller. The block diagram of PI-controller is shown in fig 2.28.

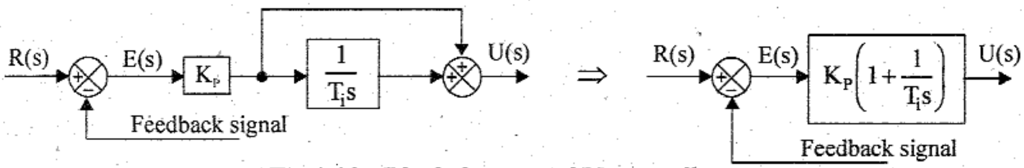


Fig 2.28 : Block diagram of PI-controller.

The advantages of both P-controller and I-controller are combined in PI-controller. The proportional action increases the loop gain and makes the system less sensitive to variations of system parameters. The integral action eliminates or reduces the steady state error.

The integral control action is adjusted by varying the integral time. The change in value of K_p affects both the proportional and integral parts of control action. The inverse of the integral time T_i is called the ***reset rate.***

EXAMPLE OF ELECTRONIC PI-CONTROLLER

The PI-controller can be realized by an op-amp integrator with gain followed by a sign changer as shown in fig 2.29.

By deriving the transfer function of the controller shown in fig (2.29) and comparing with the transfer function of PI-controller defined by equation (2.114), it can be proved that the circuit shown in fig 2.29, work as PI-controller.

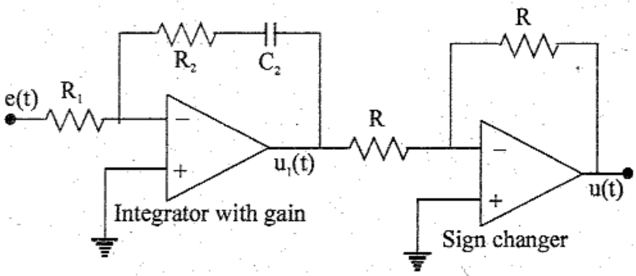


Fig 2.29 : PI-controller using op-amp.

ANALYSIS OF PI-CONTROLLER SHOWN IN FIG 2.29

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp integrator and sign changer are shown in fig 2.30. and 2.31.

From fig 2.30, $e(t) = i_1 R_1$; $\therefore\ i_1 = \frac{e(t)}{R_1}$ .....(2.114)

$$u_1(t) = -i_1 R_2 - \frac{1}{C_2}\int i_1 dt \quad .....(2.115)$$

Substitute for i_1 from equation (2.114) in equation (2.115).

$$\therefore\ u_1(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt \quad .....(2.116)$$

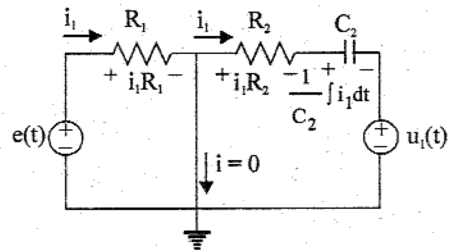


Fig 2.30 : Equivalent circuit of integrator.

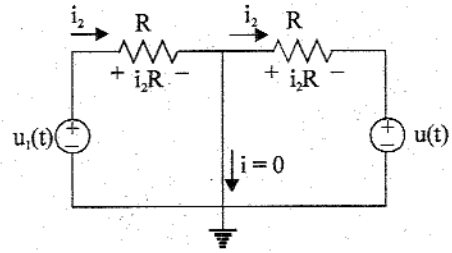


Fig 2.31 : Equivalent circuit of sign changer.

From fig 2.31, $u(t) = -i_2R, \quad \therefore \; i_2 = \frac{-u(t)}{R}$ .....(2.117)

$u_1(t) = i_2R$ .....(2.118)

Substitute for i_2 from equation (2.117) in equation (2.118),

$$\therefore \; u_1(t) = -\frac{u(t)}{R}R = -u(t) \quad .....(2.119)$$

On equating the equations (2.116) and (2.119) we get,

$$-u(t) = -\frac{e(t)}{R_1}R_2 - \frac{1}{C_2}\int \frac{e(t)}{R_1}\,dt$$

$$\therefore \; u(t) = \frac{R_2}{R_1}e(t) + \frac{1}{R_1C_2}\int e(t)\,dt \quad .....(2.120)$$

On taking Laplace transform of equation (2.120) with zero initial conditions we get,

$$U(s) = \frac{R_2}{R_1}E(s) + \frac{1}{R_1C_2}\frac{E(s)}{s}$$

$$\therefore \; \frac{U(s)}{E(s)} = \frac{R_2}{R_1}\left(1 + \frac{1}{R_2C_2s}\right) \quad .....(2.121)$$

The equation (2.121) is the transfer function of op-amp PI-controller. On comparing equation(2.121) with equation (2.113) we get,

Proportional gain, $K_p = \frac{R_2}{R_1}$; Integral time, $T_i = R_2C_2$

By varying the values of R_1 and R_2, the value of gain K_p and T_i can be adjusted.

PROPORTIONAL PLUS DERIVATIVE CONTROLLER (PD-CONTROLLER)

The proportional plus derivative controller produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

In PD - controller, $u(t) \; \alpha \left[e(t) + \frac{d}{dt}e(t)\right]$; $\therefore u(t) = K_p\,e(t) + K_pT_d\frac{d}{dt}e(t)$ .....(2.122)

where, K_p = Proportional gain

T_d = Derivative time

On taking Laplace transform of equation (2.123) with zero initial conditions we get,

$$U(s) = K_p\,E(s) + K_p\,T_d\,sE(s) \quad .....(2.123)$$

$$\therefore \text{ Transfer function of PD-controller, } \frac{U(s)}{E(s)} = K_p(1+T_d s) \quad .....(2.124)$$

The equation (2.123) gives the output of the PD-controller for the input E(s) and equation (2.124) is the transfer function of PD-controller.

The block diagram of PD-controller is shown in fig 2.32.

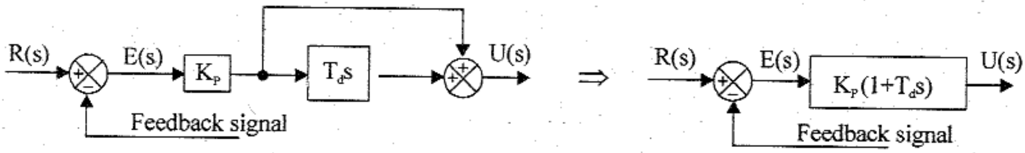


Fig 2.32 : Block diagram of PD- controller.

The derivative control acts on rate of change of error and not on the actual error signal. The derivative control action is effective only during transient periods and so it does not produce corrective measures for any constant error. Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers. The derivative controller does not affect the steady-state error directly but anticipates the error, initiates an early corrective action and tends to increase the stability of the system. While derivative control action has an advantage of being anticipatory it has the disadvantage that it amplifies noise signals and may cause a saturation effect in the actuator.

The derivative control action is adjusted by varying the derivative time. The change in the value of K_p affects both the proportional and derivative parts of control action. The derivative control is also called ***rate control.***

EXAMPLE OF ELECTRONIC PD-CONTROLLER

The PD-controller can be realized by an op-amp differentiator with gain followed by a sign changer as shown in fig 2.33.

By deriving the transfer function of the controller shown in fig 2.33 and comparing with the transfer function of PD-controller defined by equation (2.124) it can be proved that the circuit shown in fig 2.33 will work as PD-controller.

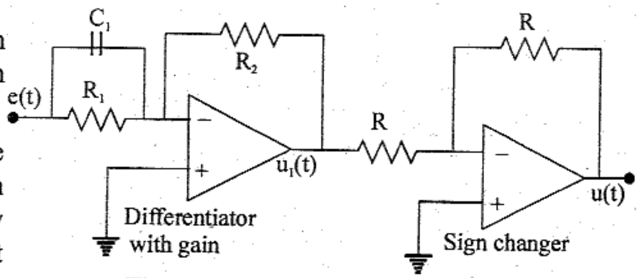


Fig 2.33 : PD controller using op-amp.

ANALYSIS OF PD-CONTROLLER SHOWN IN FIG 2.33

The assumptions made in op-amp circuit analysis are,

1. The voltages at both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp differentiator and sign changer are shown in fig 2.34 and 2.35.

From fig 2.34, $\therefore i_1 = \frac{e(t)}{R_1} + C_1 \frac{de(t)}{dt}$ .....(2.125)

$$i_1 R_2 = -u_1(t), \qquad \therefore i_1 = \frac{-u_1(t)}{R_2} \quad .....(2.126)$$

On equating the equations (2.125) and (2.126) we get,

$$-\frac{u_1(t)}{R_2} = \frac{e(t)}{R_1} + C_1 \frac{d}{dt} e(t); \qquad \therefore u_1(t) = -\left(\frac{R_2}{R_1} e(t) + R_2 C_1 \frac{d}{dt} e(t)\right) \quad .....(2.127)$$

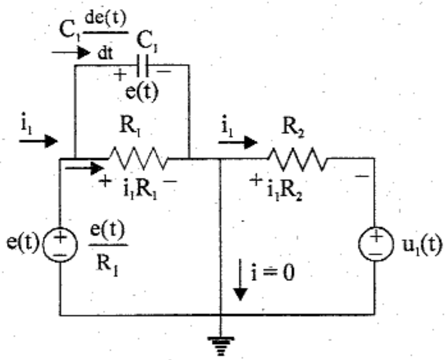


Fig 2.34 : Equivalent circuit of differentiator.

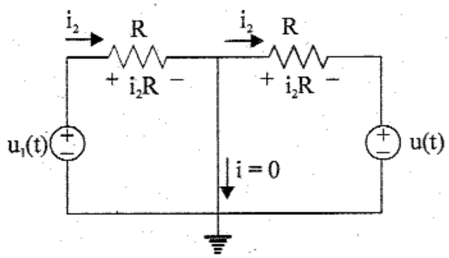


Fig 2.35 : Equivalent circuit of sign changer.

From fig 2.35, $u(t) = -i_2R$; $\therefore\ i_2 = \frac{-u(t)}{R}$ .....(2.128)

$$u_1(t) = i_2R \qquad .....(2.129)$$

Substitute for i_2 from equation (2.128) in equation (2.129).

$$\therefore\ u_1(t) = -\frac{u(t)}{R}R = -u(t) \qquad .....(2.130)$$

On equating the equations (2.127) and (2.130) we get,

$$-u(t) = -\left(\frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t)\right)$$

$$\therefore\ u(t) = \frac{R_2}{R_1}e(t) + R_2C_1\frac{d}{dt}e(t) \qquad .....(2.131)$$

On taking Laplace transform of equation (2.131) with zero initial conditions we get,

$$U(s) = \frac{R_2}{R_1}\ E(s) + R_2C_1\, s\, E(s) \qquad .....(2.132)$$

$$\therefore\ \frac{U(s)}{E(s)} = \frac{R_2}{R_1}\ (1 + R_1C_1\, s) \qquad .....(2.133)$$

The equation (2.133) is the transfer function of op-amp PD-controller. On comparing equation (2.133) with equation (2.124) we get,

Proportional gain, $K_p = \frac{R_2}{R_1}$

Derivative time, $T_d = R_1C_1$

By varying the values of R_1 and R_2, the value of gain K_p and T_d can be adjusted.

PROPORTIONAL PLUS INTEGRAL PLUS DERIVATIVE CONTROLLER (PID-CONTROLLER)

The PID-controller produces an output signal consisting of three terms : *one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.*

In PID - controller, $u(t) \;\alpha\; \left[e(t) + \int e(t)\,dt + \frac{d}{dt}e(t)\right]$

$$\therefore u(t) = K_p\, e(t) + \frac{K_p}{T_i}\int e(t)dt + K_p T_d \frac{d}{dt}e(t) \qquad .....(2.134)$$

where, K_p = Proportional gain

T_i = Integral time

T_d = Derivative time

On taking Laplace transform of equation (2.134) with zero initial conditions we get,

$$U(s) = K_p E(s) + \frac{K_p}{T_i}\frac{E(s)}{s} + K_p\, T_d\, s\, E(s) \qquad .....(2.135)$$

$$\therefore \text{ Transfer function of PID - controller, } \frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \qquad .....(2.136)$$

The equation (2.135) gives the output of the PID-controller for the input E(s) and equation (2.136) is the transfer function of the PID-controller. The block diagram of PID-contoller is shown in fig 2.36.

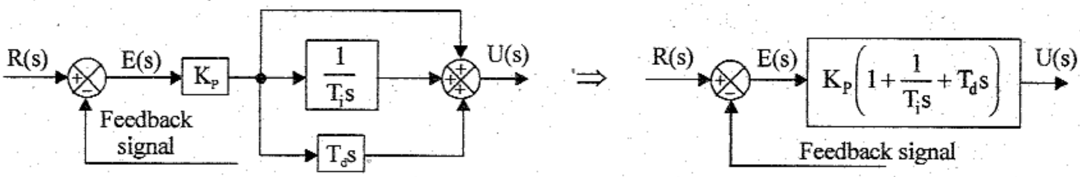


Fig 2.36: Block diagram of PID- controller.

The combination of proportional control action, integral control action and derivative control action is called PID-control action. This combined action has the advantages of the each of the three individual control actions.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error.

EXAMPLE OF ELECTRONIC PID-CONTROLLER

The PID-controller can be realized by op-amp amplifier with integral and derivative action followed by sign changer as shown in fig 2.37.

By deriving the transfer function of the controller shown in fig (2.37) and comparing with the transfer function of PID-controller defined by equation (2.136) it can be proved that the circuit shown in fig 2.37 work as PID-controller.

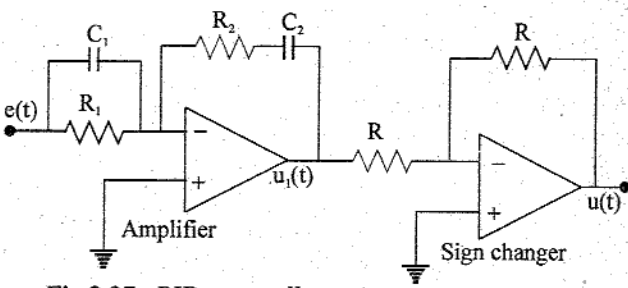


Fig 2.37 : PID- controller using op-amp.

ANALYSIS OF PID-CONTROLLER SHOWN IN FIG 2.37

The assumptions made in op-amp circuit analysis are.

1. The voltages of both inputs are equal.
2. The input current is zero.

Based on the above assumptions the equivalent circuit of op-amp amplifier and sign changer are shown in fig 2.38 and 2.39.

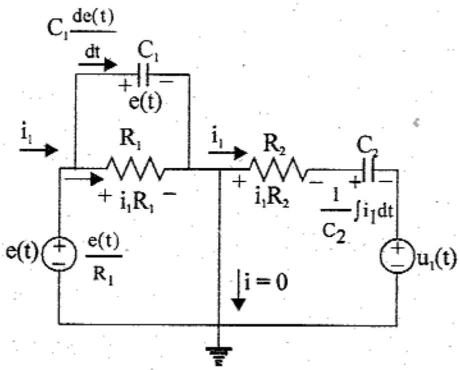


Fig 2.38 : Equivalent circuit of amplifier.

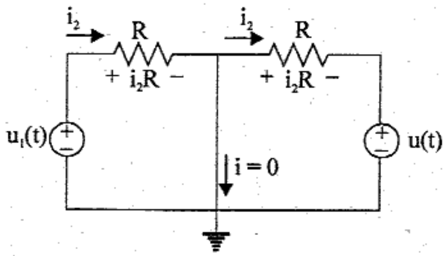


Fig 2.39 : Equivalent circuit of sign changer.

From fig 2.38, $i_1 = \frac{e(t)}{R_1} + C_1 \frac{de(t)}{dt}$ .....(2.137)

On taking Laplace transform of equation (2.137) with zero initial conditions we get,

$$I_1(s) = \frac{1}{R_1} E(s) + C_1 s E(s)$$

$$I_1(s) = \left(\frac{1}{R_1} + C_1 s\right) E(s) \quad .....(2.138)$$

From fig 2.38, $i_1 R_2 + \frac{1}{C_2}\int i_1 dt = -u_1(t)$ .....(2.139)

On taking Laplace transform of equation (2.138) with zero initial conditions we get,

$$I_1(s) R_2 + \frac{1}{C_2} \frac{I_1(s)}{s} = -U_1(s)$$

$$\therefore I_1(s)\left(R_2 + \frac{1}{C_2 s}\right) = -U_1(s) \quad .....(2.140)$$

Substitute for $I_1(s)$ from equation (2.138) in equation (2.140).

$$\therefore \left(\frac{1}{R_1} + C_1 s\right) E(s) \left(R_2 + \frac{1}{C_2 s}\right) = -U_1(s)$$

$$-\left(\frac{R_2}{R_1} + \frac{C_1}{C_2} + \frac{1}{R_1 C_2 s} + R_2 C_1 s\right) E(s) = U_1(s) \quad .....(2.141)$$

From fig 2.39, $u(t) = -i_2 R$; $\therefore i_2 = -\frac{u(t)}{R}$ .....(2.142)

$u_1(t) = i_2 R$ .....(2.143)

Substitute for i_2 from equation (2.142) in equation (2.143).

$$\therefore u_1(t) = -\frac{u(t)}{R} R = -u(t) \quad .....(2.144)$$

On taking Laplace transform of equation (2.144) we get,

$$U_1(s) = -U(s) \quad .....(2.145)$$

From equations (2.142) and (2.146) we get,

$$U(s) = \left(\frac{R_2}{R_1} + \frac{C_1}{C_2} + \frac{1}{R_1 C_2 s} + R_2 C_1 s \right) E(s)$$

$$\therefore \frac{U(s)}{E(s)} = \left(\frac{R_2 C_2 + R_1 C_1}{R_1 C_2} + \frac{1}{R_1 C_2 s} + R_2 C_1 s \right)$$

$$= \frac{R_2}{R_1} \left(\frac{R_2 C_2 + R_1 C_1}{R_2 C_2} + \frac{1}{R_2 C_2 s} + R_1 C_1 s \right) \quad .....(2.146)$$

The equation (2.146) is the transfer function of op-amp PID-controller. On comparing equation (2.146) with equation (2.136) we get,

Proportional gain, $K_p = \frac{R_2}{R_1}$

Derivative time, $T_d = R_1 C_1$; Integral time, $T_i = R_2 C_2$

Also, $\frac{R_1 C_1 + R_2 C_2}{R_2 C_2} = 1$

By varying the values of R_1 and R_2 the values of K_p, T_d and T_i are adjusted.

2.21 RESPONSE WITH P, PI, PD AND PID CONTROLLERS

In feedback control systems a controller may be introduced to modify the error signal and to achieve better control action. The introduction of controllers will modify the transient response and the steady state error of the system. The effects due to introduction of P, PI, PD and PID controllers are discussed in this section.

EFFECT OF PROPORTIONAL CONTROLLER (P-CONTROLLER)

The proportional controller produces an output signal which is proportional to error signal. The transfer function of proportional controller is given below. (Refer equation 2.83).

Transfer function of P-controller, $\frac{U(s)}{E(s)} = K_p$

The term K_p in the transfer function of proportional controller is called the gain of the controller. Hence the proportional controller amplifies the error signal and increases the loop gain of the system. The following aspects of system behaviour are improved by increasing loop gain.

* Steady state tracking accuracy.
* Disturbance signal rejection.
* Relative stability.

In addition to increase in loop gain it decreases the sensitivity of the system to parameter variations. The drawback in proportional control action is that it produces a constant steady state error.

EFFECT OF PI-CONTROLLER

The proportional plus integral controller (PI-controller) produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

Transfer function of PI-controller, $G_c(s) = K_p \left(1 + \frac{1}{T_i s} \right) = K_p \left(\frac{T_i s + 1}{T_i s} \right)$ (Refer equation 2.113)

where, K_p is proportional gain and, T_i is integral time.

The block diagram of unity feedback system with PI-controller is shown in fig 2.40.

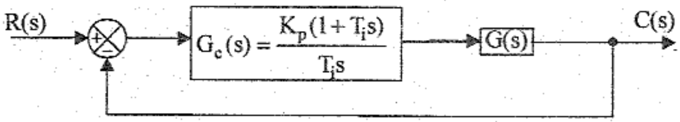


Fig 2.40 : Block diagram of feedback system with PI-controller.

Let the open loop transfer function G(s) be a second order system with transfer function, as shown in equation (2.148).

Open loop transfer function, $G(s) = \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)}$ ......(2.147)

Now, loop transfer function $= G_c(s)\,G(s)\,H(s) = G_c(s)\,G(s)$ $\boxed{H(s)=1}$

$$= K_p\left(\frac{1+T_i s}{T_i s}\right) \times \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)} = \frac{K_p\omega_n^2\,(1+T_i s)}{s^2 T_i(s+2\zeta\,\omega_n)} \quad .....(2.148)$$

Now the closed loop transfer function is given by,

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{\dfrac{K_p\omega_n^2(1+T_i s)}{s^2 T_i\,(s+2\zeta\omega_n)}}{1+\dfrac{K_p\omega_n^2(1+T_i s)}{s^2 T_i\,(s+2\zeta\omega_n)}} = \frac{K_p\omega_n^2\,(1+T_i s)}{s^2 T_i(s+2\zeta\,\omega_n)+K_p\omega_n^2\,(1+T_i s)}$$

$$= \frac{K_p\omega_n^2.(1+T_i s)}{T_i s^3 + 2\zeta\omega_n T_i s^2 + K_p\omega_n^2\,T_i\,s + K_p\omega_n^2}$$

$$= \frac{(K_p\,/\,T_i)\,\omega_n^2(1+T_i s)}{s^3 + 2\zeta\omega_n s^2 + K_p\omega_n^2\,s + \dfrac{K_p}{T_i}\omega_n^2} \qquad \boxed{K_i = \frac{K_p}{T_i}}$$

$$= \frac{K_i\omega_n^2\,(1+T_i s)}{s^3 + 2\zeta\omega_n s^2 + K_p\omega_n^2 s + K_i\omega_n^2} \quad .....(2.149)$$

From the closed loop transfer function (equation (3.149)) it is observed that the PI-controller introduces a zero in the system and increases the order by one. The increase in the order of the system results in a less stable system than the original one because higher order systems are less stable than lower order systems.

From the loop transfer function (equation (3.148)) it is observed that the PI-controller increase the type number by one. The increase in type number results in reducing the steady state error. For example if the steady state error of the original system is constant, then the integral controller will reduce the error to zero.

EFFECT OF PD-CONTROLLER

The proportional plus derivative controller produces an output signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

The transfer function of PD - controller, $G_c(s) = K_p\,(1+T_d\,s)$ (Refer equation 2.124)

where K_p is Proportional gain, T_d is Derivative time.

The block diagram of unity feedback system with PD-controller is shown in fig 2.41.

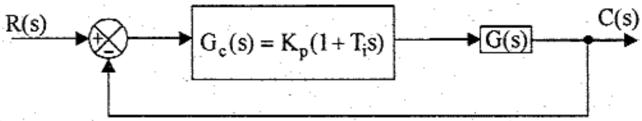


Fig 2.41 : Block diagram of feedback system with PD-controller.

Let the open loop transfer function G(s) be a second order system with transfer function as shown in equation (2.150).

Open loop transfer function, $G(s) = \dfrac{\omega_n^2}{s\,(s+2\zeta\omega_n)}$ .....(3.150)

Now, loop transfer function $= G_c(s)\,G(s)\,H(s) = G_c(s)\,G(s)$ $\boxed{H(s)=1}$

$$= K_p\big(1+T_d(s)\big) \times \frac{\omega_n^2}{s\,(s+2\zeta\omega_n)} = \frac{K_p\omega_n^2\,(1+T_d s)}{s(s+2\zeta\,\omega_n)} \quad .....(2.151)$$

Now the closed loop transfer function is given by,

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G(s)G_c(s)} = \frac{\dfrac{K_p\omega_n^2(1+T_d s)}{s\,(s+2\zeta\omega_n)}}{1+\dfrac{K_p\omega_n^2\,(1+T_d s)}{s\,(s+2\zeta\omega_n)}}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s\,(s+2\zeta\omega_n)+K_p\omega_n^2\,(1+T_d s)}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+2\zeta\omega_n s+K_p\omega_n^2+K_p\omega_n^2 T_d s}$$

$$= \frac{K_p\omega_n^2\,(1+T_d s)}{s^2+(2\zeta\omega_n+K_p\omega_n^2 T_d)\,s+K_p\omega_n^2} \qquad \boxed{K_d = K_p T_d}$$

$$= \frac{\omega_n^2\,(K_p+K_d s)}{s^2+(2\zeta\omega_n+K_d\omega_n^2)\,s+K_p\omega_n^2} \quad .....(2.152)$$

From the closed loop transfer function (equation (2.152)) it is observed that the PD-controller introduces a zero in the system and increases the damping ratio. The addition of the zero may increase the peak overshoot and reduce the rise time. But the effect of increased damping ultimately reduces the peak overshoot.

From the loop transfer function (equation (2.151)) it is observed that the PD-controller does not modify the type number of the system. Hence PD-controller will not act modify steady state error.

EFFECT OF PID-CONTROLLER

A suitable combination of the three basic modes : *proportional, integral and derivative* (PID) can improve all aspects of the system performance.

The proportional controller stabilizes the gain but produces a steady state error. The integral controller reduces or eliminates the steady state error. The derivative controller reduces the rate of change of error. The combined effect of all the three cannot be judged from the parameters K_p, K_i and K_d.

2.22 TIME RESPONSE ANALYSIS USING MATLAB

In general, the closed loop transfer function of a system is denoted as M(s).

Let, M(s) be a rational function of "s", as shown below.

$$M(s) = \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + ...+ b_{M-1}s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + ...+ a_{N-1}s + a_N}$$

For time response analysis, the coefficients of the numerator and denominator polynomials are declared as two arrays as shown below.

```
num_cof = [b0 b1 b2 ........ bM];
den_cof = [a0 a1 a2 ........ aN];
```

UNIT STEP RESPONSE

To compute step response

The unit step response can be computed and displayed using following commands.

```
syms s complex;
R = 1/s;
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
S = R*M;
disp('Unit step response of the system is,');
step_res = ilaplace(S)
```

To plot step response

Method 1 :

The unit step response can be plotted using the following command.

```
step(num_cof, den_cof);
```

Method 2 :

The unit step response of the system can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
c = step(num_cof, den_cof,t);
plot(t,c,'k');
     where, c is an array where the values of response are stored.
```

The unit step response can be computed "n" times by varying some parameter of the system (coefficient / damping ratio / natural frequency of oscillation) using the following commands.

```
t = t_start : t_step : t_end ;
for i = 1 : n
     .
     .
c(1:k, i) = step(num_cof, den_cof,t);
     .
     .
end
plot(t,c,'k');
     where, c is an array where the values of response are stored.
            k is the number of samples of response to be computed.
```

Method 3 :

The unit step response of the system can be plotted using the following commands.

```
s = tf('s');
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
t = t_start : t_step : t_end ;
sr = step(M,t);
plot(t,sr,'k');
```

IMPULSE RESPONSE

To compute impulse response

The impulse response can be computed and displayed using following commands.

```
syms s complex;
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
disp('Unit step response of the system is,');
imp_res = ilaplace(S)
```

To plot impulse response

Method 1 :

The impulse response can be plotted using the following command.

```
impulse(num_cof, den_cof);
```

Method 2 :

The impulse response of the system can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
m = impulse(num_cof, den_cof,t);
plot(t,m,'k');
     where, m is an array where the values of impulse response are stored.
```

Method 3 :

The impulse response of the system can be plotted using the following commands.

```
s = tf('s');
M = (b0*s^M+b1*s^(M-1)+...+bM)/(a0*s^N+a1*s^(N-1)+...+aN);
t = t_start : t_step : t_end ;
imp = impulse(M,t);
plot(t,imp,'k');
```

RESPONSE FOR ARBITRARY INPUT

The response of a system for an arbitrary input, r(t) can be plotted using the following commands.

```
t = t_start : t_step : t_end ;
c = Lsim(num_cof, den_cof, r, t);
plot(t,c,'k');
     where, c is an array where the values of response are stored.
```

PROGRAM 2.1

Consider the standard closed loop transfer function of the second order system given below.

$$M(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$$

Write a MATLAB program to find the unit step response for various values of damping ratio, ζ. Take, natural frequency of oscillation, $\omega_n=1$ rad/sec.

```
%Unit step response for various values of damping ratio, zeta.
%The natural frequency of oscillation, wn=1.

clc
t=0:0.2:12;                              %specify a time vector
c=zeros(61,6);                           %initialize response array as zero
zeta=[0 0.2 0.4 0.6 0.8 1];             %store zeta as an array
     for n=1:6;                          %for loop to compute c(t) 6 times
          num_cof=[0 0 1];
          den_cof=[1 2*zeta(n) 1];
          c(1:61,n)=step(num_cof,den_cof,t);
     end

plot(t,c,'k'); grid
xlabel('time,t in sec'); ylabel('Unit step response,c(t)');

text(2.8,1.86,'\zeta=0')
text(2.8,1.58,'\zeta=0.2')
text(2.8,1.30,'\zeta=0.4')
text(2.8,1.12,'\zeta=0.6')
text(2.8,0.95,'\zeta=0.8')
text(2.8,0.72,'\zeta=1.0')
```

OUTPUT

The output waveforms are shown in fig p2.1.

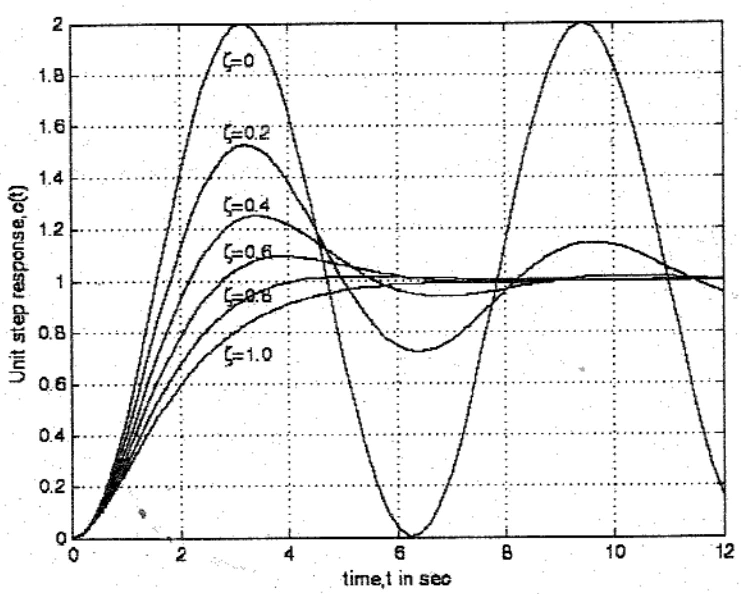


Fig P2.1 : Unit step response of second order system for various values of damping ratio.

PROGRAM 2.2

Consider the standard closed loop transfer function of the second order system given below.

$$M(s)=\omega_n^2/(s^2+2\zeta\omega_n s+\omega_n^2)$$

Write a MATLAB program to find the unit step response for various values of natural frequency of oscillation, ω_n. Take, damping ratio, ζ=0.4.

```
%Unit step response for various natural frequency of oscillation,wn.
%The damping ratio, zeta=0.4.
clc
t=0:0.1:8;                  %specify a time vector
wn=[1 2 4 6];               %store wn as an array
zeta=0.4;
c=zeros(81,4);              %initialize the response array as zeros

   for i=1:4;               %for loop to compute c(t) 4 times
   b2=wn(i)*wn(i);
   a1=2*zeta*wn(i);
   num_cof=[0 0 b2];
   den_cof=[1 a1 b2];
   c(1:81,i)=step(num_cof,den_cof,t);
   end

plot(t,c(:,1),'--k',t,c(:,2),'xk',t,c(:,3),'-k',t,c(:,4),'-.k');
grid; xlabel('time,t in sec'); ylabel('Unit step response,c(t)');
text(4.25,1.25,'wn=1')
text(1.5,1.30,'wn=2')
text(0.7,1.30,'wn=3')
text(0.1,1.25,'wn=4')
```

OUTPUT

The output waveforms are shown in fig p2.2.

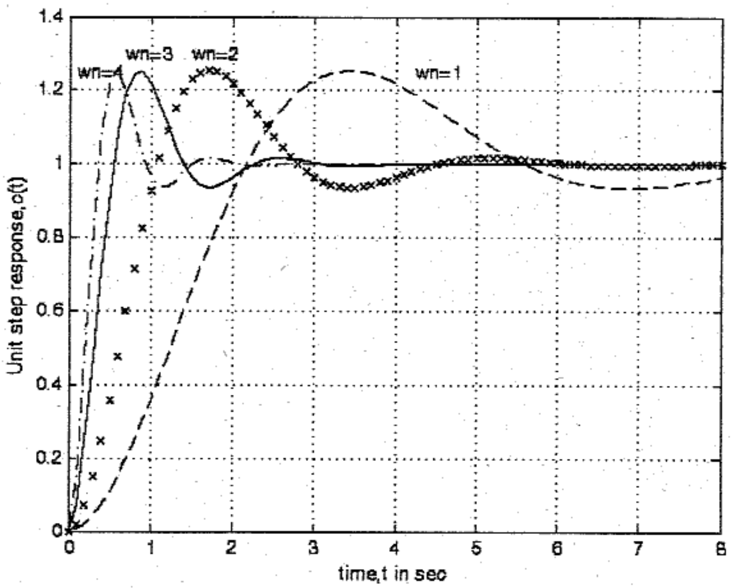


Fig P2.2 : Unit step response of second order system for various values of natural frequency of oscillation.

PROGRAM 2.3

Write a MATLAB program to find impulse response of the following systems.

a) $M_1(s)=(2s+1)/(s+1)^2$ b) $M_2(s)=s/(s+1)$ c) $M_3(s)=1/(s^2+1)$

```
%Program to find impulse response
clc
syms s complex;
M1=(2*s+1)/((s+1)^2);
disp('Impulse response of the system1 is,');
m1=ilaplace(M1)

M2=s/(s+1);
disp('Impulse response of the system2 is,');
m2=ilaplace(M2)

M3=1/(s^2+1);
disp('Impulse response of the system3 is,');
m3=ilaplace(M3)

s=tf('s');
M1=(2*s+1)/((s+1)^2);
M2=s/(s+1);
M3=1/(s^2+1);

t=0:.005:10;
m1=impulse(M1,t);
m2=impulse(M2,t);
m3=impulse(M3,t);

plot(t,m1,'--k',t,m2,'-.k',t,m3,'-k');grid
xlabel('time,t in sec');
ylabel('Impulse responses,m1(t),m2(t),m3(t)');
text(0.4,1.30,'m1(t)')
text(0.3,-0.30,'m2(t)')
text(2.2,0.90,'m3(t)')
```

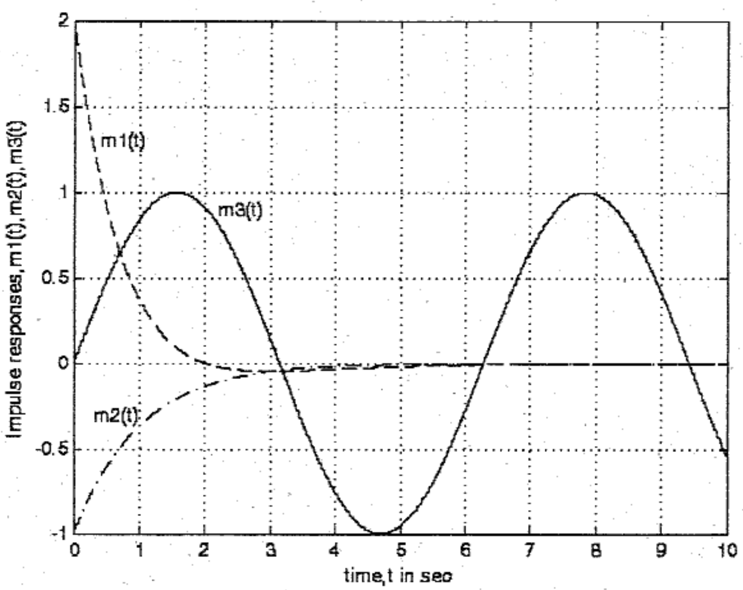


Fig P2.3 : Impulse response of systems given in program 2.3.

OUTPUT

```
Impulse response of the system1 is,
        m1 =
              (2-t)*exp(-t)

Impulse response of the system2 is,
        m2 =
               dirac(t)-exp(-t)

Impulse response of the system3 is,
        m3 =
              sin(t)
```

The output waveforms are shown in fig p2.3.

PROGRAM 2.4

Write a MATLAB program to find unit step response of the following systems.

a) $M_1(s)=4/(s^2+5s+4)$ b) $M_2(s)=100/(s^2+12s+100)$ c) $M_3(s)=600/(s^2+70s+600)$

```
%program to find unit step response
clc
syms s complex;
R=1/s;                          %Laplace of unit step input
M1=4/(s^2+5*s+4);
S1=R*M1;                        %s-domain unit step response of system1
disp('Unit step response of the system1 is,');
s1=ilaplace(S1)               %time domain unit step response of system1

M2=100/(s^2+12*s+100);
S2=R*M2;                        %s-domain unit step response of system2
disp('Unit step response of the system2 is,');
s2=ilaplace(S2)               %time domain unit step response of system2

M3=600/(s^2+70*s+600);
S3=R*M3;                        %s-domain unit step response of system3
disp('Unit step response of the system3 is,');
s3=ilaplace(S3)               %time domain unit step response of system3

s=tf('s');
M1=4/(s^2+5*s+4);
M2=100/(s^2+12*s+100);
M3=600/(s^2+70*s+600);

t=0:.005:10;
s1=step(M1,t);
s2=step(M2,t);
s3=step(M3,t);

plot(t,s1,'--k',t,s2,'-.k',t,s3,'-k');grid
xlabel('time,t in sec');
ylabel('Unit step responses,s1(t),s2(t),s3(t)');
text(2.2,0.85,'s1(t)')
text(0.2,1.15,'s2(t)')
text(0.5,0.95,'s3(t)')
```

OUTPUT

```
Unit step response of the system1 is,
        s1 =
                1/3*exp(-4*t)+1-4/3*exp(-t)

Unit step response of the system2 is,
        s2 =
                1-exp(-6*t)*cos(8*t)-3/4*exp(-6*t)*sin(8*t)

Unit step response of the system3 is,
        s3 =
                1+1/5*exp(-60*t)-6/5*exp(-10*t)

The output waveform is shown in fig p2.4.
```

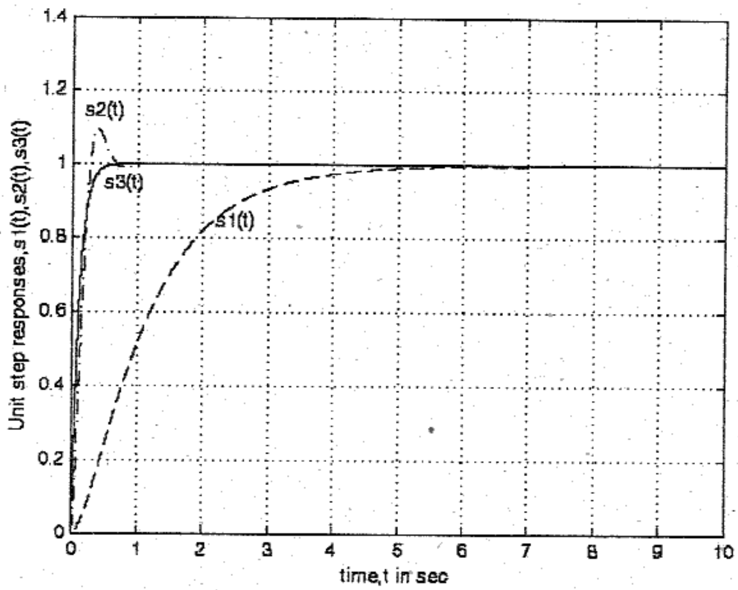


Fig P2.4 : Unit step response of systems given in program 2.4.

PROGRAM 2.5

Consider the closed loop transfer function of the following second order system,

$M(s)=16/(s^2+4s+16)$

Write a MATLAB program to find the rise time, peak time, maximum peak overshoot, and settling time from the unit step response of the system.

```
clc
t=0:0.005:5;                  %set time vector

num_cof=[0 0 16];        %store the numerator coefficients as an array
den_cof=[1 4 16];        %store denominator coefficients as an array
[c,x,t]=step(num_cof,den_cof,t);

n=1;                          %initialize count as 1
while c(n)<1.0001;      %count the time index as along as c(t)<1
     n=n+1;
end;
```

```
rise_time=(n-1)*0.005   %rise time=(count-1)*time interval

[cmax,tp]=max(c);        %determine maximum value of c(t) &
                          %corresponding time
peak_time=(tp-1)*0.005  %peak time=(tp-1)*time interval

max_overshoot=cmax-1    %compute peak overshoot

n=1001;                    %initialize count as (5/.005)+1=1001
while c(n)>0.95&c(n)<1.05;
     n=n-1;                %count time index between c(t)>0.95&c(t)<1.05
end;
settling_time_5per_err=(n-1)*0.005

n=1001;                    %initialize count as (5/.005)+1=1001
while c(n)>0.98 & c(n)<1.02;
     n=n-1;                %count time index between c(t)>0.98&c(t)<1.02
end;
settling_time_2per_err=(n-1)*0.005
```

OUTPUT

```
rise_time =
             0.6050

peak_time =
             0.9050

max_overshoot =
             0.1630

settling_time_5per_err =
                  1.3200

settling_time_2per_err =
                  2.0150
```

PROGRAM 2.6

Consider the closed loop transfer function of the following second order system,

$$M(s)=64/(s^2+8s+64)$$

Write a MATLAB program to find the response for unit step, unit ramp and unit parabolic input signals.

```
%unit step/ramp/parabolic response

clc
num_cof=[0 0 64];
den_cof=[1 8 64];

t=0:0.005:2;

r1=t;                  %unit ramp input signal
r2=0.5*t.^2;          %unit parabolic input signal

c1=step(num_cof, den_cof,t);
c2=Lsim(num_cof, den_cof,r1,t);
c3=Lsim(num_cof, den_cof,r2,t);
```

```
plot(t,c1,'--k',t,c2,'-.k',t,c3,'-k'); grid
xlabel('time,t in sec');
ylabel('Responses,c1(t),c2(t),c3(t)');
text(0.25,1.15,'c1(t)')
text(1.45,1.5,'c2(t)')
text(1.35,0.7,'c3(t)')
```

OUTPUT

The output waveform is shown in fig p2.6.

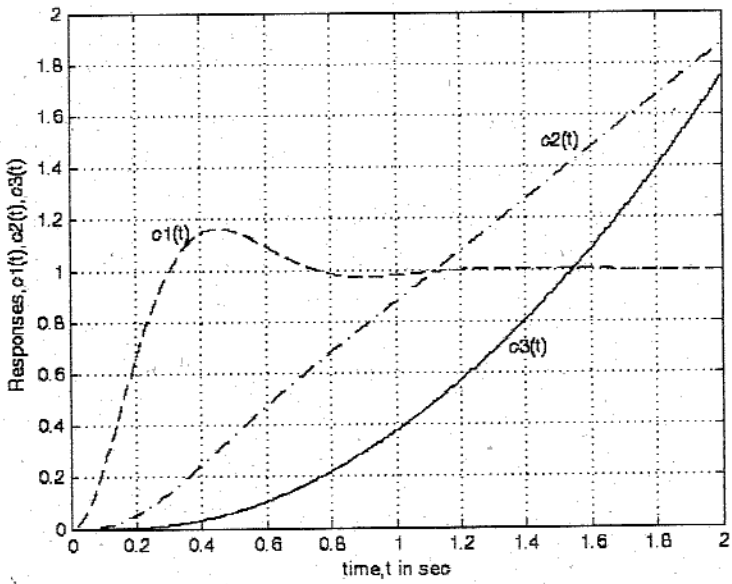


Fig P2.6 : *Step, ramp and parabolic response of system given in program 2.6.*

PROGRAM 2.7

Consider the closed loop transfer function of the following second order system,

$M(s)=5/(s^2+s+5)$

Write a MATLAB program to find the response for the input signal, $r(t)=2-2t+t^2$.

```
%program to find response for given input
clc
num_cof=[0  0  5];
den_cof=[1  1  5];
t=0:0.005:3;                          %specify a time vector

r=2-2*t+t.^2;                         %input signal

c=Lsim(num_cof,den_cof,r,t);    %compute response using Lsim function

plot(t,r,'--k',t,c,'-.k'); grid
xlabel('time,t in sec');
ylabel('Input,r(t) and output,c(t)');

text(0.25,1.65,'r(t)')
text(0.25,0.6,'c(t)')
```

OUTPUT

The output waveform is shown in fig p2.7.

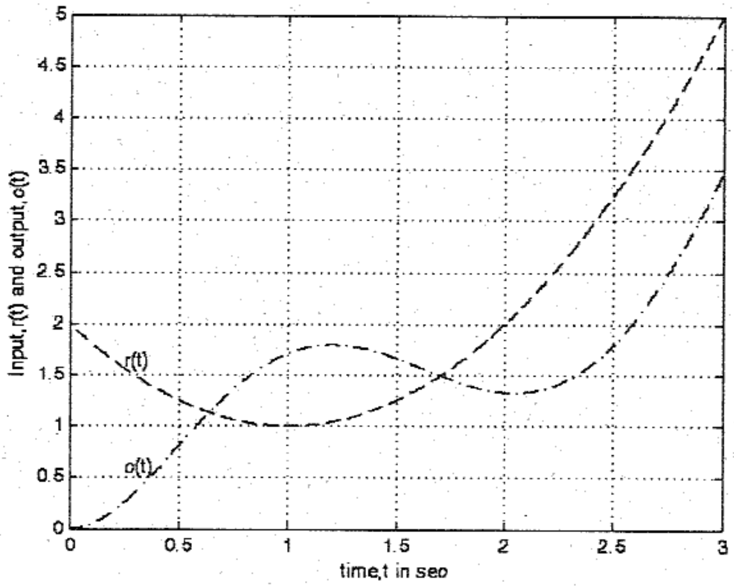


Fig P2.7 : Input and Output of the system given in program 2.7.

2.23 SHORT QUESTIONS AND ANSWERS

Q2.1 ***What is time response?***

The time response is the output of the closed loop system as a function of time. It is denoted by c(t). It is given by inverse Laplace of the product of input and transfer function of the system.

The closed loop transfer function, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$

Response in s-domain, $C(s) = \frac{R(s)\,G(s)}{1+G(s)\,H(s)}$

Response in time domain, $c(t) = \mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{R(s)\,G(s)}{1+G(s)\,H(s)}\right\}$

Q2.2 ***What is transient and steady state response?***

The transient response is the response of the system when the input changes from one state to another. The response of the system as $t \to \infty$ is called steady state response.

Q2.3 ***What is the importance of test signals?***

The test signals can be easily generated in test laboratories and the characteristics of test signals resembles, the characteristics of actual input signals. The test signals are used to predetermine the performance of the system. If the response of a system is satisfactory for a test signal, then the system will be suitable for practical applications.

Q2.4 ***Name the test signals used in control system.***

The commonly used test input signals in control system are Impulse, Step, Ramp, Acceleration and Sinusoidal signals.

Q2.5 ***Define step signal.***

The step signal is a signal whose value changes from 0 to A and remains constant at A for t > 0. The mathematical representation of step signal is,

$$r(t) = A, \; t \geq 0$$
$$= 0, \; t < 0$$

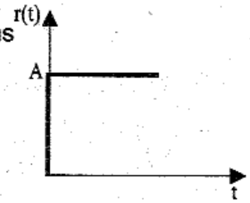


Fig Q2.5 : Step signal.

Q2.6 ***Define ramp signal.***

A ramp signal is a signal whose value increases linearly with time from an initial value of zero at t = 0. Mathematical representation of ramp signal is,

$$r(t) = At, \; t \geq 0$$
$$= 0, \; t < 0$$

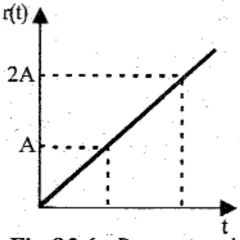


Fig Q2.6 : Ramp signal.

Q2.7 ***Define parabolic signal.***

It is a signal in which the instantaneous value varies as square of the time from an initial value of zero at t = 0. The mathematical representation of parabolic signal is,

$$r(t) = \frac{At^2}{2}, \; t \geq 0$$
$$= 0, \; t < 0$$

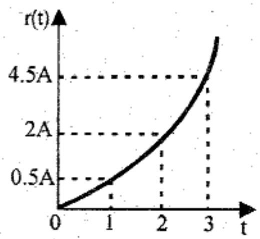


Fig Q2.4 : Parabolic signal.

Q2.8 ***What is weighing function?***

The impulse response of system is called weighing function. It is given by inverse Laplace transform of system transfer function.

Q2.9 ***What is an impulse signal?***

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all time except at t = 0. At t = 0 the magnitude becomes infinite. It is denoted by δ(t) and mathematically expressed as,

$$\delta(t) = \infty \;; \; t = 0$$
$$= 0 \;; \; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

Q2.10 ***Define pole.***

The pole of a function, F(s) is the value at which the function, F(s) becomes infinite, where F(s) is a function of complex variable s.

Q2.11 ***Define zero.***

The zero of a function, F(s) is the value at which the function, F(s) becomes zero, where F(s) is a function of complex variable s.

Q2.12 ***What is the order of a system?***

The order of the system is given by the order of the differential equation governing the system. It is also given by the maximum power of s in the denominator polynomial of transfer function. The maximum power of s also gives the number of poles of the system and so the order of the system is also given by number of poles of the transfer function.

Q2.13 ***Define damping ratio.***

The damping ratio is defined as the ratio of actual damping to critical damping.

Q2.14 ***Give the expression for damping ratio of mechanical and electrical system.***

The damping ratio of second order mechanical translational system, $\zeta = \frac{B}{2\sqrt{MK}}$

The damping ratio of second order mechanical rotational system, $\zeta = \frac{B}{2\sqrt{JK}}$

The damping ratio of second order electrical system, $\zeta = \frac{R}{2\sqrt{L/C}}$

Q2.15 ***How the system is classified depending on the value of damping?***

Depending on the value of damping, the system can be classified into the following four cases.

Case 1 : Undamped system, $\zeta = 0$

Case 2 : Underdamped system, $0 < \zeta < 1$

Case 3 : Critically damped system, $\zeta = 1$

Case 4 : Over damped system, $\zeta > 1$

Q2.16. ***Sketch the response of a second order under damped system.***

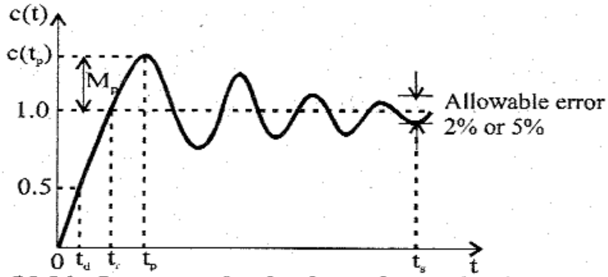


Fig Q2.16 : *Response of under damped second order system.*

Q2.17 ***What will be the nature of response of a second order system with different types of damping?***

For undamped system the response is oscillatory.
For underdamped system the response is damped oscillatory.
For critically damped system the response is exponentially rising.
For overdamped system the response is exponentially rising but the rise time will be very large.

Q2.18. ***What is damped frequency of oscillation?***

In underdamped system the response is damped oscillatory. The frequency of damped oscillation is given by, $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Q2.19. ***Give the expression for natural frequency of oscillations of electrical and mechanical system.***

The natural frequency of oscillation of second order mechanical translational system $\Big\}$ $\omega_n = \sqrt{\frac{K}{M}}$

The natural frequency of oscillation of second order mechanical rotational system $\Big\}$ $\omega_n = \sqrt{\frac{K}{J}}$

The natural frequency of oscillation of second order electrical system $\Big\}$ $\omega_n = \frac{1}{\sqrt{LC}}$

Q2.20. ***The closed loop transfer function of second order system is*** $\frac{C(s)}{R(s)} = \frac{10}{s^2+6s+10}$. ***What is the type of damping in the system?.***

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{10}{s^2+6s+10}$$

$\omega_n^2 = 10$

$\therefore \omega_n = \sqrt{10} = 3.1622 \text{ rad/sec}$

$2\zeta\omega_n = 6$

$\therefore \zeta = \frac{6}{2 \times \omega_n} = \frac{6}{2 \times \sqrt{10}} = 0.95$

Since $\zeta < 1$, the system is underdamped.

Q2.21 ***The closed loop transfer function of a second order system is given by $\frac{200}{s^2+20s+200}$. Determine the damping ratio and natural frequency of oscillation.***

Let us compare the given transfer function with the standard form of second order transfer function

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{200}{s^2+20s+200}$$

$$\therefore \omega_n^2 = 200 \qquad\qquad 2\zeta\omega_n = 20$$

$$\omega_n = \sqrt{200} = 14.14 \text{ rad/sec} \qquad\qquad \zeta = \frac{20}{2\times\omega_n} = \frac{20}{2\times 14.14} = 0.707$$

Damping ratio, $\zeta = 0.707$

Natural frequency of oscillation, $\omega_n = 14.14$ rad/sec.

Q2.22 ***A second order system has a damping ratio of 0.6 and natural frequency of oscillation is 10 rad/sec. Determine the damped frequency of oscillation.***

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-(0.6)^2} = 10\times 0.8 = 8 \text{ rad/sec}$

Q2.23 ***The open loop transfer function of a unity feedback system is $G(s) = \frac{20}{s(s+10)}$. What is the nature of response of closed loop system for unit step input.***

The closed loop transfer function,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{20/s(s+10)}{1+\frac{20}{s(s+10)}} = \frac{20}{s(s+10)+20} = \frac{20}{s^2+10s+20}$$

The standard form of second order transfer function is, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

On comparing system transfer function with standard form of second order transfer function we get,

$$\omega_n^2 = 20 \qquad\qquad 2\zeta\omega_n = 10$$

$$\therefore \omega_n = \sqrt{20} = 4.47 \text{ rad/sec} \qquad\qquad \zeta = \frac{10}{2\times\omega_n} = \frac{10}{2\times 4.47} = 1.12$$

Since damping ratio, $\zeta > 1$, the system is overdamped and the response will be exponentially rising.

Q2.24 ***List the time domain specifications.***

The time domain specifications are,

(i) Delay time (ii) Rise time (iii) Peak time

(iv) Maximum overshoot (v) Settling time.

Q2.25 ***Define delay time.***

It is the time taken for response to reach 50% of the final value, the very first time.

Q2.26 ***Define rise time.***

It is the time taken for response to raise from 0 to 100% , the very first time. For underdamped system, the rise time is calculated from 0 to 100%. But for overdamped system it is the time taken by the response to raise from 10% to 90%. For critically damped system, it is the time taken for response to raise from 5% to 95%.

Q2.27 ***Define peak time.***

It is the time taken for the response to reach the peak value, the very first time (or) It is the time taken for the response to reach peak overshoot, M_p.

Q2.28 ***Define peak overshoot.***

It is defined as the ratio of the maximum peak value to final value, where maximum peak value is measured from final value.

Let final value = $c(\infty)$, Maximum value = $c(t_p)$ $\therefore$ Peak overshoot, $M_p = \frac{c(t_p)-c(\infty)}{c(\infty)}$.

Q2.29 ***Define settling time.***

It is defined as the time taken by the response to reach and stay within a specified error and the error is usually specified as % of final value. The usual tolerable error is 2% or 5% of the final value.

Q2.30 ***The damping ratio of a system is 0.75 and the natural frequency of oscillation is 12 rad/sec. Determine the peak overshoot and the peak time.***

Peak overshoot, $M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{0.75\times\pi}{\sqrt{1-(0.75)^2}}} = 0.028$; $\therefore \%M_p = 0.028 \times 100 = 2.8\%$

Damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\zeta^2} = 12\sqrt{1-(0.75)^2} = 7.94 \text{ rad/sec}$

Peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{7.94} = 0.396 \text{ sec}$

Q2.31 ***The damping ratio of system is 0.6 and the natural frequency of oscillation is 8 rad/sec. Determine the rise time.***

Rise time, $t_r = \frac{\pi-\theta}{\omega_d}$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1}\frac{\sqrt{1-(0.6)^2}}{0.6} = 53.13^\circ = \frac{53.13}{180}\times\pi \text{ rad} = 0.927 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 8\sqrt{1-(0.6)^2} = 6.4 \text{ rad/sec}$$

$\therefore$ Rise time, $t_r = \frac{\pi-0.927}{6.4} = 0.34 \text{ sec}$

Q2.32 ***What is type number of a system? What is its significance?***

The type number is given by number of poles of loop transfer function at the origin. The type number of the system decides the steady state error.

Q2.33 ***Distinguish between type and order of a system.***

(i) Type number is specified for loop transfer function but order can be specified for any transfer function. (open loop or closed loop transfer function).

(ii) The type number is given by number of poles of loop transfer function lying at origin of s-plane but the order is given by the number of poles of transfer function.

Q2.34 ***For the system with following transfer function, determine type and order of the system.***

(i) $G(s)\,H(s) = \frac{K}{s(s+1)\,(s^2+6s+8)}$ (ii) $G(s)\,H(s) = \frac{20\,(s+2)}{s^2(s+3)\,(s+0.5)}$

(iii) $G(s)\,H(s) = \frac{(s+4)}{(s-2)\,(s+0.25)}$ (iv) $G(s)\,H(s) = \frac{10}{s^3(s^2+2s+1)}$

Ans: (i) Type - 1, order - 4 (ii) Type - 2, order - 4

(iii) Type - 0, order - 2 (iv) Type - 3, order - 5.

Q2.35 ***What is steady state error?***

The steady state error is the value of error signal e(t), when t tends to infinity . The steady state error is a measure of system accuracy. These errors arise from the nature of inputs, type of system and from non-linearity of system components.

Q2.36 ***What are static error constants?***

The K_p, K_v and K_a are called static error constants. These constants are associated with steady state error in a particular type of system and for a standard input.

Q2.37 ***Define positional error constant.***

The positional error constant $K_p = \underset{s\to 0}{Lt}\ G(s)\,H(s)$ .The steady state error in type-0 system when the input is unit step is given by$1/1+K_p$.

Q2.38 ***Define velocity error constant.***

The velocity error constant $K_v = \underset{s \to 0}{Lt}\ sG(s)\,H(s)$. The steady state error in type-1 system for unit ramp input is given by $1/K_v$.

Q2.39 ***Define acceleration error constant.***

The acceleration error constant $K_a = \underset{s \to 0}{Lt}\ s^2\,G(s)\,H(s)$. The steady state error in type-2 system for unit parabolic input is given by $1/K_a$.

Q2.40 ***A unity feedback system has a open loop transfer function of $G(s) = \frac{10}{(s+1)(s+2)}$. Determine the steady state error for unit step input.***

The steady state error for unit step input, $e_{ss} = \frac{1}{1+K_p}$, where, $K_p = \underset{s \to 0}{Lt}\ G(s)\,H(s)$.

For unity feedback system H(s) = 1.

$$\therefore K_p = \underset{s \to 0}{Lt}\ G(s) = \underset{s \to 0}{Lt}\ \frac{10}{(s+1)(s+2)} = 5 \quad \text{and} \quad e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+5} = \frac{1}{6}$$

Q2.41 ***A unity feedback system has a open loop transfer function of $G(s) = \frac{25\,(s+4)}{s(s+0.5)\,(s+2)}$. Determine the steady state error for unit ramp input.***

The steady state error for unit ramp input is, $e_{ss} = \frac{1}{K_v}$, where, $K_v = \underset{s \to 0}{Lt}\ s\,G(s)\,H(s)$. For unity feedback system H(s) = 1.

$$\therefore K_v = \underset{s \to 0}{Lt}\ sG(s) = \underset{s \to 0}{Lt}\ s\left[\frac{25\,(s+4)}{s(s+0.5)\,(s+2)}\right] = \frac{25 \times 4}{0.5 \times 2} = 100 \quad \text{and} \quad e_{ss} = \frac{1}{K_v} = \frac{1}{100} = 0.01$$

Q2.42 ***A unity feedback system has a open loop transfer function of $G(s) = \frac{20\,(s+5)}{s(s+0.1)\,(s+3)}$. Determine the steady state error for parabolic input.***

The steady state error for unit ramp input is $e_{ss} = \frac{1}{K_a}$, where, $K_a = \underset{s \to 0}{Lt}\ s^2 G(s)\,H(s)$. For unity feedback system H(s) = 1.

$$\therefore K_a = \underset{s \to 0}{Lt}\ s^2\left[\frac{20\,(s+5)}{s^2(s+0.1)\,(s+3)}\right] = \frac{20 \times 5}{0.1 \times 3} = \frac{100}{0.3} = 333.33 \quad \text{and} \quad e_{ss} = \frac{1}{K_a} = \frac{1}{333.33} = 0.003$$

Q2.43 ***What are generalized error coefficients?***

They are the coefficients of generalized error series.The generalized error series is given by,

$$e(t) = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!}\ddot{r}(t) + \frac{C_3}{3!}\dddot{r}(t) + \ldots + \frac{C_n}{n!}\overset{n}{r}(t)\ldots$$

The coefficients $C_0, C_1, C_2 \ldots C_n$ are called generalized error coefficients or dynamic error coefficients.

The n^{th} coefficient, $C_n = \underset{s \to 0}{Lt}\ \frac{d^n}{ds^n} F(s)$, where, $F(s) = \frac{1}{1 + G(s)\,H(s)}$.

Q2.44 ***Give the relation between generalized and static error coefficients.***

The following expression shows the relation between generalized and static error coefficient.

$$C_0 = \frac{1}{1+K_p}; \qquad C_1 = \frac{1}{K_v}; \qquad C_2 = \frac{1}{K_a}$$

Q2.45 ***Mention two advantages of generalized error constants over static error constants.***

(i) Generalized error series gives error signal as a function of time.

(ii) Using generalized error constants the steady state error can be determined for any type of input but static error constants are used to determine steady state error when the input is anyone of the standard input.

Q2.46. *What are the basic components of an automatic control system ?*

The basic components of an automatic control system are,

1. Error detector
2. Amplifier and controller
3. Actuator (Power actuator)
4. Plant
5. Sensor or feedback system

Q2.47 *What is automatic controller ?*

The combined unit of error detector, amplifier and controller is called automatic controller.

Q2.48 *What is the need for a controller?*

The controller is provided to modify the error signal for better control action.

Q2.49 *What are the different types of controllers?*

The different types of controller used in control system are P, PI, PD and PID controllers.

Q2.50 *What is Proportional controller and what are its advantages?*

The Proportional controller is a device that produces a control signal which is proportional to the input error signal.

The advantages in the proportional controller are improvement in steady-state tracking accuracy disturbance signal rejection and the relative stability. It also makes a system less sensitive to parameter variations.

Q2.51 *What is the drawback in P-controller?*

The drawback in P-controller is that it develop a constant steady-state error.

Q2.52 *What is integral control action?*

In integral control action, the control signal is proportional to integral of error signal.

Q2.53 *What is the advantage and disadvantage in integral controller?*

The advantage in Integral controller is that it eliminates or reduces the steady-state error. The disadvantage is that it can make a system unstable.

Q2.54 *Write the transfer function of P, PI, PD and PID controllers.*

The transfer function of P-controller, $\frac{U(s)}{E(s)} = K_p$; where, K_p = Proportional gain.

The transfer function of PI-controller, $\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}\right)$; where, T_i = Integral time constant.

The transfer function of PD-controller, $\frac{U(s)}{E(s)} = K_p(1+T_d s)$; where, T_d = Derivative time constant.

The transfer function of PID-controller, $\frac{U(s)}{E(s)} = K_p\left(1+\frac{1}{T_i s}+T_d s\right)$

Q2.55 *What is Reset rate?*

The Reset rate is the reciprocal of integral time or reset time. The reset rate is the number of times per minute that the proportional part of the control action is duplicated and it is measured in terms of repeats/minute.

Q2.56 *Why derivative control is not employed in isolation?*

A derivative control mode in isolation produces no corrective efforts for any constant errors. Because it acts only on rate of change of error.

Q2.57 *What is PI-controller?*

The PI-controller is a device which produces a control signal consisting of two terms : *one proportional to error signal and the other proportional to the integral of error signal.*

Q2.58 What is PD-controller?

The PD-controller is a device which produces a control signal consisting of two terms : *one proportional to error signal and the other proportional to the derivative of error signal.*

Q2.59 What is PID-Controller?

The PID-controller is a device which produces a control signal consisting of three terms : *one proportional to error signal, another one proportional to integral of error signal and the third one proportional to derivative of error signal.*

Q2.60 Give an example of electronic PID-controller

The electronic PID-controller can be realized by an op-amp amplifier with integral and derivative action followed by sign changer, as shown in figure Q2.60.

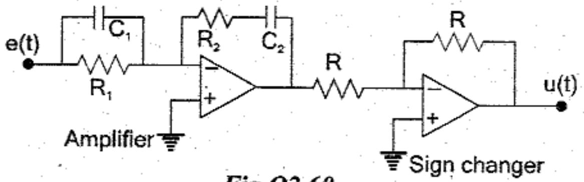


Fig Q2.60

Q2.61 Sketch the step response of a P and PI-controller ?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.61.

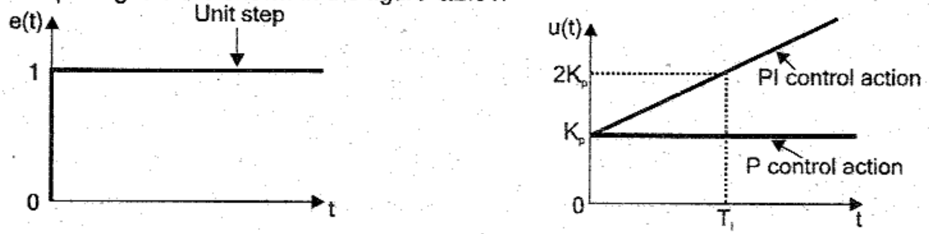


Fig Q2.61

Q2.62 Sketch the ramp response of P, PD and PID-controller?

Let e(t) be the input signal to the controller and u(t) be the output signal to the controller. The input and output signals are shown in the figure Q2.62.

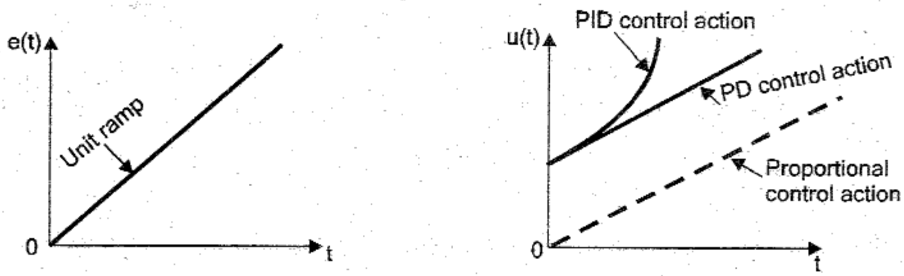


Fig Q2.62

Q2.63 What is the effect on system performance when a proportional controller is introduced in a system?

The proportional controller improves the steady-state tracking accuracy, disturbance signal rejection and relative stability of the system. It also increases the loop gain of the system which results in reducing the sensitivity of the system to parameter variations.

Q2.64 What is the disadvantage in proportional controller?

The disadvantage in proportional controller is that it produces a constant steady state error.

Q2.65 ***What is the effect of PI-controller on the system performance?***

The PI - controller increases the order of the system by one, which results in reducing, the steady state error. But the system becomes less stable than the original system.

Q2.66 ***What is the effect of PD-controller on the system performance?***

The effect of PD - controller is to increase the damping ratio of the system and so the peak overshoot is reduced.

Q2.67 ***Why derivative controller is not used in control systems?***

The derivative controller produces a control action based on rate of change of error signal and it does not produce corrective measures for any constant error. Hence derivative controller is not used in control systems.

Q2.68 ***Determine the impulse response of the feedback system governed by the closed loop transfer function,*** $M(s) = \frac{2s+1}{(s+1)^2}$.

By partial fraction expansion the given closed loop transfer function can be expressed as,

$$\therefore M(s) = \frac{2s+1}{(s+1)^2} = \frac{A}{(s+1)^2} + \frac{B}{s+1}$$

$$A = \frac{2s+1}{(s+1)^2} \times (s+1)^2 \bigg|_{s=-1} = 2s+1\big|_{s=-1} = 2(-1)+1 = -1$$

$$B = \frac{d}{ds}\left[\frac{2s+1}{(s+1)^2} \times (s+1)^2\right]\bigg|_{s=-1} = \frac{d}{ds}[2s+1]\bigg|_{s=-1} = 2$$

$$\therefore M(s) = \frac{-1}{(s+1)^2} + \frac{2}{(s+1)}$$

The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{-1}{(s+1)^2} + \frac{2}{s+1}\right\} = -t\,e^{-t} + 2\,e^{-t}$$

Q2.69 ***Determine the impulse response of the feedback systems governed by the following closed loop transfer functions,***

a) $M(s) = \frac{s}{s+1}$; **b)** $M(s) = \frac{1}{s^2+1}$.

a) The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{s}{s+1}\right\} = \mathcal{L}^{-1}\left\{1 - \frac{1}{s+1}\right\} = \delta(t) - e^{-t}$$

b) The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

Q2.70 ***Determine the impulse response of the feedback system governed by the closed loop transfer function,*** $M(s) = \frac{2(s+3)}{(s+3)^2+1}$.

The impulse response is given by inverse Laplace transform of closed loop transfer function.

$$\therefore \text{Impulse response, } m(t) = \mathcal{L}^{-1}\left\{\frac{2(s+3)}{(s+3)^2+1}\right\} = 2\,e^{-3t}\cos t$$

2.24 EXERCISES

E2.1 What is the unit-step response of the system shown in fig E2.1

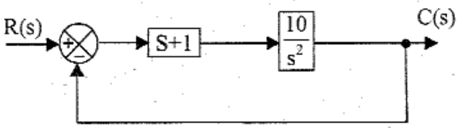


Fig E2.1.

E2.2 Obtain the unit-step response of a unity-feedback system

whose open-loop transfer function is $G(s)=\dfrac{5(s+20)}{s(s+4.59)\,(s^2+3.41s+16.35)}$.

E2.3 The open loop transfer function of an unity feedback control system is given by $G(s)=\dfrac{100}{s(s+2)\,(s+5)}$.

For unit step input,find the time response of the closed loop system and determine % over shoot and the rise time.

E2.4 A Servomechanism has its moment of inertia J = 10×10^{-6} Kg-m², retarding friction, B = 400×10^{-6} N-m/(rad/sec) and elasticity coefficient, K = 0.004 N-m/rad. Find the natural frequency and damping factor of the system.

E2.5 For a second order system whose open loop transfer function $G(s)=\dfrac{4}{s(s+2)}$, determine the maximum over shoot, the time to reach the maximum overshoot when a step displacement of 18° is given to the system. Find the rise time, time constant and the settling time for an error of 7% .

E2.6 Consider the unity feedback closed loop system where the forward transfer function is $G(s)=\dfrac{25}{s(s+5)}$.

Obtain the rise time, Peak time, Maximum overshoot and the settling time when the system is subjected to a unit-step input.

E2.7 Consider the system shown in fig E2.7, where $\zeta = 0.6$ and $\omega_n = 0.5$ rad/sec. Determine the rise time, peak time, maximum overshoot and settling time, when the system is subjected to a unit-step input.

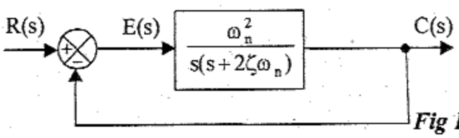


Fig E2.7.

E2.8 For the system shown in fig E2.8, determine the values of K and K_h so that the maximum overshoot in the unit step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain rise time and settling time.

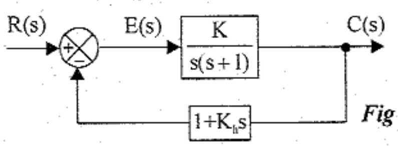


Fig E2.8.

E2.9 The system shown in fig E2.9 subjected to a unit-step input. Determine the values of K and T, where the Maximum overshoot of the system is 25.4% corresponding to $\zeta = 0.4$.

E2.10 Determine the values of K and T of the closed-loop system shown in Fig E2.10, so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that J=1 Kg-m².

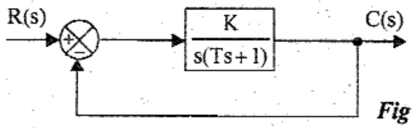


Fig E2.9.

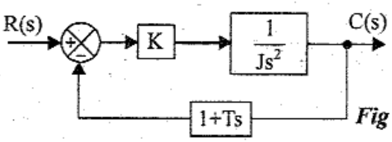


Fig E2.10.

E2.11 A unity-feedback system is characterized by the open-loop transfer function $G(s)=\frac{1}{s(0.5s+1)\,(0.2s+1)}$.

a) Determine the steady-state errors to unit-step, unit-ramp and unIt parabolic inputs.

b) Determine rise time, peak time, peak overshoot and settling time of the unit-step respone of the system.

E2.12 For a system whose $G(s)=\frac{10}{s(s+1)\,(s+2)}$, find the steady state error when it is subjected to the input, $r(t) = 1+2t +1.5\, t^2$.

E2.13 A unity feedback system has $G(s)=\frac{1}{s(1+s)}$. The input to the system is described by $r(t) = 4+6t+2t^3$. Find the generalized error coefficients and steady state error.

E2.14 A unity feedback system has the forward path transfer function $G(s)=\frac{10}{(s+1)}$.

Find the steady state error and the generalised error coefficient for $r(t) = t$.

E2.15 Find out the position, velocity and acceleration error coefficients for the following unity feedback systems having forward loop transfer function G(s) as,

(a) $\frac{100}{(1+0.5s)\,(1+2s)}$ (b) $\frac{K}{s(1+0.1s)\,(1+s)}$

(c) $\frac{K}{s^2(s^2+8s+100)}$ (d) $\frac{K(1+s)\,(1+2s)}{s^2(s^2+4s+20)}$

E2.16 The open loop transfer function of a unity feedback control system is $G(s) = 9/(s+1)$, using the generalized error series determine the error signal and steady state error of the system when the system is excited by,

(i) $r(t)=2$ (ii) $r(t)=t$

(iii) $r(t) = 3t^2/2$ (iv) $r(t) = 1+2t +3t^2/2$

E2.17 For unity feedback system having open loop transfer function as $G(s)=\frac{K(s+2)}{s^2(s^2+7s+12)}$. Determine,

(i) type of system,

(ii) error constants K_p, K_v and K_a

(iii) steady state error for parabolic input.

ANSWER FOR EXERCISE PROBLEMS

E2.1 $c(t) = -1.1455\, e^{-8.87t} + 0.1455 e^{-1.13t} + 1$

E2.2 $c(t) = 1 + \frac{3}{8} e^{-t} \cos 3t - \frac{17}{24} e^{-t} \sin 3t - \frac{11}{8} e^{-3t} \cos t - \frac{13}{8} e^{-3t} \sin t$

E2.3 $c(t) = \left[1 - 0.186\, e^{-7.45t} - 0.88\, e^{0.225t} \cos(3.65t - 22°)\right]$

As t tends to infinity, c(t) tends to infinity and so the system is unstable. Therefore % over shoot and rise time are not defined.

E2.4 Natural frequency, $\omega_n = 20$ rad/sec, Damping factor, $\zeta = 1$.

E2.5 Maximum overshoot = 0.16, when input is 18%, $M_p = 2.88\%$
Peak time, t_p=1.81sec. Rise time, t_r=1.21 sec
Time constant, T=1 sec, Settling time for 7% error = 2.66 sec.

E2.6 Rise time, t_r = 0.55sec, %Peak overshoot, M_p = 9.5%
Peak time, t_p = 0.785sec, Settling time, t_s = 1.33sec (for 2% error); t_s =1 sec(for 5% error)

E2.7 Rise time, t_r = 0.55sec, Maximum overshoot, M_p = 0.095
Peak time, t_p = 0.785sec, Settling time, t_s = 1 sec (for 5% criterion)

E2.8 K=12.5, Rise time, t_r = 0.65sec
K_h = 0.178 ; Settling time, t_s = 2.48 sec (for 2% error) ; t_s = 1.86 sec (for 5% error)

E2.9 K=1.42, T=1.09

E2.10 K=2.95 N-m T=0.471 sec

E2.10 (a) $e_{ss}\big|_{\text{unit step}} = 0$
$e_{ss}\big|_{\text{unit ramp}} = 1$
$e_{ss}\big|_{\text{unit parabola}} = \infty$

(b) Rise time, $t_r = 1.91$ sec
Peak time, $t_p = 2.79$ sec
Peak overshoot, $M_p = 0.1265$
Settling time, $t_s = 5.4$ sec

E2.12 The total steady state error is ∞.

E2.13 $C_0 = 0$; $C_1 = 1$; $C_2 = 0$; $C_3 = -6$; $e_{ss} = \infty$

E2.14 C_0=1/11; C_1=10/121; $e_{ss} = \infty$.

E2.15

Question	K_p	K_v	K_a
(a)	100	0	0
(b)	∞	K	0
(c)	∞	∞	K/100
(d)	∞	∞	K/20

E2.16 (i) e(t) = 0.2 ; e_{ss} = 0.2
(ii) e(t) = 0.1t+0.09 ; $e_{ss} = \infty$.
(iii) $e(t) = 0.15t^2 + 0.27\,t - 0.054$; $e_{ss} = \infty$.
(iv) $e(t) = 0.15t^2 + 0.77\,t + 0.226$; $e_{ss} = \infty$.

E2.17 (i) It is type-2 system
(ii) $K_p = \infty$; $K_v = \infty$; K_a = K/6
(iii) e_{ss} = 6/K